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THE RADON-NIKODYM THEOREM FOR NON-COMMUTATIVE L^p-SPACES

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Abstract. Let $\mathcal{M}$ be a von Neumann algebra. We will show that for two normal semifinite faithful weights φ, ψ on $\mathcal{M}$, the corresponding non-commutative L^p-spaces $L^p(\mathcal{M},\varphi)$ and $L^p(\mathcal{M},\psi)$ are isometrically isomorphic.

Regarding a von Neumann algebra $\mathcal{M}$ and its predual $\mathcal{M}_*$ as a non-commutative version of L^∞-space and L^1-space, respectively, the author [5] interpolated the above two Banach spaces by applying Calderón's complex method [1, 2] and obtained non-commutative L^p-spaces $L^p_{(\alpha)}(\mathcal{M},\varphi)$, $1 < p < \infty$, parametrized by a complex number α arising from the modular action of a normal semifinite faithful weight φ on $\mathcal{M}$. This construction includes both Kosaki's one ([7]), which is equivalent to our case where φ is a state and $\alpha = \pm 1/2$ and Terp's one ([10]), which is equivalent to our case where $\alpha = 0$ and φ is possibly unbounded, namely a weight (cf. [5, Remark in p.1036]).

The weight φ plays a rôle similar to a measure in the commutative case. The classical Radon-Nikodým theorem tells us that the L^p-spaces for two mutually absolutely continuous measures on a measure space are mutually isometrically isomorphic. Indeed, the isomorphism is given by the multiplication by a suitable power of the Radon-Nikodým derivative.

In this paper, we will prove the non-commutative analogue of the Radon-Nikodým theorem: for any given two n.s.f. weights φ, ψ on $\mathcal{M}$, we will construct a natural isometric map between the corresponding L^p-spaces. In the case where φ, ψ are states, Kosaki tried to construct such an isometric map [7, Theorem 4.4]. His map essentially consists of the multiplication by Connes' Radon-Nikodým cocycles, the non-commutative analogue of Radon-Nikodým derivative. To realize this, he first considers "reiterated" compatible pair of L^2- and L^1-spaces and define the isomorphic map between L^p-spaces ($1 < p < 2$) as the evaluation map of isomophism between the two function spaces arising form the reiterated pairs for φ and ψ, and

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by using duality between L^p- and L^q-spaces $(1/p + 1/q = 1)$ isomorphisms for all p, $1 < p < \infty$, are obtained. His idea is clear and reasonable enough, but it is often hard to obtain analytic elements for the Radon-Nikodým derivative enough to show that the evaluation maps are well-defined, unless good conditions are posed on the states φ and ψ. In order to avoid this difficulty, we will make use of Connes' trick of 2×2 matrices and bimodule actions established in [6], and obtain the desired map directly, without recourse to reiteration. Note that our L^p-spaces corresponding to the weights φ and ψ are isometrically isomorphic to Haagerup's universal one [4], as is mentioned in [5, p.1059 l.2 from bottom and Theorem 3.8], and hence isomorphic to each other, but it is much more desirable to construct isomorphisms in a more explicit way and independently of Haagerup's result.

We briefly describe the construction of L^p-spaces [5]. First, we sketch the modular theory (for details, see [8, 9]). Let $\mathcal{M}$ be a von Neumann algebra and φ an n.s.f. weight on $\mathcal{M}$. Let $\{\pi_\varphi, \mathfrak{n}_\varphi, \Lambda_\varphi\}$ be the semi-cyclic representation induced from $(\mathcal{M}, \varphi)$. We define the associated left Hilbert algebra $\mathfrak{A}_\varphi$ by

$$\mathfrak{A}_\varphi = \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*.$$

Next, we define an anti-linear operator S_0 on $\mathfrak{A}_\varphi$ by

$$S_0 \Lambda_\varphi(x) = \Lambda_\varphi(x^*), \ x \in \mathfrak{A}_\varphi.$$

Then S_0 is preclosed. Let S be the closure of S_0, and $S = J_\varphi \Delta_\varphi$ be its polar decomposition.

Then by [9, Chapter VI, Theorem 1.19], we have

$$\Delta_\varphi^{it} \pi_\varphi(\mathcal{M}) \Delta_\varphi^{-it} = \pi_\varphi(\mathcal{M}), \ t \in \mathbb{R},$$

and hence we can define a one-parameter automorphism group $\{\sigma_t^\varphi\}_{t \in \mathbb{R}}$ on $\mathcal{M}$ by $\pi_\varphi(\sigma_t^\varphi(x)) = \Delta_\varphi^{it} \pi_\varphi(x) \Delta_\varphi^{-it}$, $x \in \mathcal{M}$, $t \in \mathbb{R}$. It can be extended to a *complex* one-parameter automorphism group on $\mathfrak{a}_0^\varphi$, where $\mathfrak{a}_0^\varphi = \Lambda_\varphi^{-1}(\mathfrak{A}_0^\varphi)$, $\mathfrak{A}_0^\varphi = \{\xi \in \bigcap_{n=-\infty}^{\infty} \mathcal{D}(\Delta_\varphi^n) \mid \Delta_\varphi^n \xi \in \mathfrak{A}_\varphi, \ n \in \mathbb{Z}\}$ ($\mathcal{D}(T)$ means the domain of a linear operator T, and $\mathfrak{A}_0^\varphi$ is called the full Tomita algebra).

For $\alpha \in \mathbb{C}$, we put

$$L_{(\alpha)}^\varphi = \left\{ x \in \mathcal{M} \;\middle|\; \begin{array}{l} \text{there exist a unique } \varphi_x^{(\alpha)} \in \mathcal{M}_* \text{ such that} \\ \varphi_x^{(\alpha)}(y^* z) = (\pi_\varphi(x) J_\varphi \Delta_\varphi^{\overline{\alpha}} \Lambda_\varphi(y) | J_\varphi \Delta_\varphi^{-\alpha} \Lambda_\varphi(z)) \\ \text{for all } y, z \in \mathfrak{a}_0^\varphi \end{array} \right\}.$$

We define two maps $i_{(\alpha)}^\varphi : L_{(\alpha)}^\varphi \to \mathcal{M}$ and $j_{(\alpha)}^\varphi : L_{(\alpha)}^\varphi \to \mathcal{M}_*$ by $i_{(\alpha)}^\varphi(x) = x$, $j_{(\alpha)}^\varphi(x) = \varphi_x^{(\alpha)}$ for $x \in L_{(\alpha)}^\varphi$, and together with their adjoint maps, we define a compatible pair $(\mathcal{M}, \mathcal{M}_*)_{(\alpha)}^\varphi$ by Figure 1. Then we apply Calderón's complex interpolation method to the pair $(\mathcal{M}, \mathcal{M}_*)_{(\alpha)}^\varphi$, and obtain a non-commutative L^p-space $L_{(\alpha)}^p(\mathcal{M}, \varphi)$ as the interpolation spaces $C_{1/p}(\mathcal{M}, \mathcal{M}_*)_{(\alpha)}^\varphi$.

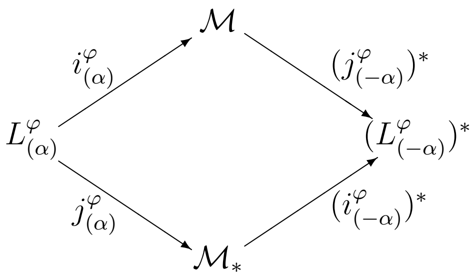

FIGURE 1. a compatible pair $(\mathcal{M}, \mathcal{M}_*)^{\varphi}_{(\alpha)}$

Next, we explain the theory of balanced weight (see [8, §3] for details). Let φ and ψ be two n.s.f. weights on $\mathcal{M}$. We consider the balanced weight χ on $\mathcal{N} = M_2(\mathbb{C}) \otimes \mathcal{M}$ by

$$\chi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \varphi(a) + \psi(d), \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{N}_+.$$

Then χ is an n.s.f. weight on $\mathcal{N}$. Since

$$\chi \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \varphi(a^*a) + \psi(b^*b) + \varphi(c^*c) + \psi(d^*d),$$

we have

$$\mathfrak{n}_\chi = \begin{pmatrix} \mathfrak{n}_\varphi & \mathfrak{n}_\psi \\ \mathfrak{n}_\varphi & \mathfrak{n}_\psi \end{pmatrix}$$

and the standard Hilbert space $\mathcal{H}_\chi$ is canonically identified with $\mathcal{H}_\varphi \oplus \mathcal{H}_\psi \oplus \mathcal{H}_\varphi \oplus \mathcal{H}_\psi$ via the map

$$\Lambda_\chi \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \Lambda_\varphi(a) \\ \Lambda_\psi(b) \\ \Lambda_\varphi(c) \\ \Lambda_\psi(d) \end{pmatrix}, \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{n}_\chi.$$

Under this identification, J_χ, Δ_χ and π_χ are described as follows:

$$J_\chi = \begin{pmatrix} J_\varphi & 0 & 0 & 0 \\ 0 & 0 & J_{\psi,\varphi} & 0 \\ 0 & J_{\varphi,\psi} & 0 & 0 \\ 0 & 0 & 0 & J_\psi \end{pmatrix}, \tag{1}$$

$$\Delta_\chi = \begin{pmatrix} \Delta_\varphi & 0 & 0 & 0 \\ 0 & \Delta_{\varphi,\psi} & 0 & 0 \\ 0 & 0 & \Delta_{\psi,\varphi} & 0 \\ 0 & 0 & 0 & \Delta_\psi \end{pmatrix}, \tag{2}$$

$$\pi_\chi(x) = \begin{pmatrix} \pi_\varphi(x_{11}) & 0 & \pi_\varphi(x_{12}) & 0 \\ 0 & \pi_\psi(x_{11}) & 0 & \pi_\psi(x_{12}) \\ \pi_\varphi(x_{21}) & 0 & \pi_\varphi(x_{22}) & 0 \\ 0 & \pi_\psi(x_{21}) & 0 & \pi_\psi(x_{22}) \end{pmatrix}, \ x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \mathcal{N}. \tag{3}$$

By [8, (3.16)], we have

$$J_{\varphi,\psi}\pi_\psi(a)J_{\psi,\varphi} = J_\varphi\pi_\varphi(a)J_\varphi, \tag{4}$$

$$J_\psi\pi_\psi(a)J_{\psi,\varphi} = J_{\psi,\varphi}\pi_\varphi(a)J_\varphi \tag{5}$$

for $a \in \mathcal{M}$. Since, for $t \in \mathbb{R}$,

$$\begin{aligned} &\pi_\chi(\sigma_t^\chi(x)) \\ &= \Delta_\chi^{it}\pi_\chi(x)\Delta_\chi^{-it} \\ &= \begin{pmatrix} \Delta_\varphi^{it}\pi_\varphi(x_{11})\Delta_\varphi^{-it} & 0 & \Delta_\varphi^{it}\pi_\varphi(x_{12})\Delta_{\psi,\varphi}^{-it} & 0 \\ 0 & \Delta_{\varphi,\psi}^{it}\pi_\psi(x_{11})\Delta_{\varphi,\psi}^{-it} & 0 & \Delta_{\varphi,\psi}^{it}\pi_\psi(x_{12})\Delta_\psi^{-it} \\ \Delta_{\psi,\varphi}^{it}\pi_\varphi(x_{21})\Delta_\varphi^{-it} & 0 & \Delta_{\psi,\varphi}^{it}\pi_\varphi(x_{22})\Delta_{\psi,\varphi}^{-it} & 0 \\ 0 & \Delta_\psi^{it}\pi_\psi(x_{21})\Delta_{\varphi,\psi}^{-it} & 0 & \Delta_\psi^{it}\pi_\psi(x_{22})\Delta_\psi^{-it} \end{pmatrix} \end{aligned}$$

belongs to $\mathcal{N}$, equations (4) and (5) yield

$$J_{\varphi,\psi}\Delta_{\varphi,\psi}^{it}\pi_\psi(a)\Delta_{\varphi,\psi}^{-it}J_{\psi,\varphi} = J_\varphi\Delta_\varphi^{it}\pi_\varphi(a)\Delta_\varphi^{-it}J_\varphi, \tag{6}$$

$$J_\psi\Delta_\psi^{it}\pi_\psi(a)\Delta_{\varphi,\psi}^{-it}J_{\psi,\varphi} = J_{\psi,\varphi}\Delta_{\psi,\varphi}^{it}\pi_\varphi(a)\Delta_\varphi^{-it}J_\varphi. \tag{7}$$

Since J_χ and Δ_χ^{it} commute, from (1) and (2) we have

$$\Delta_{\psi,\varphi}^{it}J_{\varphi,\psi}\pi_\psi(a)J_{\psi,\varphi}\Delta_{\psi,\varphi}^{-it} = \Delta_\varphi^{it}J_\varphi\pi_\varphi(a)J_\varphi\Delta_\varphi^{-it}, \tag{8}$$

$$\Delta_\psi^{it}J_\psi\pi_\psi(a)J_{\psi,\varphi}\Delta_{\psi,\varphi}^{-it} = \Delta_{\varphi,\psi}^{it}J_{\psi,\varphi}\pi_\varphi(a)J_\varphi\Delta_\varphi^{-it}. \tag{9}$$

Next, we examine the relationship between comptible pairs $(\mathcal{N},\mathcal{N}_*)_{(\alpha)}^\chi$, $(\mathcal{M},\mathcal{M}_*)_{(\alpha)}^\varphi$ and $(\mathcal{M},\mathcal{M}_*)_{(\alpha)}^\psi$. Note that, $\mathcal{N}_*$ can be identified with $M_2(\mathbb{C}) \otimes \mathcal{M}_*$ via

$$\left\langle \begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{pmatrix}, \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \right\rangle_{\mathcal{N}_*,\mathcal{N}} = \sum_{i,j=1}^{2} \langle \kappa_{ij}, x_{ij} \rangle_{\mathcal{M}_*,\mathcal{M}}$$

for $\kappa_{ij} \in \mathcal{M}_*$ and $x_{ij} \in \mathcal{M}$. Moreover, we put

$$\mathfrak{a}_0^{\varphi,\psi} = \left\{ x \in \mathfrak{n}_\psi \mid \Lambda_\psi(x) \in \bigcap_{n\in\mathbb{Z}} \mathcal{D}(\Delta_{\varphi,\psi}^n) \right\}$$

and

$$\mathfrak{a}_0^{\psi,\varphi} = \left\{ x \in \mathfrak{n}_\varphi \mid \Lambda_\varphi(x) \in \bigcap_{n\in\mathbb{Z}} \mathcal{D}(\Delta_{\psi,\varphi}^n) \right\}.$$

Then we can express the full Tomita algebra $\mathfrak{a}_0^\chi$ as follows.

$$\mathfrak{a}_0^\chi = \left\{ a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathfrak{n}_\chi \cap \mathfrak{n}_\chi^* \,\middle|\, \Lambda_\chi(a) \in \bigcap_{n\in\mathbb{Z}} \mathcal{D}(\Delta_\chi^n) \right\}$$

$$= \begin{pmatrix} \mathfrak{a}_0^{\varphi} & \mathfrak{a}_0^{\varphi,\psi} \\ \mathfrak{a}_0^{\psi,\varphi} & \mathfrak{a}_0^{\psi} \end{pmatrix} \tag{10}$$

Finally, we define

$$L_{(\alpha)}^{\psi,\varphi} = \left\{ x \in \mathcal{M} \;\middle|\; \begin{array}{l} \text{there exist a unique } (\psi\varphi)_x^{(\alpha)} \in \mathcal{M}_* \text{ such that} \\ (\psi\varphi)_x^{(\alpha)}(y^*z) = (\pi_\varphi(x)J_{\varphi,\psi}\Delta_{\varphi,\psi}^{\overline{\alpha}}\Lambda_\psi(y)|J_\varphi\Delta_\varphi^{-\alpha}\Lambda_\varphi(z)) \\ \text{for all } y \in \mathfrak{a}_0^{\varphi,\psi},\ z \in \mathfrak{a}_0^{\varphi} \end{array} \right\}$$

and put $L_{(\alpha)}^{\varphi,\psi}$ in a symmetric way.

Lemma 1.

(i) *For* $y,\ z \in \mathfrak{a}_0^{\psi,\varphi}$*, we have* $y^*z \in L_{(\alpha)}^{\varphi}$ *and*

$$\varphi_{y^*z}^{(-\alpha)}(x) = (\pi_\psi(x)J_{\psi,\varphi}\Delta_{\psi,\varphi}^{\overline{\alpha}}\Lambda_\varphi(y)|J_{\psi,\varphi}\Delta_{\psi,\varphi}^{-\alpha}\Lambda(z)),\ x \in \mathcal{M}.$$

(ii) *If* $a \in L_{(\alpha)}^{\psi,\varphi}$*, then* $a^* \in L_{(\overline{\alpha})}^{\varphi,\psi}$ *and* $(\varphi\psi)_{a^*}^{(\overline{\alpha})} = (\psi\varphi)_a^{(\alpha)*}$.

Proof. (i) Since $y,\ z \in \mathfrak{a}_0^{\varphi}$, we find that $y^*z \in L_{(\alpha)}^{\varphi}$ and

$$\varphi_{y^*z}^{(it)}(x) = (\pi_\varphi(x)J_\varphi\Delta_\varphi^{it}\Lambda_\varphi(y)|J_\varphi\Delta_\varphi^{it}\Lambda(z)),\ x \in \mathcal{M} \tag{11}$$

for all $t \in \mathbb{R}$ (replace $\mathfrak{a}_0^{\varphi}$ by $\mathfrak{n}_\varphi^*\mathfrak{n}_\varphi$ in [5, Proposition 2.3], see also [5, Remark in p. 1037]). On the other hand, for $a,\ b \in \mathfrak{a}_0^{\varphi}$, we have

$$\begin{aligned} &(\pi_\varphi(y^*z)J_\varphi\Delta_\varphi^{-it}\Lambda_\varphi(a)|J_\varphi\Delta_\varphi^{-it}\Lambda(b)) \\ &= \varphi_{y^*z}^{(it)}(a^*b) \text{ (by the definition of } L_{(\alpha)}^{\varphi}) \\ &= (\pi_\varphi(a^*b)J_\varphi\Delta_\varphi^{it}\Lambda_\varphi(y)|J_\varphi\Delta_\varphi^{it}\Lambda(z)) \quad \text{(by (11))} \\ &= (\pi_\psi(a^*b)J_{\psi,\varphi}\Delta_{\psi,\varphi}^{it}\Lambda_\varphi(y)|J_{\psi,\varphi}\Delta_{\psi,\varphi}^{it}\Lambda(z)) \quad \text{(by (8))}. \end{aligned}$$

By analytic continuation, we have

$$(\pi_\varphi(y^*z)J_\varphi\Delta_\varphi^{-\overline{\alpha}}\Lambda_\varphi(a)|J_\varphi\Delta_\varphi^{\alpha}\Lambda(b)) = (\pi_\psi(a^*b)J_{\psi,\varphi}\Delta_{\psi,\varphi}^{\overline{\alpha}}\Lambda_\varphi(y)|J_{\psi,\varphi}\Delta_{\psi,\varphi}^{-\alpha}\Lambda(z)).$$

This means that $y^*z \in L_{(-\alpha)}^{\varphi}$ and

$$\varphi_{y^*z}^{(-\alpha)}(x) = (\pi_\psi(x)J_{\psi,\varphi}\Delta_{\psi,\varphi}^{\overline{\alpha}}\Lambda_\varphi(y)|J_{\psi,\varphi}\Delta_{\psi,\varphi}^{-\alpha}\Lambda(z)),\ x \in \mathcal{M}.$$

(ii) The assertion also follows from the analytic continuation of equation (9), so the details will be omitted. □

Lemma 2. *For the balanced weight* χ *of* φ *and* ψ*, we have*

$$L_{(\alpha)}^{\chi} = \begin{pmatrix} L_{(\alpha)}^{\varphi} & L_{(\alpha)}^{\psi,\varphi} \\ L_{(\alpha)}^{\varphi,\psi} & L_{(\alpha)}^{\psi} \end{pmatrix}$$

and

$$\chi_a^{(\alpha)} = \begin{pmatrix} \varphi_{a_{11}}^{(\alpha)} & (\psi\varphi)_{a_{12}}^{(\alpha)} \\ (\varphi\psi)_{a_{21}}^{(\alpha)} & \psi_{a_{22}}^{(\alpha)} \end{pmatrix}$$

for $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in L^{\chi}_{(\alpha)}$.

Proof. Let $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in L^{\chi}_{(\alpha)}$. For any

$$y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}, \quad z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \in \mathfrak{a}_0^{\chi},$$

we have

$$\begin{aligned}
&\chi_a^{(\alpha)}(y^*z) \\
&= (\pi_\chi(a)J_\chi\Delta_\chi^{\overline{\alpha}}\Lambda_\chi(y)|J_\chi\Delta_\chi^{-\alpha}\Lambda_\chi(z)) \\
&= \left(\begin{pmatrix} \pi_\varphi(a_{11})J_\varphi\Delta_\varphi^{\overline{\alpha}}\Lambda_\varphi(y_{11}) + \pi_\varphi(a_{12})J_{\varphi,\psi}\Delta_{\varphi,\psi}^{\overline{\alpha}}\Lambda_\psi(y_{12}) \\ \pi_\psi(a_{11})J_{\psi,\varphi}\Delta_{\psi,\varphi}^{\overline{\alpha}}\Lambda_\varphi(y_{21}) + \pi_\psi(a_{12})J_\psi\Delta_\psi^{\overline{\alpha}}\Lambda_\psi(y_{22}) \\ \pi_\varphi(a_{21})J_\varphi\Delta_\varphi^{\overline{\alpha}}\Lambda_\varphi(y_{11}) + \pi_\varphi(a_{22})J_{\varphi,\psi}\Delta_{\varphi,\psi}^{\overline{\alpha}}\Lambda_\psi(y_{12}) \\ \pi_\psi(a_{21})J_{\psi,\varphi}\Delta_{\psi,\varphi}^{\overline{\alpha}}\Lambda_\varphi(y_{21}) + \pi_\psi(a_{22})J_\psi\Delta_\psi^{\overline{\alpha}}\Lambda_\psi(y_{22}) \end{pmatrix} \middle| \begin{pmatrix} J_\varphi\Delta_\varphi^{-\alpha}\Lambda_\varphi(z_{11}) \\ J_{\psi,\varphi}\Delta_{\psi,\varphi}^{-\alpha}\Lambda_\varphi(z_{21}) \\ J_{\varphi,\psi}\Delta_{\varphi,\psi}^{-\alpha}\Lambda_\psi(z_{12}) \\ J_\psi\Delta_\psi^{-\alpha}\Lambda_\psi(z_{22}) \end{pmatrix} \right) \\
&= (\pi_\varphi(a_{11})J_\varphi\Delta_\varphi^{\overline{\alpha}}\Lambda_\varphi(y_{11})|J_\varphi\Delta_\varphi^{-\alpha}\Lambda_\varphi(z_{11})) \\
&\quad +(\pi_\psi(a_{11})J_{\psi,\varphi}\Delta_{\psi,\varphi}^{\overline{\alpha}}\Lambda_\varphi(y_{21})|J_{\psi,\varphi}\Delta_{\psi,\varphi}^{-\alpha}\Lambda_\varphi(z_{21})) \\
&\quad +(\pi_\varphi(a_{12})J_{\varphi,\psi}\Delta_{\varphi,\psi}^{\overline{\alpha}}\Lambda_\psi(y_{12})|J_\varphi\Delta_\varphi^{-\alpha}\Lambda_\varphi(z_{11})) \\
&\quad +(\pi_\psi(a_{12})J_\psi\Delta_\psi^{\overline{\alpha}}\Lambda_\psi(y_{22})|J_{\psi,\varphi}\Delta_{\psi,\varphi}^{-\alpha}\Lambda_\varphi(z_{21})) \\
&\quad +(\pi_\varphi(a_{21})J_\varphi\Delta_\varphi^{\overline{\alpha}}\Lambda_\varphi(y_{11})|J_{\varphi,\psi}\Delta_{\varphi,\psi}^{-\alpha}\Lambda_\psi(z_{12})) \\
&\quad +(\pi_\psi(a_{21})J_{\psi,\varphi}\Delta_{\psi,\varphi}^{\overline{\alpha}}\Lambda_\varphi(y_{21})|J_\psi\Delta_\psi^{-\alpha}\Lambda_\psi(z_{22})) \\
&\quad +(\pi_\varphi(a_{22})J_{\varphi,\psi}\Delta_{\varphi,\psi}^{\overline{\alpha}}\Lambda_\psi(y_{12})|J_{\varphi,\psi}\Delta_{\varphi,\psi}^{-\alpha}\Lambda_\psi(z_{12})) \\
&\quad +(\pi_\psi(a_{22})J_\psi\Delta_\psi^{\overline{\alpha}}\Lambda_\psi(y_{22})|J_\psi\Delta_\psi^{-\alpha}\Lambda_\psi(z_{22})).
\end{aligned}$$

On the other hand, if we put

$$\chi_a^{(\alpha)} = \begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{pmatrix} \in \mathcal{N}_*,$$

then we have

$$\begin{aligned}
\chi_a^{(\alpha)}(y^*z) &= \kappa_{11}(y_{11}^*z_{11}) + \kappa_{12}(y_{12}^*z_{11}) + \kappa_{11}(y_{21}^*z_{21}) + \kappa_{12}(y_{22}^*z_{21}) \\
&\quad +\kappa_{21}(y_{11}^*z_{12}) + \kappa_{22}(y_{12}^*z_{12}) + \kappa_{21}(y_{21}^*z_{22}) + \kappa_{22}(y_{22}^*z_{22}).
\end{aligned}$$

Hence, by putting $y_{12} = y_{21} = y_{22} = z_{12} = z_{21} = z_{22} = 0$, we have

$$\kappa_{11}(y_{11}^*z_{11}) = (\pi_\varphi(a_{11})J_\varphi\Delta_\varphi^{\overline{\alpha}}\Lambda_\varphi(y_{11})|J_\varphi\Delta_\varphi^{-\alpha}\Lambda_\varphi(z_{11}))$$

for all $y_{11}, z_{11} \in \mathfrak{a}_0^{\varphi}$. This means $a_{11} \in L^{\varphi}_{(\alpha)}$ and $\varphi^{(\alpha)}_{a_{11}} = \kappa_{11}$. Similarly, we can deduce

$$\begin{aligned}
&a_{12} \in L^{\psi\varphi}_{(\alpha)} \text{ and } (\psi\varphi)^{(\alpha)}_{a_{12}} = \kappa_{12}, \\
&a_{21} \in L^{\varphi\psi}_{(\alpha)} \text{ and } (\varphi\psi)^{(\alpha)}_{a_{21}} = \kappa_{21}, \\
&a_{22} \in L^{\psi}_{(\alpha)} \text{ and } \psi^{(\alpha)}_{a_{22}} = \kappa_{22}.
\end{aligned}$$

Conversely, let $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \begin{pmatrix} L^{\varphi}_{(\alpha)} & L^{\psi,\varphi}_{(\alpha)} \\ L^{\varphi,\psi}_{(\alpha)} & L^{\psi}_{(\alpha)} \end{pmatrix}$. We claim $a \in L^{\chi}_{(\alpha)}$. Let y, z be as above. Then we have

$$\begin{aligned}
&(\pi_\psi(a_{11}) J_{\psi,\varphi} \Delta^{\overline{\alpha}}_{\psi,\varphi} \Lambda_\varphi(y_{21}) | J_{\psi,\varphi} \Delta^{-\alpha}_{\psi,\varphi} \Lambda_\varphi(z_{21})) \\
&\quad = \varphi^{(-\alpha)}_{y_{21}^* z_{21}}(a_{11}) \quad \text{(by Lemma 1 (i))} \\
&\quad = \varphi^{(\alpha)}_{a_{11}}(y_{21}^* z_{21}) \quad \text{(by [5, Theorem 2.5]).}
\end{aligned} \tag{12}$$

Similarly, we have

$$\begin{aligned}
&(\pi_\psi(a_{12}) J_\psi \Delta^{\overline{\alpha}}_\psi \Lambda_\psi(y_{22}) | J_{\psi,\varphi} \Delta^{-\alpha}_{\psi,\varphi} \Lambda_\varphi(z_{21})) \\
&\quad = \overline{(\pi_\psi(a_{12}^*) J_{\psi,\varphi} \Delta^{-\alpha}_{\psi,\varphi} \Lambda_\varphi(z_{21}) | J_\psi \Delta^{\overline{\alpha}}_\psi \Lambda_\psi(y_{22}))} \\
&\quad = \overline{(\varphi\psi)^{(\alpha)}_{a_{12}^*}(z_{21}^* y_{22})} \\
&\quad = (\psi\varphi)^{(\overline{\alpha})}_{a_{12}}(y_{22}^* z_{21}) \quad \text{(by Lemma 1 (ii)),}
\end{aligned}$$

$$(\pi_\varphi(a_{22}) J_{\varphi,\psi} \Delta^{\overline{\alpha}}_{\varphi,\psi} \Lambda_\psi(y_{12}) | J_{\varphi,\psi} \Delta^{-\alpha}_{\varphi,\psi} \Lambda_\psi(z_{12})) = \varphi^{(\alpha)}_{a_{22}}(y_{12}^* z_{12})$$

and

$$(\pi_\varphi(a_{21}) | J_{\varphi,\psi} \Delta^{-\alpha}_{\varphi,\psi} \Lambda_\psi(z_{12})) = (\varphi\psi)^{(\overline{\alpha})}_{a_{21}}(y_{11}^* z_{12}).$$

Consequently, we have

$$\begin{aligned}
&(\pi_\chi(a) J_\chi \Delta^{\overline{\alpha}}_\chi \Lambda_\chi(y) | J_\chi \Delta^{-\alpha}_\chi \Lambda_\chi(z)) \\
&\quad = \varphi^{(\alpha)}_{a_{11}}(y_{11}^* z_{11}) + (\psi\varphi)^{(\alpha)}_{a_{12}}(y_{12}^* z_{11}) + \varphi^{(\alpha)}_{a_{11}}(y_{21}^* z_{21}) + (\psi\varphi)^{(\alpha)}_{a_{12}}(y_{22}^* z_{21}) \\
&\qquad + (\varphi\psi)^{(\alpha)}_{a_{21}}(y_{11}^* z_{12}) + \psi^{(\alpha)}_{a_{22}}(y_{12}^* z_{12}) + (\varphi\psi)^{(\alpha)}_{a_{21}}(y_{21}^* z_{22}) + \psi^{(\alpha)}_{a_{22}}(y_{22}^* z_{22}) \\
&\quad = \left\langle \begin{pmatrix} \varphi^{(\alpha)}_{a_{11}} & (\psi\varphi)^{(\alpha)}_{a_{12}} \\ (\varphi\psi)^{(\alpha)}_{a_{21}} & \psi^{(\alpha)}_{a_{22}} \end{pmatrix}, y^* z \right\rangle_{\mathcal{N}_*, \mathcal{N}}.
\end{aligned}$$

Hence $a \in L^{\chi}_{(\alpha)}$. □

As a sub-compatible pair [6, Definition 6.4] of $(\mathcal{N}, \mathcal{N}_*)^{\chi}_{(\alpha)}$, we take

$$\left(\begin{pmatrix} \mathcal{M} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathcal{M}_* & 0 \\ 0 & 0 \end{pmatrix} \right)^{\chi}_{(\alpha)}.$$

We compare the sub-compatible pair

$$\left(\begin{pmatrix} \mathcal{M} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathcal{M}_* & 0 \\ 0 & 0 \end{pmatrix} \right)^{\chi}_{(\alpha)}$$

and $(\mathcal{M}, \mathcal{M}_*)^{\varphi}_{(\alpha)}$. Let $x \in \mathcal{M}$ and $\kappa \in \mathcal{M}_*$. Suppose

$$(j^{\chi}_{(-\alpha)})^* \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = (i^{\chi}_{(-\alpha)})^* \begin{pmatrix} \kappa & 0 \\ 0 & 0 \end{pmatrix}.$$

Then, by the calculations in the proof of (2), we have

$$\kappa(a^* b) = (\pi_\varphi(x) J_\varphi \Delta^{\overline{\alpha}}_\varphi \Lambda_\varphi(a) | J_\varphi \Delta^{-\alpha}_\varphi \Lambda_\varphi(b)) \text{ for all } y, z \in \mathfrak{a}^{\varphi}_0, \tag{13}$$

and

$$\kappa(c^*d) = (\pi_\psi(x)J_{\psi,\varphi}\Delta_{\psi,\varphi}^{\overline{\alpha}}\Lambda_\varphi(c)|J_{\psi,\varphi}\Delta_{\psi,\varphi}^{-\alpha}\Lambda_\varphi(d)) \text{ for all } y,z \in \mathfrak{a}_0^{\psi,\varphi}. \tag{14}$$

Hence $x \in L_{(\alpha)}^\varphi$ and $\varphi_x^{(\alpha)} = \kappa$, and consequently,

$$(j_{(-\alpha)}^\varphi)^*(x) = (i_{(-\alpha)}^\varphi)^*(\kappa)$$

(*cf.* [5, Proposition 3.6]). Conversely, suppose that $(j_{(-\alpha)}^\varphi)^*(x) = (i_{(-\alpha)}^\varphi)^*(\kappa)$. Then, by the same argument as in (12), we have (14) as well as (13). Hence

$$(j_{(-\alpha)}^\chi)^* \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = (i_{(-\alpha)}^\chi)^* \begin{pmatrix} \kappa & 0 \\ 0 & 0 \end{pmatrix}.$$

This equivalence of conditions tells us that by identifying

$$\begin{pmatrix} \mathcal{M} & 0 \\ 0 & 0 \end{pmatrix} \qquad \left(\text{resp.} \begin{pmatrix} \mathcal{M}_* & 0 \\ 0 & 0 \end{pmatrix}\right)$$

with $\mathcal{M}$ (resp. $\mathcal{M}_*$), the sub-compatible pair

$$\left(\begin{pmatrix} \mathcal{M} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathcal{M}_* & 0 \\ 0 & 0 \end{pmatrix}\right)_{(\alpha)}^\chi$$

is equivalent to $(\mathcal{M},\mathcal{M}_*)_{(\alpha)}^\varphi$ in the sense of [6, Definition 6.17]. By [6, Proposition 6.18], the interpolation space

$$C_{1/p}\left(\begin{pmatrix} \mathcal{M} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathcal{M}_* & 0 \\ 0 & 0 \end{pmatrix}\right)_{(\alpha)}^\chi$$

is isometrically isomorphic to $L_{(\alpha)}^p(\mathcal{M},\varphi)$ via the map

$$(j_{(-\alpha)}^\chi)^* \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} + (i_{(-\alpha)}^\chi)^* \begin{pmatrix} \kappa & 0 \\ 0 & 0 \end{pmatrix} \mapsto (j_{(-\alpha)}^\varphi)^*(x) + (i_{(-\alpha)}^\varphi)^*(\kappa)$$

for all

$$\xi = (j_{(-\alpha)}^\chi)^* \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} + (i_{(-\alpha)}^\chi)^* \begin{pmatrix} \kappa & 0 \\ 0 & 0 \end{pmatrix} \in C_{1.p}\left(\begin{pmatrix} \mathcal{M} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathcal{M}_* & 0 \\ 0 & 0 \end{pmatrix}\right)_{(\alpha)}^\chi$$

with $x \in \mathcal{M}, \kappa \in \mathcal{M}_*$. In a similar way, we can construct a natural isometric map between

$$C_{1/p}\left(\begin{pmatrix} 0 & 0 \\ 0 & \mathcal{M} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{M}_* \end{pmatrix}\right)_{(\alpha)}^\chi \qquad \text{and} \qquad L_{(\alpha)}^p(\mathcal{M},\psi).$$

Then, by [5, Theorem 2.14],

$$\overline{(j_{(-\alpha)}^\chi)^* \begin{pmatrix} L_{(\alpha)}^\varphi & 0 \\ 0 & 0 \end{pmatrix}}^{\text{norm}} = C_{1/p}\left(\begin{pmatrix} \mathcal{M} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathcal{M}_* & 0 \\ 0 & 0 \end{pmatrix}\right)_{(\alpha)}^\chi \tag{15}$$

and

$$\overline{(j^{\chi}_{(-\alpha)})^* \begin{pmatrix} 0 & 0 \\ 0 & L^{\psi}_{(\alpha)} \end{pmatrix}}^{\text{norm}} = C_{1/p}\left(\begin{pmatrix} 0 & 0 \\ 0 & \mathcal{M} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{M}_* \end{pmatrix}\right)^{\chi}_{(\alpha)}$$

By [6, Proposition 6.22], the set $(j^{\chi}_{(-\alpha)})^*((\mathfrak{a}^{\chi}_0)^2)$ is norm dense in $L^p_{(\alpha)}(\mathcal{N},\chi)$. We put

$$L_1 = \overline{(j^{\chi}_{(-\alpha)})^* \begin{pmatrix} (\mathfrak{a}^{\varphi}_0)^2 & 0 \\ 0 & 0 \end{pmatrix}}^{\text{norm}} \subset L^p_{(\alpha)}(\mathcal{N},\chi) \tag{16}$$

and

$$L_2 = \overline{(j^{\chi}_{(-\alpha)})^* \begin{pmatrix} 0 & 0 \\ 0 & (\mathfrak{a}^{\psi}_0)^2 \end{pmatrix}}^{\text{norm}} \subset L^p_{(\alpha)}(\mathcal{N},\chi).$$

Again by [6, Proposition 6.22], L_1 (resp. L_2) equals

$$C_{1/p}\left(\begin{pmatrix} \mathcal{M} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathcal{M}_* & 0 \\ 0 & 0 \end{pmatrix}\right)^{\chi}_{(\alpha)} \quad \left(\text{resp. } C_{1/p}\left(\begin{pmatrix} 0 & 0 \\ 0 & \mathcal{M} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{M}_* \end{pmatrix}\right)^{\chi}_{(\alpha)}\right)$$

and can be naturally identified with $L^p_{(\alpha)}(\mathcal{M},\varphi)$ (resp. $L^p_{(\alpha)}(\mathcal{M},\psi)$).

Next, we recall the bimodule structure of $L^p_{(\alpha)}(\mathcal{N},\chi)$ (see [6, §7]). For $a \in \mathcal{N}$, we put the left and the right actions by

$$\pi^{\chi}_{p,(\alpha)}(a) = (U^{\chi}_{p,(-1/2,\alpha)})^{-1} \circ \pi^{\chi}_{p,L}(a) \circ U^{\chi}_{p,(-1/2,\alpha)}$$

and

$$\pi^{\chi}_{p,(\alpha)}{}'(a) = (U^{\chi}_{p,(1/2,\alpha)})^{-1} \circ \pi^{\chi}_{p,R}{}'(a) \circ U^{\varphi}_{p,(1/2,\alpha)}.$$

Here, $U^{\chi}_{p,(-1/2,\alpha)}$ is an isometric isomorphism of $L^p_{(\alpha)}(\mathcal{N},\chi)$ onto the left L^p-space $L^p_{(-1/2)}(\mathcal{N},\chi)$ satisfying

$$U^{\chi}_{p,(-1/2,\alpha)}((j^{\chi}_{(-\alpha)})^*(y)) = j^*_{\chi,(1/2)}(\sigma^{\chi}_{s-i(1+2r)/2p}(y))$$

for all $y \in (\mathfrak{a}^{\chi}_0)^2$, where $\alpha = r + is$, and $\pi^{\chi}_{p,L}(a)$ is a bounded linear operator on $L^p_{(-1/2)}(\mathcal{N},\chi)$ defined by

$$\pi^{\chi}_{p,L}(a)(j^*_{\chi,(1/2)}(x) + i^*_{\chi,(1/2)}(\kappa)) = j^*_{\chi,(1/2)}(ax) + i^*_{\chi,(1/2)}(a\kappa)$$

for all $\xi = j^*_{\chi,(1/2)}(x) + i^*_{\chi,(1/2)}(\kappa) \in L^p_{(-1/2)}(\mathcal{N},\chi)$, $x \in \mathcal{N}, \kappa \in \mathcal{N}_*$.

Similarly, $U^{\chi}_{p,(1/2,\alpha)}$ is an isometric isomorphism of $L^p_{(\alpha)}(\mathcal{N},\chi)$ onto the right L^p-space $L^p_{(1/2)}(\mathcal{N},\chi)$ satisfying

$$U^{\chi}_{p,(1/2,\alpha)}((j^{\chi}_{(-\alpha)})^*(y)) = j^*_{\chi,(1/2)}(\sigma^{\chi}_{s+i(1-2r)/2p}(y))$$

for all $y \in (\mathfrak{a}^{\chi}_0)^2$, and $\pi^{\chi}_{p,R}{}'(b)$ is a bounded linear operator on $L^p_{(1/2)}(\mathcal{N},\chi)$ defined by

$$\pi^{\chi}_{p,R}{}'(b)(j^*_{\chi,(1/2)}(x) + i^*_{\chi,(1/2)}(\kappa)) = j^*_{\chi,(1/2)}(xb) + i^*_{\chi,(1/2)}(\kappa b)$$

for all $\eta = j^*_{\chi,(-1/2)}(x) + i^*_{\chi,(-1/2)}(\kappa) \in L^p_{(1/2)}(\mathcal{N},\chi)$, $x \in \mathcal{N}, \kappa \in \mathcal{N}_*$.

Next, we put

$$p_1 = \pi^{\chi}_{p,(\alpha)}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \circ {\pi^{\chi}_{p,(\alpha)}}'\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},$$

and

$$p_2 = \pi^{\chi}_{p,(\alpha)}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \circ {\pi^{\chi}_{p,(\alpha)}}'\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}.$$

Since the left and the right actions commute, p_1 and p_2 are idempotent operators on $L^p_{(\alpha)}(\mathcal{N},\chi)$.

Proposition 1.

(i) *The range of* p_1 *is* L_1.
(ii) *The range of* p_2 *is* L_2.

Proof. The proofs of (i) and (ii) are similar, so we will prove only (i). Let

$$y = \begin{pmatrix}y_{11} & y_{12}\\ y_{21} & y_{22}\end{pmatrix} \in \mathcal{N}.$$

Then, by [8, 3.10], there exist strongly* continuous one-parameter groups $\{\sigma^{\psi,\varphi}_t\}_{t\in\mathbb{R}}$, $\{\sigma^{\varphi,\psi}_t\}_{t\in\mathbb{R}}$ of isometries of $\mathcal{M}$ onto $\mathcal{M}$ such that

$$\sigma^{\chi}_t(y) = \begin{pmatrix}\sigma^{\varphi}_t(y_{11}) & \sigma^{\varphi,\psi}_t(y_{12})\\ \sigma^{\psi,\varphi}_t(y_{21}) & \sigma^{\psi}_t(y_{22})\end{pmatrix}$$

for all $t \in \mathbb{R}$. Moreover, suppose that $y \in \mathfrak{a}^{\chi}_0$. By analytic continuation, the one-parameter groups $\sigma^{\psi,\varphi}$ and $\sigma^{\varphi,\psi}$ can be uniquely extended to complex one-parameter groups on $\mathfrak{a}^{\psi,\varphi}_0$ and $\mathfrak{a}^{\varphi,\psi}_0$, respectively, such that

$$\sigma^{\chi}_\alpha(y) = \begin{pmatrix}\sigma^{\varphi}_\alpha(y_{11}) & \sigma^{\varphi,\psi}_\alpha(y_{12})\\ \sigma^{\psi,\varphi}_\alpha(y_{21}) & \sigma^{\psi}_\alpha(y_{22})\end{pmatrix}$$

for all $\alpha \in \mathbb{C}$. Let

$$A = \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}, B = \begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix} \in \mathfrak{a}^{\chi}_0.$$

Then $(j^\chi_{(-\alpha)})^*(AB) \in L^p_{(\alpha)}(\mathcal{N},\chi)$ and we have

$$
\begin{aligned}
&p_1(j^\chi_{(-\alpha)})^*(AB) \\
&= \pi^\chi_{p,(\alpha)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} {\pi^\chi_{p,(\alpha)}}' \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (j^\chi_{(-\alpha)})^*(AB) \\
&= \pi^\chi_{p,(\alpha)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (U^\chi_{p,(1/2,\alpha)})^{-1} {\pi^\chi_{p,R}}' \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U^\chi_{p,(1/2,\alpha)} (j^\chi_{(-\alpha)})^*(AB) \\
&= \pi^\chi_{p,(\alpha)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (U^\chi_{p,(1/2,\alpha)})^{-1} {\pi^\chi_{p,R}}' \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} j^*_{\chi,(-1/2)}(\sigma^\chi_{i(1-2r)/2p+s}(AB)) \\
&= \pi^\chi_{p,(\alpha)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (U^\chi_{p,(1/2,\alpha)})^{-1} {\pi^\chi_{p,R}}' \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
&\qquad j^*_{\chi,(-1/2)}(\sigma^\chi_{i(1-2r)/2p+s}(A)\sigma^\chi_{i(1-2r)/2p+s}(B)) \\
&= \pi^\chi_{p,(\alpha)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (U^\chi_{p,(1/2,\alpha)})^{-1} {\pi^\chi_{p,R}}' \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
&\qquad j^*_{\chi,(-1/2)} \left(\sigma^\chi_{i(1-2r)/2p+s}(A) \begin{pmatrix} \sigma^\varphi_{i(1-2r)/2p+s}(b_{11}) & \sigma^{\varphi,\psi}_{i(1-2r)/2p+s}(b_{12}) \\ \sigma^{\psi,\varphi}_{i(1-2r)/2p+s}(b_{21}) & \sigma^{\psi}_{i(1-2r)/2p+s}(b_{22}) \end{pmatrix} \right) \\
&= \pi^\chi_{p,(\alpha)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (U^\chi_{p,(1/2,\alpha)})^{-1} \\
&\qquad j^*_{\chi,(-1/2)} \left(\sigma^\chi_{i(1-2r)/2p+s}(A) \begin{pmatrix} \sigma^\varphi_{i(1-2r)/2p+s}(b_{11}) & 0 \\ \sigma^{\psi,\varphi}_{i(1-2r)/2p+s}(b_{21}) & 0 \end{pmatrix} \right) \\
&= \pi^\chi_{p,(\alpha)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} j^*_\chi \left(A \begin{pmatrix} b_{11} & 0 \\ b_{21} & 0 \end{pmatrix} \right) \\
&= (U^\chi_{p,(-1/2,\alpha)})^{-1} \pi^\chi_{p,L} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U^\chi_{p,(-1/2,\alpha)} j^*_\chi \left(A \begin{pmatrix} b_{11} & 0 \\ b_{21} & 0 \end{pmatrix} \right) \\
&= (U^\chi_{p,(-1/2,\alpha)})^{-1} \pi^\chi_{p,L} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
&\qquad j^*_{\chi,(1/2)} \left(\sigma^\chi_{-i(1+2r)/2p+s}(A) \begin{pmatrix} \sigma^\varphi_{-i(1+2r)/2p+s}(b_{11}) & 0 \\ \sigma^{\psi,\varphi}_{-i(1+2r)/2p+s}(b_{21}) & 0 \end{pmatrix} \right) \\
&= (U^\chi_{p,(-1/2,\alpha)})^{-1} \pi^\chi_{p,L} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
&\qquad j^*_{\chi,(1/2)} \left(\begin{pmatrix} \sigma^\varphi_{-i(1+2r)/2p+s}(a_{11}) & \sigma^{\varphi,\psi}_{-i(1+2r)/2p+s}(a_{12}) \\ \sigma^{\psi,\varphi}_{-i(1+2r)/2p+s}(a_{21}) & \sigma^{\psi}_{-i(1+2r)/2p+s}(a_{22}) \end{pmatrix} \begin{pmatrix} \sigma^\varphi_{-i(1+2r)/2p+s}(b_{11}) & 0 \\ \sigma^{\psi,\varphi}_{-i(1+2r)/2p+s}(b_{21}) & 0 \end{pmatrix} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= (U^{\chi}_{p,(-1/2,\alpha)})^{-1} \\
&\quad j^{*}_{\chi,(1/2)}\left(\begin{pmatrix}\sigma^{\varphi}_{-i(1+2r)/2p+s}(a_{11}) & \sigma^{\varphi,\psi}_{-i(1+2r)/2p+s}(a_{12})\\ 0 & 0\end{pmatrix}\begin{pmatrix}\sigma^{\varphi}_{-i(1+2r)/2p+s}(b_{11}) & 0\\ \sigma^{\psi,\varphi}_{-i(1+2r)/2p+s}(b_{21}) & 0\end{pmatrix}\right) \\
&= (j^{\chi}_{(-\alpha)})^{*}\left(\begin{pmatrix}a_{11} & a_{12}\\ 0 & 0\end{pmatrix}\begin{pmatrix}b_{11} & 0\\ b_{21} & 0\end{pmatrix}\right) \\
&= (j^{\chi}_{(-\alpha)})^{*}\begin{pmatrix}a_{11}b_{11}+a_{12}b_{21} & 0\\ 0 & 0\end{pmatrix}.
\end{aligned}
$$

Since $AB \in (\mathfrak{a}^{\chi}_0)^2 \subset L^{\chi}_{(\alpha)}$, we find that $a_{11}b_{11}+a_{12}b_{21} \in L^{\varphi}_{(\alpha)}$ by Lemma 2. Moreover, $a_{11}b_{11} \in (\mathfrak{a}^{\varphi}_0)^2$ by (10). Hence we have

$$
(j^{\chi}_{(-\alpha)})^{*}\begin{pmatrix}(\mathfrak{a}^{\varphi}_0)^2 & 0\\ 0 & 0\end{pmatrix} \subset p_1 (j^{\chi}_{(-\alpha)})^{*}((\mathfrak{a}^{\chi}_0)^2) \subset (j^{\chi}_{(-\alpha)})^{*}\begin{pmatrix}L^{\varphi}_{(\alpha)} & 0\\ 0 & 0\end{pmatrix}.
$$

Taking norm closures, we have

$$
\overline{p_1 (j^{\chi}_{(-\alpha)})^{*}((\mathfrak{a}^{\chi}_0)^2)}^{\text{norm}} = L_1
$$

by (15) and (16). Since p_1 is idempotent, its range is closed. Hence we get the assertion. □

Next, we put

$$
u_1 = \pi^{\chi}_{p,(\alpha)}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \circ {\pi^{\chi}_{p,(\alpha)}}'\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},
$$

and

$$
u_2 = \pi^{\chi}_{p,(\alpha)}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \circ {\pi^{\chi}_{p,(\alpha)}}'\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.
$$

Then we state our main result.

Theorem 1. *With the notations above, for $\alpha \in \mathbb{C}$, $u_1|_{L_1}$ is an isometric isomorphism of L_1 onto L_2. By the natural identification of L_1 (*resp. *L_2) with $L^{p}_{(\alpha)}(\mathcal{M},\varphi)$ (*resp. *$L^{p}_{(\alpha)}(\mathcal{M},\psi)$), $u_1|_{L_1}$ and $u_2|_{L_2}$ give rise to isometric isomorphisms*

$$
\begin{aligned}
U^{\psi,\varphi}_{p,(\alpha)} &: L^{p}_{(\alpha)}(\mathcal{M}_{\varphi}) \to L^{p}_{(\alpha)}(\mathcal{M},\psi), \quad \text{and} \\
U^{\varphi,\psi}_{p,(\alpha)} &: L^{p}_{(\alpha)}(\mathcal{M}_{\psi}) \to L^{p}_{(\alpha)}(\mathcal{M},\varphi).
\end{aligned}
$$

These two maps are mutually inverse.

Moreover, let θ be another n.f.s. weight on $\mathcal{M}$. Then we have the chain rule:

$$
U^{\theta,\psi}_{p,(\alpha)} \circ U^{\psi,\varphi}_{p,(\alpha)} = U^{\theta,\varphi}_{p,(\alpha)}.
$$

Proof. By simple computations, we have $u_2u_1 = p_1$ and $u_1u_2 = p_2$. By [6, Theorem 7.1], u_1 and u_2 are contractions. Hence we conclude that $u_1|_{L_1}$ is an isometric isomorphism of L_1 onto L_2, and that its inverse is given by $u_2|_{L_2}$, which is identified with $U_p^{\varphi,\psi}$.

To prove the chain rule, we consider $\mathcal{S} = M_3(\mathbb{C}) \otimes \mathcal{M}$, and a weight δ on $\mathcal{S}$ defined by

$$\delta \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \varphi(a_{11}) + \psi(a_{22}) + \theta(a_{33}), \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathcal{S}_+.$$

Then δ is an n.f.s. weight on $\mathcal{S}$. Similarly as in the 2×2 case, we can identify

$$\overline{(j^{\delta}_{(-\alpha)})^* \begin{pmatrix} (\mathfrak{a}_0^{\varphi})^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}^{\text{norm}} \qquad \text{with } L^p_{(\alpha)}(\mathcal{M}, \varphi),$$

$$\overline{(j^{\delta}_{(-\alpha)})^* \begin{pmatrix} 0 & 0 & 0 \\ 0 & (\mathfrak{a}_0^{\psi})^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}}^{\text{norm}} \qquad \text{with } L^p_{(\alpha)}(\mathcal{M}, \psi)$$

and

$$\overline{(j^{\delta}_{(-\alpha)})^* \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (\mathfrak{a}_0^{\theta})^2 \end{pmatrix}}^{\text{norm}} \qquad \text{with } L^p_{(\alpha)}(\mathcal{M}, \theta).$$

Since the modular action of δ is given by

$$\sigma^{\delta}_{\beta} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} \sigma^{\varphi}_{\beta}(a_{11}) & \sigma^{\varphi,\psi}_{\beta}(a_{12}) & \sigma^{\varphi,\theta}_{\beta}(a_{13}) \\ \sigma^{\psi,\varphi}_{\beta}(a_{21}) & \sigma^{\psi}_{\beta}(a_{22}) & \sigma^{\psi,\theta}_{\beta}(a_{23}) \\ \sigma^{\theta,\varphi}_{\beta}(a_{31}) & \sigma^{\theta,\psi}_{\beta}(a_{32}) & \sigma^{\theta}_{\beta}(a_{33}) \end{pmatrix}$$

for all

$$a = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathfrak{a}_0^{\delta} \quad \text{and} \quad \beta \in \mathbb{C},$$

under the above identifications, $U^{\psi,\varphi}_{p,(\alpha)}$ $\left(\text{resp. } U^{\theta,\psi}_{p,(\alpha)},\ U^{\theta,\varphi}_{p,(\alpha)}\right)$ is given by

$$v_1 = \pi^\delta_{p,(\alpha)}\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix} \circ {\pi^\delta_{p,(\alpha)}}'\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},$$

$$\left(\text{resp. } v_2 = \pi^\delta_{p,(\alpha)}\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix} \circ {\pi^\delta_{p,(\alpha)}}'\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\right.$$

$$\left. v_3 = \pi^\delta_{p,(\alpha)}\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix} \circ {\pi^\delta_{p,(\alpha)}}'\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}\right).$$

Since $v_3 = v_2 v_1$, the chain rule is proved. □

References

[1] J. Bergh and J. Löfström, *Interpolation Spaces: an Introduction*, Springer-Verlag, 1976.

[2] A. P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. **24** (1964), 113–190.

[3] A. Connes, *Caractérisation des espaces vectoriels ordonnés sous-jacent aux algèbres de von Neumann*, Ann. Inst. Fourier **24** (1974), 121–155.

[4] U. Haagerup, *L^p-spaces associated with an arbitrary von Neumann algebra*, Colloques Internationaux CNRS, No.274, 175–184.

[5] H. Izumi, *Constructions of non-commutative L^p-spaces with a complex parameter arising from modular actions*, Int. J. Math. **8** (1997), 1029–1066.

[6] H. Izumi, *Natural bilinear forms, natural sesquilinear forms and the associated duality of non-commutative L^p-spaces*, Int. J. Math. **9** (1998), 975–1039.

[7] H. Kosaki, *Applications of the Complex Interpolation Method to a von Neumann Algebra: Non-commutative L^p-Spaces*, J. Funct. Anal. **56** (1984), 29–78.

[8] S. Strătilă, *Modular Theory in Operator Algebras*, Abacus Press, 1981.

[9] M. Takesaki, *Theory of Operator Algebras II*, Springer-Verlag, 2003.

[10] M. Terp, *Interpolation spaces between a von Neumann algebra and its predual*, J. Operator Theory **8** (1982), 327–360.

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