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A SUBSTITUTION RULE FOR THE PENROSE TILING

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Abstract. We study the structure of the Penrose tiling (PT, in short) constructed by the matching rule, and deduce directly a substitution rule from that, which gives us (i) local configuration of the tiles, (ii) elementary proofs of the aperiodicity, the locally isomorphic property, and the uncountability, (iii) alternative proof of the fact that all PT's obtained by the matching rule can be constructed via the up-down generation.

1. Introduction

Penrose tiling (PT) is one of the remarkable aperiodic tilings consisting of only two prototiles (e.g., two rhombs) and is known to be aperiodic, to have locally isomorphic property, and to have uncountable family of mutually distinct ones [8]. To construct PT, following three methods are well-known:

(i) matching rule,

(ii) inflation rule (up-down generation, UD in short), and

(iii) projection method.

Letting $\mathcal{P}_n$, $n = 1, 2, 3$ be the set of PT's obtained by these methods, we have ([1, 2, 6, 10]) $\mathcal{P}_2 \subset \mathcal{P}_1 \subset \mathcal{P}_3$ and for the opposite inclusion ($\mathcal{P}_1 \subset \mathcal{P}_2$) Robinson's argument using the notion of supertile is known [5]. In this paper, we would like to study the structure of PT using de Bruijn's matching rule only[1]; most of our results are well-known but our argument is, we believe, more elementary and straightforward.

It is known that there are eight allowed patterns around each vertices (Figure 2.1.1). In section 2, we study the unique patches[2] ($P_1, \ldots, P_8$) determined by each patterns (Figures 2.2.1, …, 2.2.8), and show that PT is the superposition of both P_6 and P_8 (Theorem 2.1, Corollary 2.1). This fact implies that the patches P_6 and P_8 are "the local charts" in a sense, describing the local configuration of tiles in PT, and in other words, we can regard them as "prototiles" on the construction of

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[1]The matching rule is to draw single and double arrows on edges of each prototiles and arrange these tiles on the plane by matching those arrows (Figure 1.1).

[2]They are equal to the connected components of the empires [5].

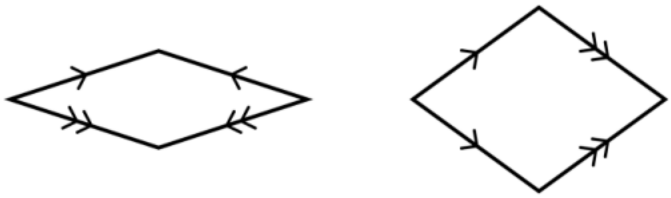

Figure 1.1: Two prototiles with arrows which composes PT.

PT. We remark that essentially the same fact is found by [7], but our P_6 is smaller than the decagon in [7] and is minimal in the sense that no proper subset of which can cover P_8. In section 3, we show that if we regard P_6, P_8 as "prototiles" then their "matching rule" coincides (Theorem 3.1). From which we find, in section 4, a substitution rule which gives us the family $\{A_n\}_{n=1}^{\infty}$ of local charts of PT with diam $(A_n) \to \infty$ (Theorem 4.1), which easily proves the aperiodicity and the locally isomorphic property. This substitution rule is essentially the same as well-known inflation rule: our rule is similar to applying the inflation rule twice. However, ours is deduced differently from the matching rule. In section 5, we show there are uncountably many PT's which are not congruent each other, using the idea of the proof of uncountability of PT by UD. In section 6, we compare our substitution rule with UD and show that all PT's constructed by the matching rule are also obtained by UD. In Appendix 1, we provide detailed arguments omitted in the proof of Theorem 2.1. In Appendix 2, we consider some curves determined by the matching rules, which is essentially the same as Conway's curves[4], and study the basic properties of them. Due to the self-similar structure of PT, they have fractal structure. We compute the Hausdorff dimension of those objects obtained by "the thermodynamic procedure" on these curves, which shows the difference of the density of overlaps between upper and lower sides of A_n.

2. Local chart

2.1. The global rule

It is straightforward to see that at most eight patterns are allowed on each vertices in PT, as named $p_1, p_2, \ldots, p_8$ in Figure 2.1.1.

When we start to put tiles following the matching rule, it turns out that we sometimes fail to tile the plane, as is seen in the next two examples.

(1) Two joints of p_5 : The pattern p_5 has two "joints" which allow both p_4 and p_8 (Figure 2.1.2). If we put p_4 on both of them, we fail to tile the plane (Figure 2.1.3): if we put p_4 on one of the joints, then we must put p_8 on the other side. This property plays an important role to determine the global distribution of tiles in PT.

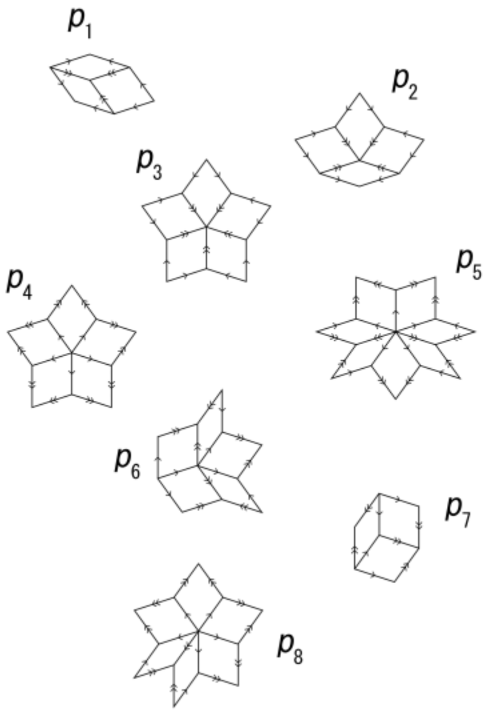

Figure 2.1.1: Eight patterns around each vertices in PT[9, p. 178, Figure 6.8].

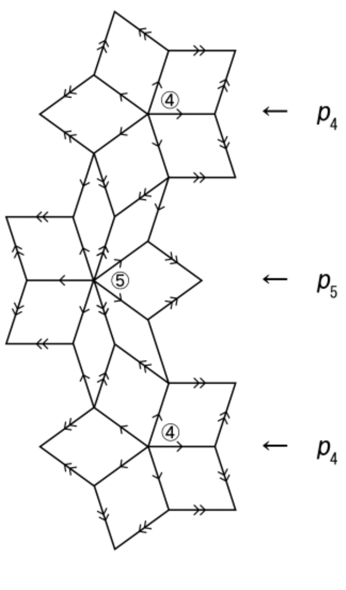

Figure 2.1.2: If we put p_4 on both of two joints in p_5,

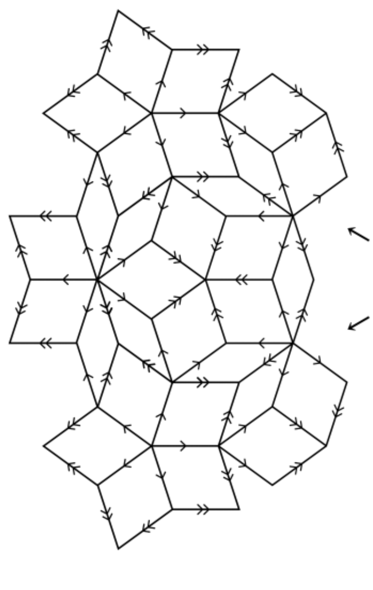

Figure 2.1.3: then we cannot put tiles on one of two places (pointed by arrows).

(2) Five "joints" around p_4: the pattern p_4 has five "joints" (a, b, c, d, e in Figure 2.2.4) which allows both p_5 and p_8. If we put p_5's on four of them, then p_8 must be put on the fifth one (Figure 2.1.4). In other words, if we put p_5 on all of them, we fail to tile the plane. In fact, the number of p_5's on these joints must be either 0 or 2 or 4. These facts imply that, certain local configuration of tiles may determine that of wider region, so that there may be some rules on the global distribution of tiles, and henceforth we call these rules "the global rules."

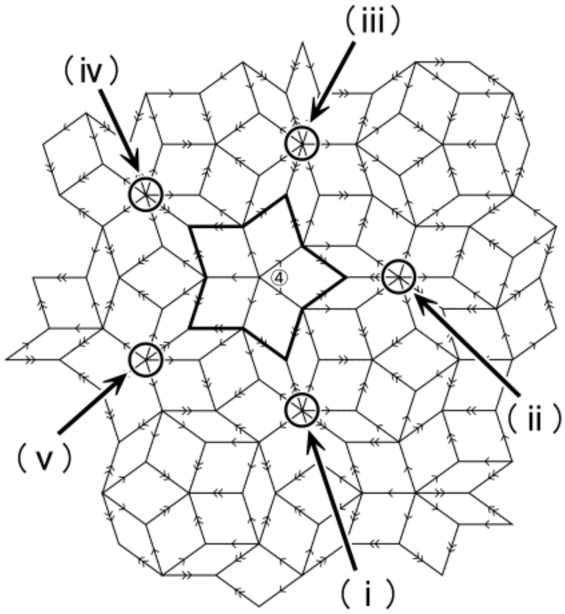

Figure 2.1.4: The five joints around p_4, (i) if we put p_5 here, (ii) then p_5 must be put here. (iii) if we next put p_5 here, (iv) then p_5 must be put here and (v) p_8 must be put here.

2.2. The unique patches determined by each patterns

The facts in former subsection imply that we should not put tiles as we like: there must be some global rules. For instance, suppose we begin to tile the plane with p_1 as the starting point. Then the configuration of the tiles near p_1 must be the one shown in Figure 2.2.1. This patch is maximal in the sense that beyond which we have more than two choices of putting tiles: for instance, we can put either p_5 or p_8 on a in Figure 2.2.1. We denote by P_1 this unique patch determined by p_1. Similarly, we have $P_2, \ldots, P_8$ as shown in Figures 2.2.2, ..., 2.2.8.

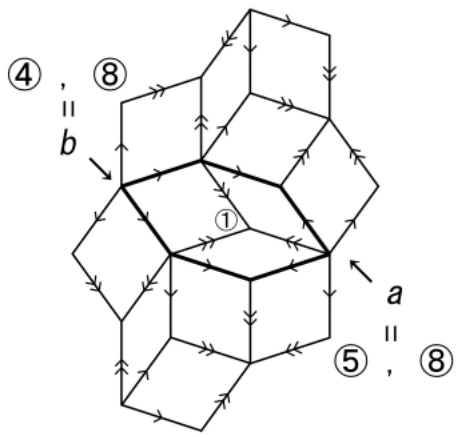

Figure 2.2.1: The patch P_1 determined uniquely by p_1.

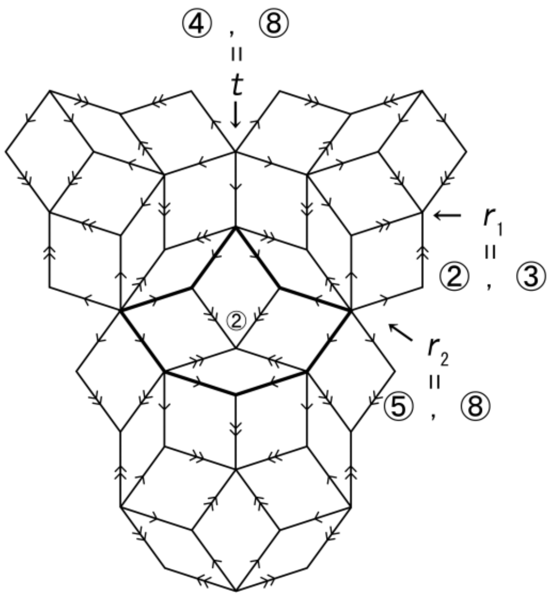

Figure 2.2.2: The patch P_2 determined uniquely by p_2.

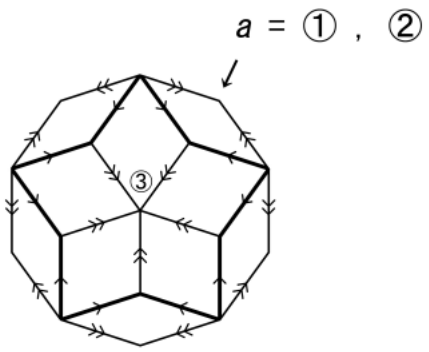

Figure 2.2.3: The patch P_3 determined uniquely by p_3.

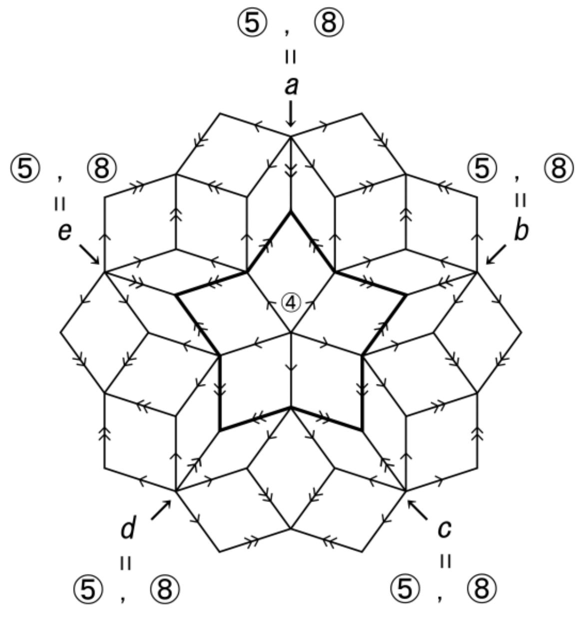

Figure 2.2.4: The patch P_4 determined uniquely by p_4.

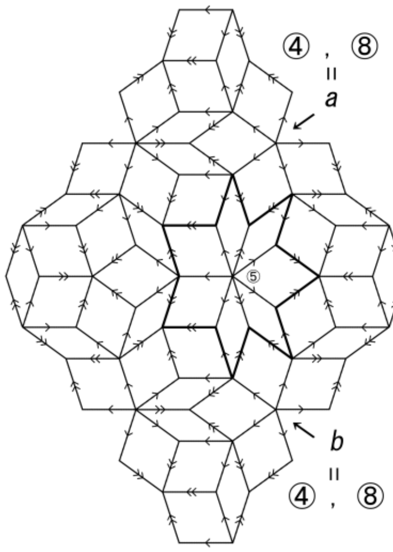

Figure 2.2.5: The patch P_5 determined uniquely by p_5.

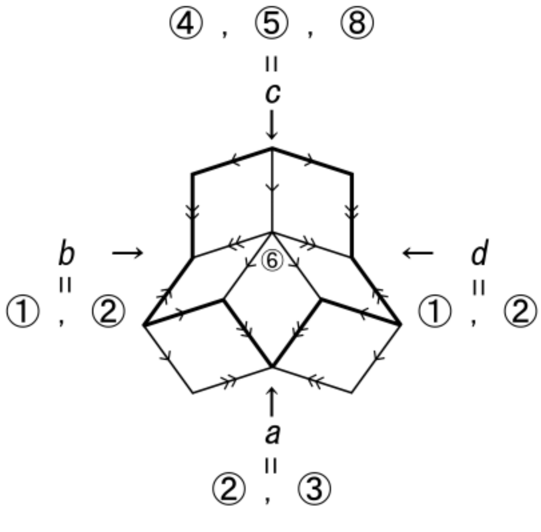

Figure 2.2.6: The patch P_6 determined uniquely by p_6.

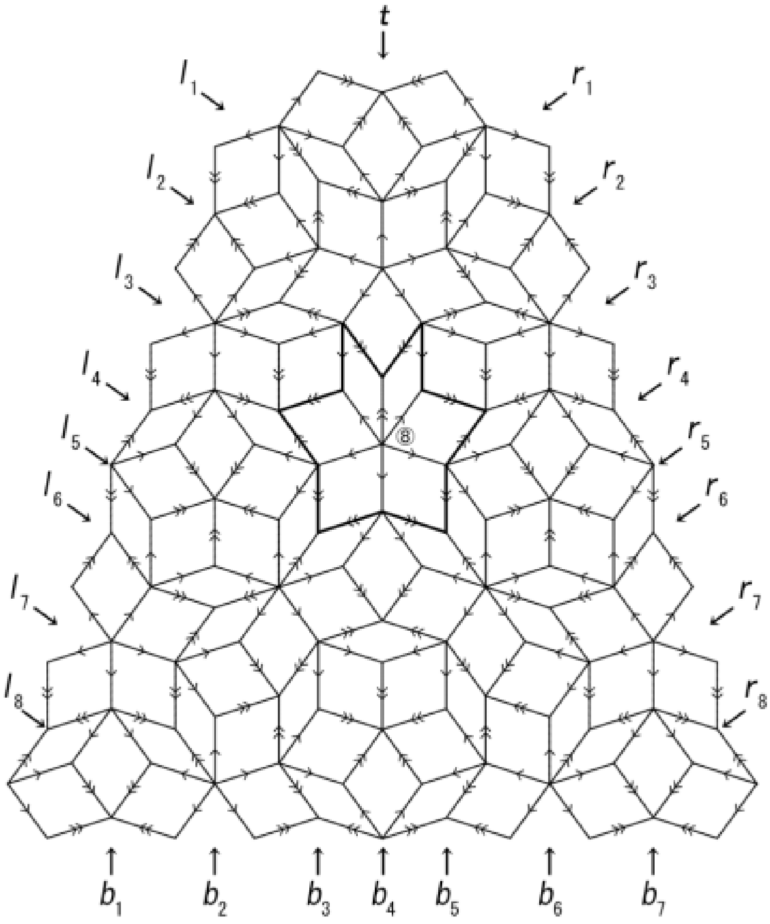

Figure 2.2.8: The patch P_8 determined uniquely by p_8.

P_7 is omitted and so is Figure 2.2.7, for this is the same as that shown in Figure 2.1.1; p_7 does not determine any local configurations around it. If for instance we start to put tiles from p_1, we first obtain P_1, and to proceed, we put one of the allowed tiles on the boundary. Then this tile would determine further the configuration of tiles to some extent. For instance, if we determine $a = p_5$ in P_1, then we obtain P_5. Hence putting p_5 on a boundary point means putting the patch P_5 there, and putting tiles along the matching rule on the plane is done by repeating this process. In other words, to tile the plane along the matching rule is to superimpose the copies of patches $P_1, \ldots, P_8$ compatibly.

2.3. The local chart

By definition, $\{P_1, P_2, \ldots, P_8\}$ is a ordered set. Moreover, by Figures 2.2.1, ..., 2.2.8, we notice that

$$P_j \subset P_8, \quad j = 1, 2, \ldots, 7. \tag{2.1}$$

In fact, we find all patterns $p_1, \ldots, p_8$ on P_8. Therefore, if we meet p_8 frequently enough as we tile the plane, then it would determine larger area than the other ones. Hence if the density of p_8's is high enough, the PT should be the superposition of P_8, which turns out to be true.

Theorem 2.1 *PT is the superposition of* P_8. *In other words, for any vertex* p *in PT, we can find* P_8 *which contains* p.

Figure 2.3.1 is a typical example of PT where the location of P_8's is shown[3].

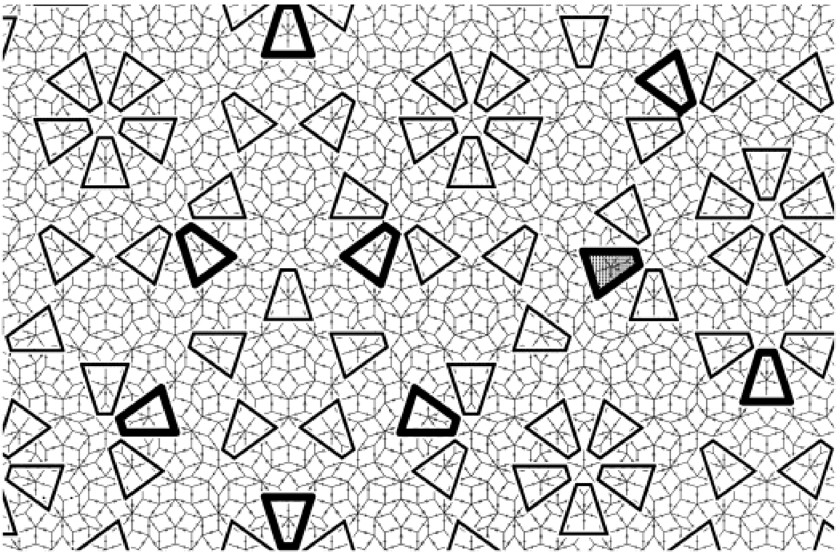

Figure 2.3.1: A typical example of PT. The location of P_8's are indicated by the trapezoid-like figures. The thick ones are the centers of $P_8'(= A_2)$ and the shaded one is the center of A_3 (P_8', A_2 and A_3 are defined in Section 4).

By Theorem 2.1, we can regard P_8 as a "prototile" of PT; PT is constructed by "tiling" the copies of P_8. Hence P_8 is the "local chart" which gives us the local configuration of tiles in PT; to know how tiles are distributed near a given vertex p, we only have to find a p_8 near p and identify it on a chart P_8.

Proof. It suffices to show the following fact: when we start to put tiles from P_j ($j \neq 8$), then p_8 appears whose corresponding P_8 contains P_j.

(1) P_5 : as is seen in the global rule in §2.1, one of two joints (a, b in Figure 2.2.5) must be p_8 and thus P_5 is contained by a copy of P_8.

(2) P_4 : as is seen in the global rule in §2.1, one of the five joints (a, b, c, d, e in Figure 2.2.4) must be p_8, and thus P_4 is contained by a copy of P_8.

[3]Figure 2.3.1 is drawn by "tilings.exe" made by V. C. Gulyaev.

(3) P_2 : in any case, either p_5 or p_8 appears and P_2 is contained by P_5 or P_8 respectively (Lemma 7.2).
(4) P_1 : as the case (3), it is seen to be contained by either P_4, P_5 or P_8 (Lemma 7.1).
(5) P_3 : it is seen to be contained by either P_2 or P_5 (Lemma 7.3).
(6) P_6 : it is seen to be contained by either P_1, P_2, P_4, P_5 or P_8 (Lemma 7.4).
(7) P_7 : P_7 does not determine any vertices around it. By putting some allowed tiles on its boundary, it becomes

$$P_7 \subset P_j, \quad j = 1, 2, \ldots, 6.$$

P_j $(j = 1, 2, \ldots, 6)$ is already shown to be contained by P_8. □

Remark 2.1 The global rule discussed in §2.1 can be seen in P_8. In fact, by Theorem 2.1 (Theorem 2.1 can be proved without using these rules), PT is the superposition of P_8 and there are two P_5's in P_8 which are connected to p_8 through their joints. Hence one of two joints in P_5 must be connected to p_8 in PT. The second rule can be seen similarly.

Corollary 2.1 *PT is the superposition of P_6. In other words, for any vertex p in PT, we can find P_6 which contains p.*

Proof. It suffices to see that P_8 is the superposition of P_6 (Figure 2.3.2). □

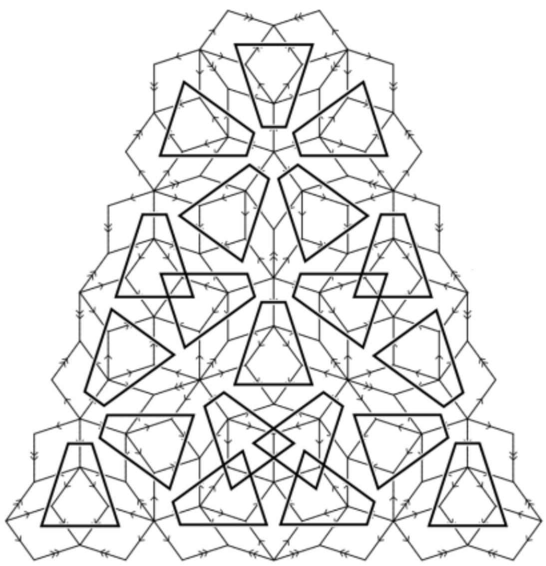

Figure 2.3.2: P_8 is the superposition of P_6. The location of P_6 is indicated by the trapezoid-like figures.

3. Coincidence of matching rules of P_6, P_8

The next question is how P_8's are distributed in PT. By Theorem 2.1 and Corollary 2.1, PT is constructed by regarding P_6 or P_8 as the prototile and hence we would like to know the "matching rules (overlapping rules)" of those. Then, we find

Theorem 3.1 *The matching rule of P_6 and that of P_8 are the same.*

Proof. We first study the matching rule of P_6. We can put either p_1 or p_2 on the left and right side (b, d in Figure 2.2.6) of P_6, and can put either p_2 or p_3 on the bottom (a in Figure 2.2.6). We write $L = 1$ (resp. $L = 2$) if we put p_1 (resp. p_2) on the left side, and similarly for the right side. We write $B = I$ (resp. $B = II$) if we put p_2 (resp. p_3) on the bottom. Thus we can describe the configuration of tiles near P_6 by the triple (L, R, B), and then we find there are five possibilities: $(1, 1, I), (1, 1, II), (1, 2, II), (2, 1, II), (2, 2, II)$. In Figure 3.0, we show labels for these P_6's which compose P_8.

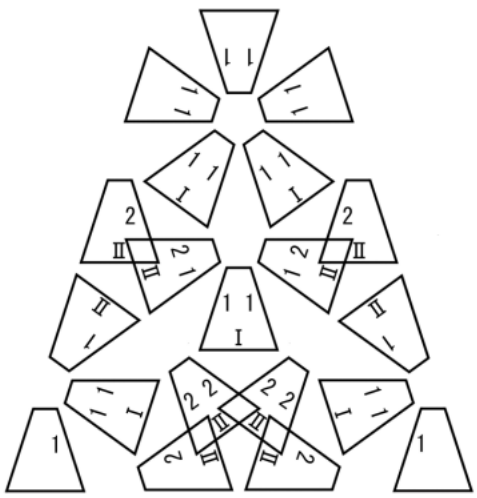

Figure 3.0:[4]The labelling of P_6's which compose P_8.

If we let $L = 1$, then the arrangement of P_6 is shown in Figure 3.1: another P_6 is put on the left (this patch is equal to P_1). In Figure 3.1 (and similarly for the other ones), the location of P_6's is indicated as trapezoid-like figures, with the original one $(1, *, *)$ having the double lines. If furthermore we let $B = I$ (resp. $B = II$), then the arrangement of P_6 are given in Figure 3.2 (resp. Figure 3.3) (these patches are equal to P_2 (resp. P_5)). If we put $L = 2$, we necessarily have $B = II$ and the arrangement of P_6 is given in Figure 3.4 (this patch is equal to P_2).

[4]Since Figures 3.1–3.4 are related to Figures 3.5–3.8, we number the figures of this section in this way.

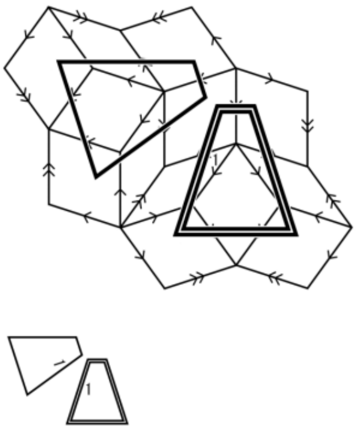

Figure 3.1: The configuration of P_6's in $(1, *, *)$.

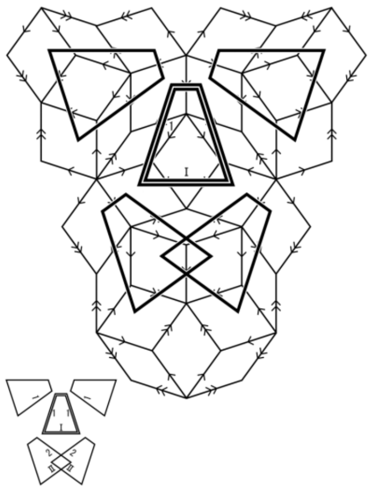

Figure 3.2: The configuration of P_6 in $(1, *, I)$.

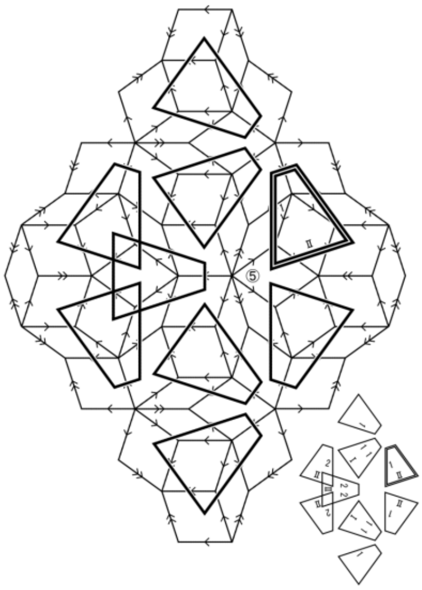

Figure 3.3: The configuration of P_6 in $(1, *, II)$.

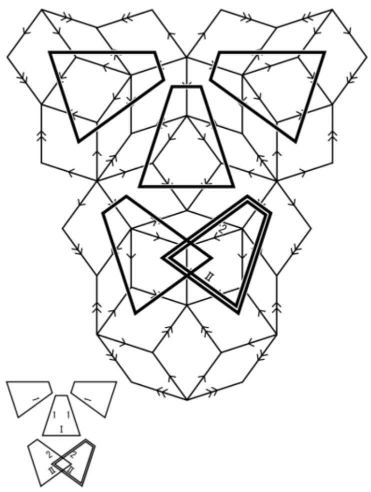

Figure 3.4: The configuration of P_6 in $(2, *, II)$.

We next study the matching rule of P_8. There are two possibilities for each of putting tiles on the sides and bottom of P_8. We write [5] $L = 1$, if $(l_2, \ldots, l_8) = (p_2, p_8, p_1, p_5, p_1, p_4, p_1)$, and write $L = 2$, if $(l_2, \ldots, l_8) = (p_3, p_5, p_2, p_6, p_2, p_8, p_2)$. For

[5]The definition of l_j's, r_j's and b_j's are given in Figure 2.2.8.

the bottom, we write $B = I$, if $(b_1, \ldots, b_7) = (p_2, p_8, p_1, p_5, p_1, p_8, p_2)$ and $B = II$, if $(b_1, \ldots, b_7) = (p_3, p_5, p_2, p_6, p_2, p_5, p_3)$. Then we only have five possibilities for (L, R, B): $(1, 1, I), (1, 1, II), (1, 2, II), (2, 1, II), (2, 2, II)$. Figures 3.5, 3.6, 3.7, and 3.8 show the configurations of $(1, *, *)$, $(1, *, I)$, $(1, *, II)$, $(2, *, II)$ respectively from which we see that the arrangement of P_8 in these figures are the same as that of P_6 in Figures 3.1,. . . , 3.4. Therefore the statement follows clearly from these observations. □

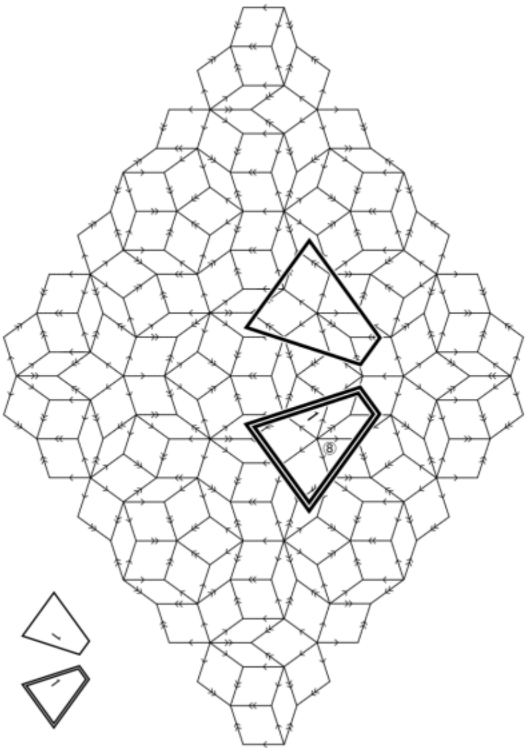

Figure 3.5: The configuration of P_8 in $(1, *, *)$.

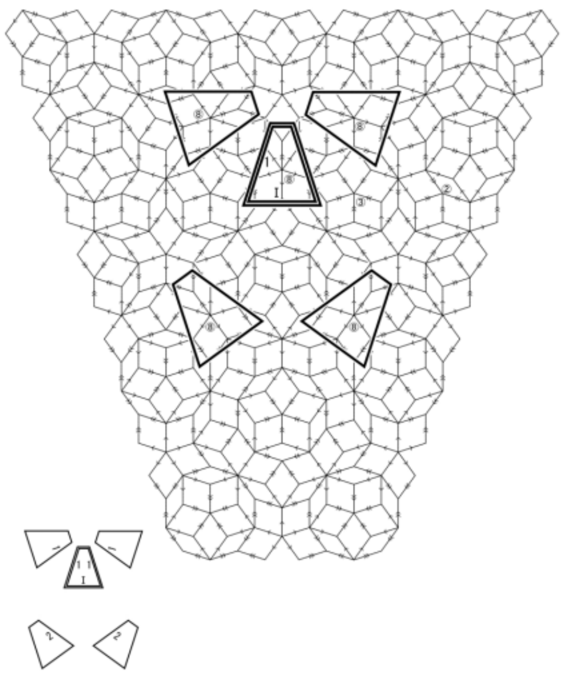

Figure 3.6: The configuration of P_8 in $(1, *, I)$.

Remark 3.1 Alternative proof for Theorem 3.1 is possible. By Corollary 2.1, all configurations of tiles in PT must be explained by the matching rule of P_6. In fact, the arrangement of P_6 in P_8 (Figure 3.0), the matching rule of P_8, and the global rule in §2.1 are all explained by this rule only.

4. The substitution rule of P_6, P_8

By Theorem 2.1, the local arrangement of P_6 can be seen in P_8 (Figure 2.3.2). Furthermore, by Theorem 3.1, it also shows the local arrangement of P_8, if we replace P_6's in Figure 2.3.2 by P_8, which is done in Figure 4.1.

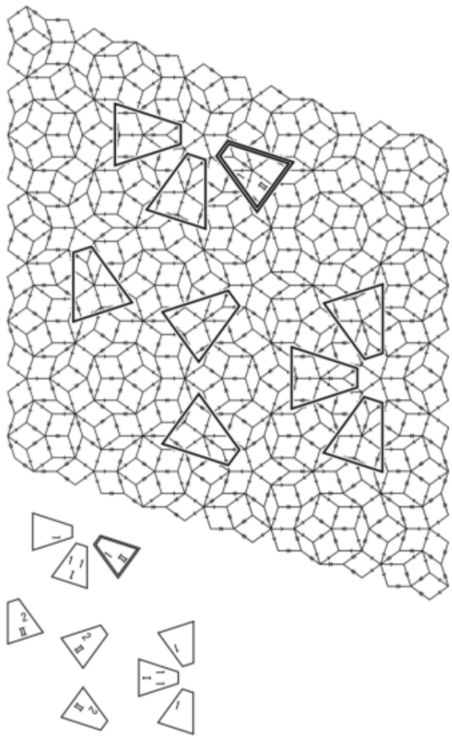

Figure 3.7: The configuration of P_8 in $(1, *, II)$.

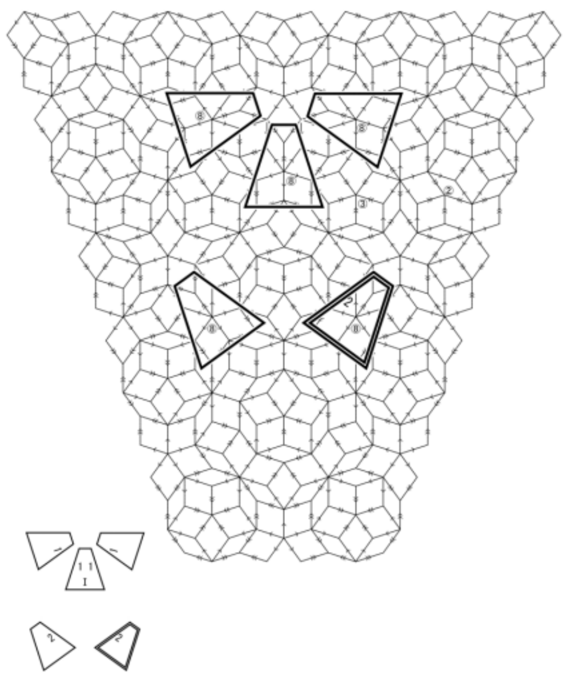

Figure 3.8: The configuration of P_8 in $(2, *, II)$.

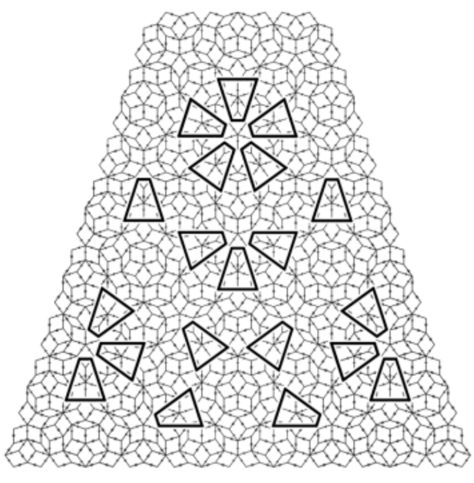

Figure 4.1: The patch obtained from P_8 by replacing P_6's by P_8.

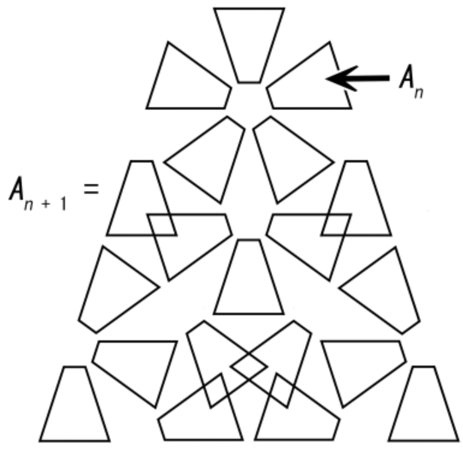

Figure 4.2: The composition of A_{n+1} by A_n's.

We denote this patch in Figure 4.1 by P_8'. Hence we have shown that the PT is the superposition of P_8'. This process can be repeated. We denote this substitution by σ and let $A_0 = P_6, A_1 = P_8, A_2 = \sigma(P_8) = P_8', \ldots, A_{n+1} := \sigma(A_n)$ and so on. A_{n+1} is obtained by arranging A_n's as Figure 4.2 [6]. Thus we obtain the following theorem.

Theorem 4.1 *For any $n = 0, 1, \ldots$, PT is the superposition of A_n. In other words, for any vertex p in PT, we can find a patch congruent to A_n containing p.*

Figure 2.3.1 shows an example of PT in which the locations of A_1, A_2 and A_3 are shown. So we have obtained the sequence $\{A_n\}$ of local charts with diam $(A_n) \to \infty$ which are related via the substitution described above. Theorem 4.1 gives elementary proof of important properties of PT.

Corollary 4.1 (Aperiodicity) *PT is aperiodic.*

Proof. Suppose we have a translation of length l under which the PT is invariant. Pick A_n in PT whose diameter is more than l. The patches congruent to A_{n-1} which are contained in A_n must be mapped by this translation each other which leads us to a contradiction: the local distribution of these patches are distinct. $\square$

Corollary 4.2 (Locally isomorphic property) *In PT, any finite patterns appear infinitely many times.*

Corollary 4.3 *Any PT's are locally isomorphic.*

Proofs of Corollaries 4.2, 4.3 are omitted.

Remark 4.1 By applying the substitution rule to the global rule discussed in §2.1, we obtain global rules of arbitrary large size. For instance, Figure 3.7 represents a rule obtained by applying the substitution once to the rule (1): if the P_8 in the upper left is $(1, 1, *)$, then the one in the lower right must be $(1, 2, *)$.

5. Uncountability

In this section, we show that there are uncountably many PT's which are not congruent each other. In order to do this, we number the twenty patches congruent to A_n in A_{n+1} as $1, 2, \ldots, 20$ (Figure 5.1).

By Theorem 4.1, the configuration of tiles in PT can be determined by specifying the location of A_n in A_{n+1} in each n, which in turn is determined by a sequence

$$\rho = (a_1, a_2, \ldots, a_n, \ldots), \quad a_n \in I := \{1, 2, \ldots, 20\}.$$

Remark 5.1 Since A_n's have overlaps in PT, the A_{n+1} which contain this particular A_n is not uniquely determined and so is a_n in this embedding procedure. We thus define a_n to be the minimum among those numbers.

[6] The "matching rule" of A_n as a prototile is inductively proved to be the same as that of P_6.

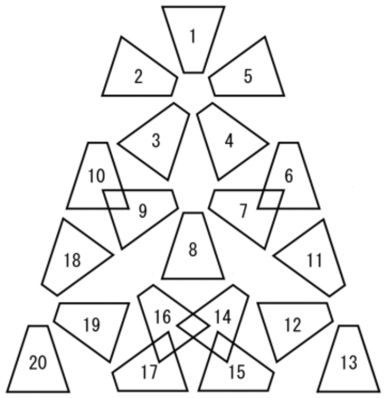

Figure 5.1: The numbering of A_{n-1} which compose A_n.

Since this correspondence $\{a_n\} \mapsto PT(\rho)$ is not one to one, we introduce the notion of the cofinal sequences. We say that two sequences $\{a_n\}, \{b_n\}$ is cofinal iff

$$a_n = b_n, \quad n \geq N$$

for some N.

Lemma 5.1 (cofinal)
Two sequences ρ, ρ' are cofinal iff $PT(\rho)$, $PT(\rho')$ are congruent.

Proof. (i) Suppose ρ, ρ' are cofinal and let

$$\begin{aligned} \rho &= (a_1, a_2, \ldots, a_k, b_{k+1}, b_{k+2}, \ldots) \\ \rho' &= (a'_1, a'_2, \ldots, a'_k, b_{k+1}, b_{k+2}, \ldots) \end{aligned}$$

Then the patch A_{k+1} corresponding to b_{k+1} contains both of that corresponding to a_k, a'_k. Hence PT(ρ) and PT(ρ') coincide from the $(k+1)$-th step so that they are congruent.
(ii) Suppose $PT(\rho)$,$PT(\rho')$ are congruent. We let them coincide by a translation. Letting $A_{\rho,1}, A'_{\rho,1}$ be their starting patch of ρ, ρ', we can find A_n containing both of them for large n. Then we have

$$a_k = a'_k, \quad k \geq n$$

so that ρ and ρ' are cofinal. □

Theorem 5.1 (uncountability)
We have uncountably many PT's which are not congruent each other.

Proof. We write $\rho \sim \rho'$ if they are cofinal. Since it is an equivalence relation, it suffices to show that

$$S := \{\{a_k\}_{k=1}^{\infty} : a_k \in I\} / \sim$$

is uncountable. Suppose S is countable and let $S = \{[\rho^{(n)}]\}_{n=1}^{\infty}$ be its enumeration. We choose representatives $\rho^{(n)}$ from each $[\rho^{(n)}]$. Let $\{p_n\}$ be the set of prime numbers and let $Q(n) := \{nk : k = 1, 2, \ldots\}$ be the set of multiples of n. We construct a sequence $\delta = \{\delta_n\}, \delta_n \in I$ by the following procedure.

$$\begin{aligned}
&\delta_k \neq a_k^{(1)}, \quad k \in Q(2) \\
&\delta_k \neq a_k^{(2)}, \quad k \in Q(3) \setminus Q(2) \\
&\delta_k \neq a_k^{(3)}, \quad k \in Q(5) \setminus (Q(1) \cup Q(2)) \\
&\cdots \\
&\delta_n \neq a_k^{(n)}, \quad k \in Q(p_n) \setminus \bigcup_{j=1}^{n-1} Q(p_j) \\
&\cdots
\end{aligned}$$

Then it is clear that

$$[\delta] \neq [\rho^{(n)}]$$

for any n. □

6. Comparison with the inflation rule

Theorem 6.1 *All PT's constructed by the matching rule can also be obtained by the up-down generation (UD).*

Proof. By Theorem 4.1, it suffices to show that, for any n, we can construct A_n by UD starting from anywhere in A_n.

Lemma 6.1 *Let I_k be the patch made by applying to P_6 the inflation rule k times. For any n, we can find some $k = k(n)$ such that*

$$A_n \subset I_{k(n)}.$$

Lemma 6.1 follows immediately from Lemma 6.2 below.

Lemma 6.2 *Let $B_0 := P_6$, and let B_n be the patch obtained from B_0 by applying the inflation rule $2n$ times. Then for any n, we have*

$$A_{n-1} \subset B_n \subset A_n.$$

and thus

$$\bigcup_n B_n = \bigcup_n A_n.$$

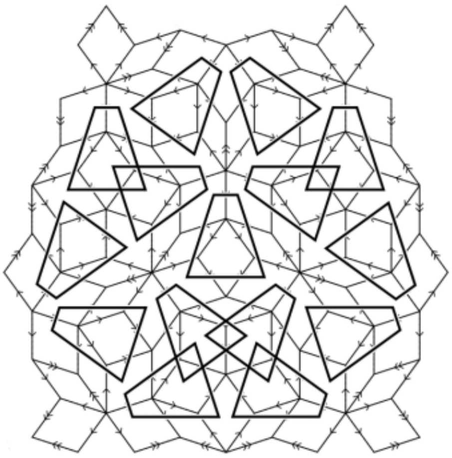

Figure 6.1: The patch obtained by applying the inflation rule twice to P_6.

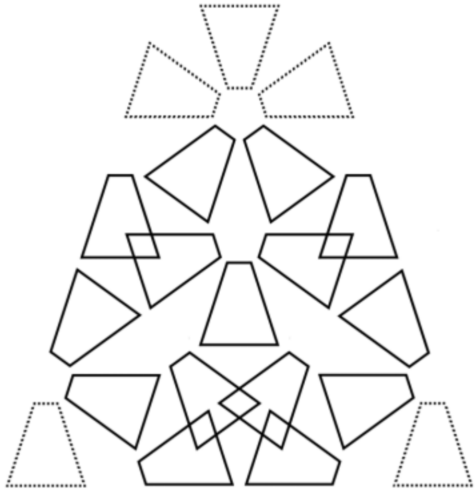

Figure 6.2: The patch in Figure 6.1 is the one made by eliminating five P_6's (located in $1, 2, 5, 13, 20$ in Figure 5.1) in P_8.

Proof. B_1 is equal to the patch by eliminating five P_6's (located in $1, 2, 5, 13, 20$ in Figure 5.1) from P_8 (Figures 6.1, 6.2).

B_2 is obtained by replacing P_6's in B_1 by B_1 so that $B_2 \subset A_2$. Furthermore, the patch congruent to B_1 lying in the center of B_2 has p_8 in its center, and surrounded by B_1's. Hence $A_1 \subset B_2$. By repeating this procedure inductively, we have

$$A_{n-1} \subset B_n \subset A_n$$

for any n. □

Lemma 6.1 implies that for any n we can find large "P_6" which contains A_n (Figure 6.3 shows the case for $n = 1$).

In the construction of PT by UD, we transform PT to a tiling by triangles by dividing the big (resp. small) rhomb along its long (resp. short) diagonal, and then compose triangles along the given composition sequence.

Lemma 6.3 *For any two triangles s, S in PT with $s \subset S$, we can find a composition sequence in UD starting from s such that we obtain S by following this composition sequence.*

Proof. Divide S once into triangles along the substitution atlas and let S_1 be the one which contains s. Then divide S_1 again and let S_2 be the one which contains s. We can repeat this procedure until we arrive at s. □

We pick and fix any triangle s in A_n. Take $k(n)$ such that $A_n \subset I_{k(n)}$. Regard $I_{k(n)}$ as the patch of larger tiling, and take sufficiently large triangle S which contains

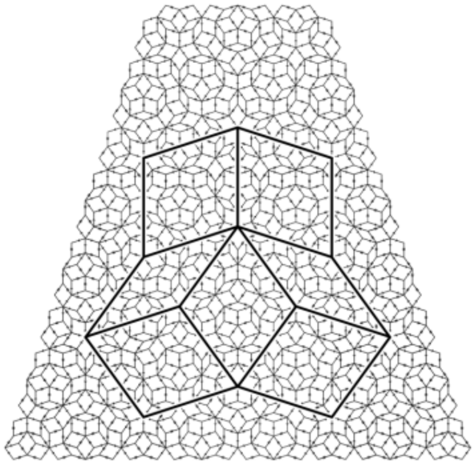

Figure 6.3: A large P_6 containing $A_1 (= P_8)$.

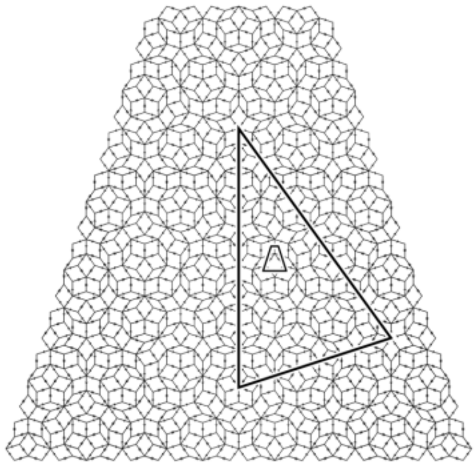

Figure 6.4: An example of P_6 and the triangle which contains it.

$I_{k(n)}$. An example of such P_6 and triangle S containing it is given in Figure 6.4. The big P_6 in Figure 6.3 and small P_6 in Figure 6.4 should be identified. Since

$$s \subset A_n \subset I_{k(n)} \subset S,$$

we can use Lemma 6.3 and find a composition sequence in UD starting from s to construct S, which gives us A_n as its subset. The proof of Theorem 6.1 is thus completed. $\square$

7. Appendix 1

This section provides the argument omitted in the proof of Theorem 2.1.

Lemma 7.1 *If we start tiling from P_1, it is contained by one of P_4, P_5, P_8.*

Proof. Let a, b be the vertices on the boundary of P_1, as is shown in Figure 2.2.1. If we put tiles on the boundary of P_1 and proceed until we can not determine the configuration of tiles uniquely any more, we will find either p_4, p_5 or p_8 somewhere and our patch then becomes P_4, P_5, P_8 respectively. We show this fact by specifying the location of the original p_1 in the patch P_4, P_5, P_8. The conclusion is:
(1) If we put $a = p_5$, we have P_5, that is, P_1 is contained by P_5 and p_1 corresponds to either A or B in Figure 7.1.2.
(2) If $a = p_8$, we have P_8 (Figure 7.1.3, A).
(3) If $b = p_4$, we have P_4 (Figure 7.1.1, A).
(4) If $b = p_8$, we have P_8 (Figure 7.1.3, B or C).
All possibilities are exhausted by those considerations. $\square$

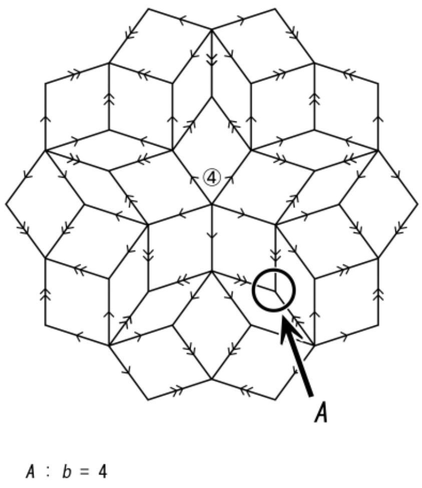

Figure 7.1.1: Case $a = p_4$: when we put $a = p_4$ in P_1, then we have P_4 and the original p_1 is located in A.

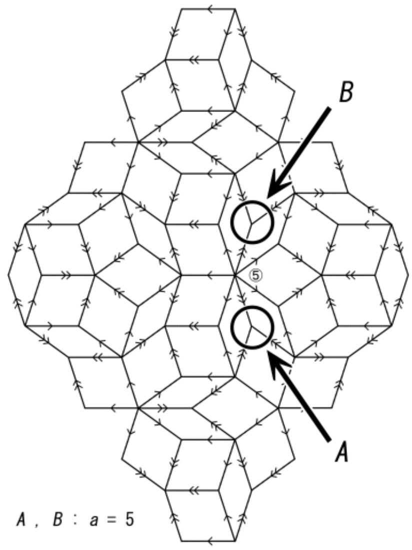

Figure 7.1.2: Case $a = p_5$ in P_1.

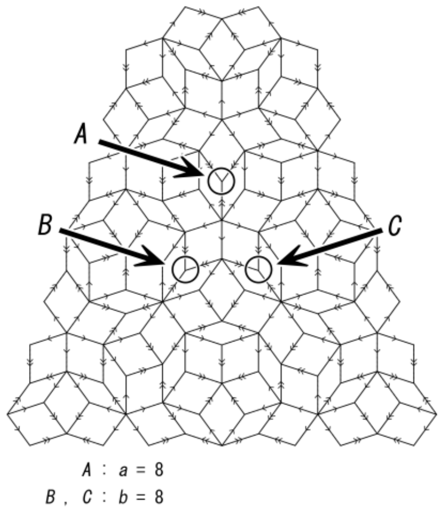

Figure 7.1.3. Case $a = p_8$ or $d = p_8$ in P_1.

Lemma 7.2 *If we start tiling from P_2, it is contained by either P_5 or P_8.*

Proof. We argue as in the proof of Lemma 7.1. Let t, r_1, r_2 be vertices on the boundary of P_2, being shown in Figure 2.2.2.
(1) If $(t, r_1) = (p_4, p_2)$, we have P_8 (Figure 7.2.2, A)
(2) If $(t, r_1) = (p_4, p_3)$ we have P_8 (Figure 7.2.2, B)
(3) If $t = p_8$, we have P_8 (Figure 7.2.2, C)
(4) If $r_2 = p_5$, we have P_5 (Figure 7.2.1, A) □

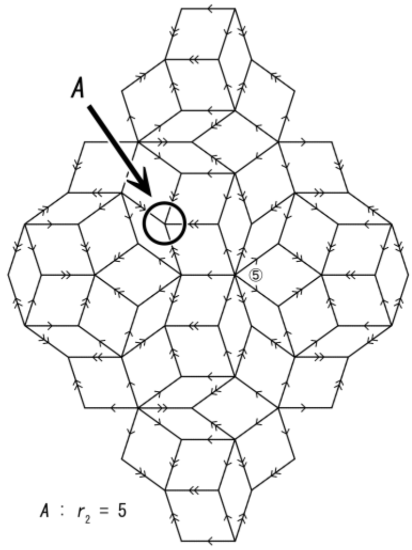

Figure 7.2.1: Case $r_2 = p_5$ in P_2.

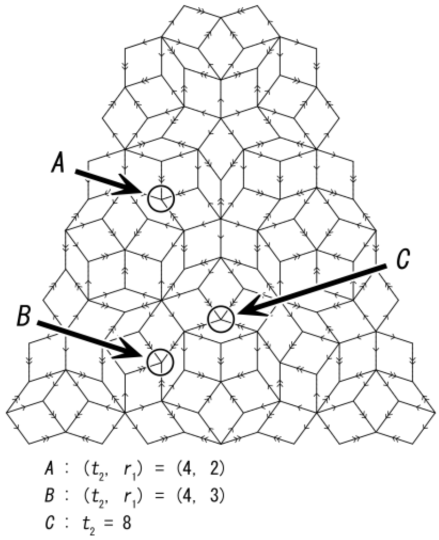

Figure 7.2.2: Case (t, r_1)
$= (p_4, p_2), (p_4, p_3)$, $t = p_8$ in P_2.

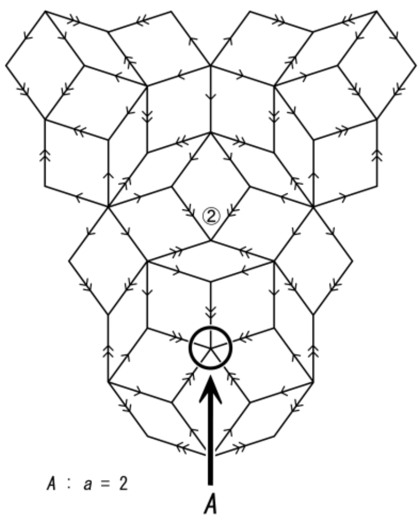

Figure 7.3.1: Case $a = p_2$ in P_3.

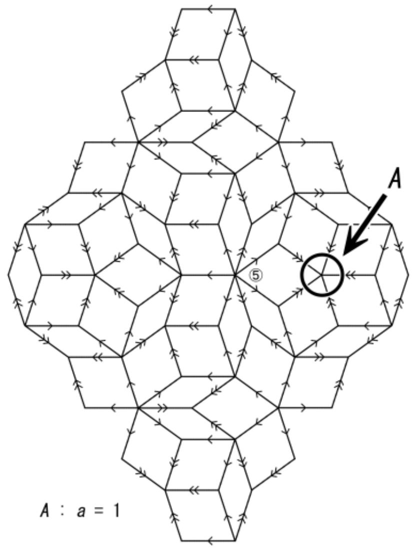

Figure 7.3.2: Case $a = p_1$ in P_3.

Lemma 7.3 *If we start tiling from P_3, it is contained by either P_2 or P_5.*

Proof. Letting a be the vertex shown in Figure 2.2.3, we have

(1) If $a = p_1$, we have P_5 (Figure 7.3.2, A).
(2) If $a = p_2$, we have P_2 (Figure 7.3.1, A). □

Lemma 7.4 *If we start tiling from P_6, it is contained by one of P_1, P_2, P_4, P_5, P_8.*

Proof. Letting a, b, c, d be the vertices shown in Figure 2.2.6, we have
(1) If $a = p_2$, we have P_2 (Figure 7.4.2, A).
(2) If $(a, b) = (p_3, p_1)$, we have P_5 (Figure 7.4.3, A).
(3) If $(a, b) = (p_3, p_2)$, we have P_2 (Figure 7.4.2, B).
(4) If $b = p_1$, we have P_1 (Figure 7.4.1, A).
(5) If $c = p_4, p_5, p_8$, we have P_4, P_5, P_8 respectively (figures are omitted). □

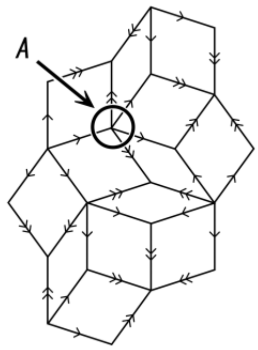

Figure 7.4.1: Case $b = p_1$ in P_6.

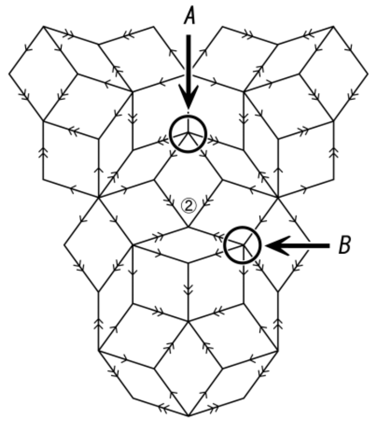

Figure 7.4.2: Case $a = p_2$ or $(a, b) = (p_3, p_2)$ in P_6.

8. Appendix 2

If we connect two single arrows in two rhombs by a curve (Figure 8.1), we have some curves in the Penrose tiling (Figures 8.2, 8.3, 8.4).

In this section, we study some basic properties of these curves [7]. These curves are topologically equivalent to what is found by Conway [4], and in fact the statement of Theorem 8.1 is the same as his observation, which treats the problem on how many curves are of infinite length. We recall that, as was discussed in Section 5, each congruence class of PT are characterized by the sequence $\rho = (a_1, a_2, \ldots), a_n \in I = \{1, 2, \ldots, 20\}$ which indicates how smaller patches are embedded to larger ones.

[7] The curves obtained by connecting double arrows are always closed around each vertices, and are not studied here.

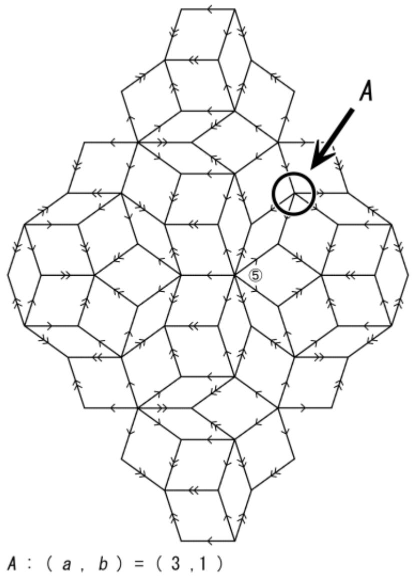

Figure 7.4.3: Case $(a, b) = (p_3, p_1)$ in P_6.

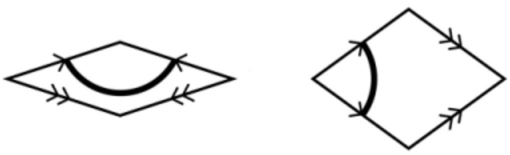

Figure 8.1: curves connecting single arrows in two rhomb tiles.

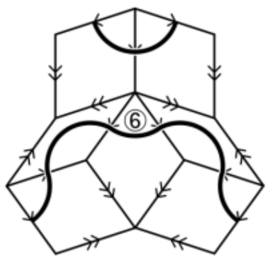

Figure 8.2: curves in P_6.

Theorem 8.1 (i) *If $a_n = 8$ for large n, then PT has two curves of infinite length.*

(ii) *If $a_n \in \{3, 4, 6, 7, 9, 10\}$ or $a_n \in \{12, 14, 15, 16, 17, 19\}$ for large n, then PT has a curve of infinite length.*

(iii) *Otherwise, all curves are closed.*

Proof. Figures 8.2, 8.3, 8.4 show the curves on P_6, P_8, P_8' respectively. By combining two curves in P_6 suitably, we have two long curves, four closed ones, and seven ones with open ends in P_8 (Figure 8.3), which in turn glues together to make two long curves, four closed ones, and seven ones with open ends in P_8' (Figure 8.4). Because the overlapping rules of $A_n's$ are all equivalent, we can show that every A_n has two long curves and four closed ones on the same places. Since the Patch 8 in Figure 5.1 is the only one which includes both of two long curves, and since patches $\{3, 4, 6, 7, 9, 10\}$ include the upper one, and patches $\{12, 14, 15, 16, 17, 19\}$ include the lower one, we have the statement of Theorem 8.1. $\square$

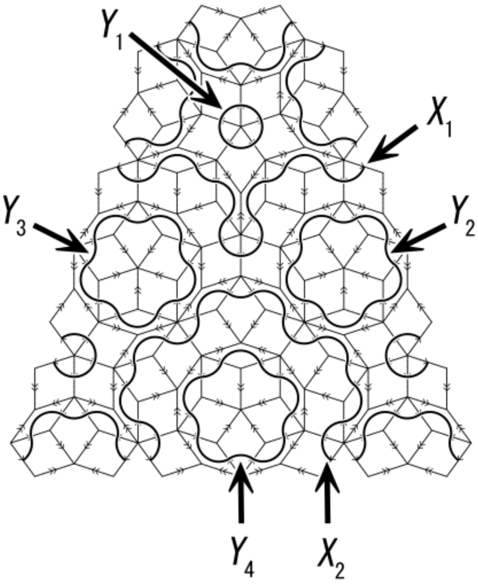

Figure 8.3: curves in P_8.

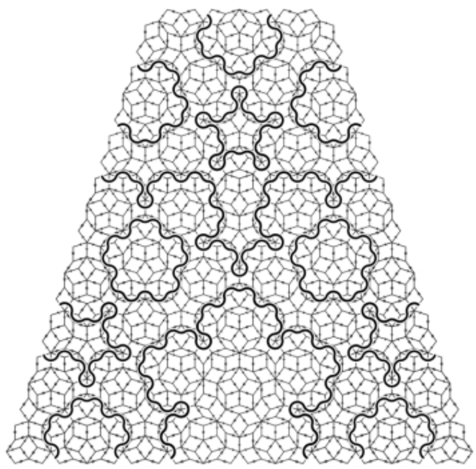

Figure 8.4: curves in P'_8.

The nature of curves (closed or extended) directly reflects how the patch A_n are embedded to A_{n+1} so that we can reconstruct PT from the distribution of them. Some concrete investigation of combination of curves further show that an finite area of PT has five-fold symmetry if and only if they are surrounded by a closed curve.

The self-similar nature of PT implies that these curves have some fractal structure. To have some quantitative statement, we would like to consider "the thermodynamic limit" of A_n and to compute the Hausdorff dimension of that. Since the inflation rule says the size of A_n is proportional to τ^{2n} ($\tau = \frac{1+\sqrt{5}}{2}$), we overlap $\tau^{-2n}A_n$ so that each two long curve and four closed ones in $A'_n s$ defines the fractal set denoted by $X_1, X_2, Y_1, \ldots, Y_4$ in Figure 8.3 [8].By the standard method of computing the Hausdorff dimension (e.g., [3]), we have the following fact.

Theorem 8.2

$$\begin{aligned}
\rho_1 &:= \dim_H(X_1) = \dim_H(Y_1) = \frac{\log 2}{\log \tau}. \\
\rho_2 &:= \dim_H(X_2) = \dim_H(Y_j) = \frac{\log \alpha}{2 \log \tau}, \quad j = 2, 3, 4, \quad \alpha = \frac{3 + \sqrt{17}}{2}.
\end{aligned}$$

[8]To be precise, we define the similarity transformation between $\tau^{-2n}A_n$ and $\tau^{-2(n+1)}A_{n+1}$ and consider the unique invariant set determined by these transformations. For instance, in X_1, the similarity transformation is like that in the snowflake curve but we have also to attach some pieces taken from the lower long curve.

The fact $1 < \rho_2 < \rho_1$ reflects the difference of the density of A_{n-1}'s in A_n between the upper and lower sides of A_n.

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