Nihonkai Math. J.
Vol.19(2008), 105–110

STEINER RATIOS FOR LENGTH SPACES HAVING ENDS

SHINETSU TAMURA AND NOBUHIRO INNAMI

ABSTRACT. We prove that the Steiner ratios for complete locally compact length spaces having n ends are less than or equal to $n/2(n-1)$. In particular, the Steiner ratio of a complete simply connected surface with a pole satisfying the Visibility axiom is $1/2$.

1. Introduction

Let M be a complete locally compact length space. Then, any two points p and q in M can be joined by a minimal geodesic segment whose length is the distance between p and q. Let P be a finite set of points in M. A shortest network interconnecting P is called a *Steiner minimum tree* which is denoted as SMT(P). An SMT(P) may have vertices which are not in P. Such vertices are called *Steiner points*. A spanning tree on P is a tree with vertex set P where all edges are minimal geodesic segments. A shortest spanning tree on P is called a *minimum spanning tree* on P which is denoted as MST(P). Let $L(T)$ be the total length of edges in a tree T. In a complete locally compact length space M two minimal geodesic segments can have a subsegment in common. In that case we consider that all edges in MST(P) connect two points in P even if they contain their subsegment in common. This does not occur if M is an Alexandrov space with curvature bounded below. The *Steiner ratio* is given by

$$\rho = \rho(M) = \inf_{P} \frac{L(\mathrm{SMT}(P))}{L(\mathrm{MST}(P))}.$$

where P runs over all finite sets of points in M. Du and Hwang ([2]) have stated that $\rho(M) = \sqrt{3}/2$ if M is a Euclidean plane. Their statement was the affirmative answer of a famous conjecture of Gilbert and Pollak ([4]). However their proof is not complete because the length of minimum inner spanning trees is not continuous

2000 *Mathematics Subject Classification.* Primary 53C20; Secondary 05C05.
Key words and phrases. Differential geometry, geometry of geodesics, Steiner tree, Steiner ratio.

([6]). Rubinstein and Weng ([8]) have stated that $\rho(M) = \sqrt{3}/2$ if M is a 2-dimensional sphere of constant Gaussian curvature. Ivanov, Tuzhilin and Cieslik ([7]) have estimated some Steiner ratios for manifolds. In particular, they proved that $\rho(M)$ is less than or equal to $\rho(\mathbb{E}^k)$ if M is a smooth Riemannian manifold with dimension k where $\mathbb{E}^k$ is a k-dimensional Euclidean space, and that $\rho(M) \geq \rho(\widetilde{M})$ if $\widetilde{M}$ is a covering manifold over a smooth Riemannian manifold M. Innami and Kim ([5]) have recently proved that $\rho(M) = 1/2$ if M ia a complete simply connected 2-dimensional Riemannian manifold having constant negative Gaussian curvature, namely, a Poincaré disk.

We say that M has n *ends* if there exists a compact set K in M such that $M - K$ consists of n connected components which are unbounded and there is no compact set K' such that $M - K'$ has $n+1$ connected components which are unbounded.

In the present note we will prove the following theorem.

Theorem 1.1. *Let M be a complete locally compact length space with n ends. Then, the Steiner ratio of M is less than or equal to $n/2(n-1)$.*

We do not assume in Theorem 1.1 that the dimension of M is two. If the dimension of M is two, then a certain set of straight lines surrounds a domain with ends, and hence, we have the following corollary.

Corollary 1.1. *Let M be a complete Alexandrov surface homeomorphic to a plane with curvature bounded below. If M has n straight lines which are mutually disjoint such that each straight line has all other straight lines in one hand side of it, then we have that $1/2 \leq \rho(M) \leq n/2(n-1)$.*

Let M be a complete Alexandrov surface with curvature bounded below. We can see the definition of Alexandrov spaces in [1]. A point p in M is called a *pole* of M if all the geodesics emanating from p are minimal geodesic rays in M. Hence, if M has a pole, then M is homeomorphic to a plane and its singular point is possibly the pole p, since no singular points exist in the interior of any minimal geodesic segment. We say that a point p in M satisfies the *Visibility axiom* if for any two distinct rays $q(s)$, $0 \leq s < \infty$, and $r(t)$, $0 \leq t < \infty$, from $p = q(0) = r(0)$ any sequence of minimal geodesic segments $T(q(s), r(t))$ have a subsequence $T(q(s_i), r(t_i))$ converging to a minimal geodesic line which is called a *straight line* as $s_i \longrightarrow \infty$ and $t_i \longrightarrow \infty$. If all points in M satisfy the Visibility axiom, then M is said to satisfy the Visibility axiom. The Visibility axiom was first defined for complete Riemannian manifolds with nonpositive curvature in [3]. There are many examples of Riemannian manifolds satisfying the Visibility axiom in the class of complete simply connected Riemannian manifolds with nonpositive curvature. However we use the axiom in the above sense.

Corollary 1.2. *Let M be a complete Alexandrov surface homeomorphic to a plane with curvature bounded below. If M has a point p such that there are infinitely many rays from the point p and it satisfies the Visibility axiom, then $\rho(M) = 1/2$.*

This corollary implies that the Steiner ratio of Poincaré disks is 1/2 ([5]).

2. Preliminaries

Let M be a complete locally compact length space with distance $d(\cdot, \cdot)$. Then, there exists a minimal geodesic segment $T(p, q)$ connecting any points p and q in M. Two minimal geodesic segments can have their subsegment in common. Then, SMT(P) satisfies the following property.

Lemma 2.1. *Let M be a complete locally compact length space and let P be a finite set of points in M. Then,*

(i) *all terminal points of* SMT*(P) are points in P.*

In additon, if M is an Alexandrov surface with curvature bounded below, then SMT(P) satisfies the following properties.

Lemma 2.2. *Let M be a complete Alexandrov surface with curvature bounded below and P a set of n points in M. Then,* SMT(P) *satisfies the following properties.*

(ii) *Any two edges meet at an angle of at least $2\pi/3$.*

(iii) *Every Steiner point has degree exactly three, and hence, is not a singular point. The edges emanating from it to three neighboring vertices are unique minimal geodesic segments.*

(iv) *There are at most $n - 2$ Steiner points in* SMT(P).

Proof. The properties (i) and (iv) are proved in the same way for the Euclidean case ([2]). We prove (ii). We can see the notation and some properties of Alexanvrov spaces in [1]. Suppose there exists a vertex p where two edges meet at the angle $\theta < 2\pi/3$. Let $T(p, q)$ and $T(p, r)$ be those edges. Let q_1 and r_1 be points in $T(p, q)$ and $T(p, r)$ with $a = d(p, q_1) = d(p, r_1)$. For a sufficiently small positive ϵ the part of the circle $C(p, \epsilon) = \{y \in M \mid d(p, y) = \epsilon\} \cap \triangle(pq_1r_1)$ contains a point s with $b = d(s, q_1) = d(s, r_1)$, since $d(r_1, q_2) > d(r_1, r_2)$ and $d(q_1, r_2) > d(q_1, q_2)$ where q_2 and r_2 are the endpoints of the subarc of $C(p, \epsilon)$ in $T(p, q_1)$ and $T(p, r_1)$, respectively. The quadrangle $\tilde{p}\tilde{q}_1\tilde{s}\tilde{r}_1$ consisting of comparison triangle domains $\triangle(\tilde{p}\tilde{q}_1\tilde{s})$ and $\triangle(\tilde{p}\tilde{r}_1\tilde{s})$ has the angle $\angle(\tilde{q}_1\tilde{p}\tilde{r}_1)$ less than $2\pi/3$ with $\theta_1 = \angle(\tilde{q}_1\tilde{p}\tilde{s}) = \angle(\tilde{r}_1\tilde{p}\tilde{s}) < \theta/2$. Hence, we have that

$$\tau = 2\cos\theta_1 > 1 \quad \text{and} \quad b^2 = a^2 + \epsilon^2 - \tau a\epsilon.$$

Then, we have that

$$a^2 - \left(b + \frac{\epsilon}{2}\right)^2 = b\epsilon \left(\frac{\tau a}{b} - 1 - \frac{5}{4}\frac{\epsilon}{b}\right).$$

Therefore, we can get a sufficiently small positive ϵ such that

$$a > b + \frac{\epsilon}{2}.$$

Thus, we have that

$$\begin{aligned} d(p, q_1) + d(p, r_1) &= d(\tilde{p}, \tilde{q}_1) + d(\tilde{p}, \tilde{r}_1) \\ &= 2a > 2\left(b + \frac{\epsilon}{2}\right) \\ &= d(\tilde{q}_1, \tilde{s}) + d(\tilde{r}_1, \tilde{s}) + d(\tilde{p}, \tilde{s}) \\ &= d(q_1, s) + d(r_1, s) + d(p, s), \end{aligned}$$

contaradicting that $T(p, q)$ and $T(p, r)$ are edges in $\mathrm{SMT}(P)$.

The property (iii) follows (ii) because there are at least three edges emanating from this point, each angle is at least $2\pi/3$ and the total angle around this point is at most 2π. This completes the proof. □

The following is proved in the same way as was seen in [5].

Lemma 2.3. *Let M be a complete Alexandrov surface with curvature bounded below. Then,*

$$\rho(M) \geq 1/2.$$

This lemma shows the lower bound 1/2 in Corollary 1.1.

3. Proofs of Theorem 1.1, Corollaries 1.1 and 1.2

Proof of Theorem 1.1. Let K be a compact set such that $M - K$ consists of n connected components $U_1, U_2, \ldots, U_n$ which are unbounded. Let $p \in K$. For a sufficiently large number s we take a set $P(s)$ of n points $p_1(s), p_2(s), \ldots, p_n(s)$ such that $p_i(s) \in U_i$ and $d(p, p_i(s)) = s$ for all $i = 1, 2, \ldots, n$. Let $r_{ij}(s)$ be a foot of p on a minimal geodesic segment $T(p_i(s), p_j(s))$ for any $i \neq j$, nemaly, $r_{ij}(s) \in T(p_i(s), p_j(s))$ and $d(p, r_{ij}(s)) = d(p, T(p_i(s), p_j(s)))$. Since each $T(p_i(s), p_j(s))$ passes through K, there exists a positive R such that $d(p, r_{ij}(s)) < R$ for all $i \neq j$ and for all s. By the triangle inequality we have

$$\begin{aligned} d(p_i(s), p_j(s)) &\leq d(p, p_i(s)) + d(p, p_j(s)) = 2s \\ &\leq 2d(p, r_{ij}(s)) + d(r_{ij}(s), p_i(s)) + d(r_{ij}(s), p_j(s)) \\ &\leq 2d(p, r_{ij}(s)) + d(p_i(s).p_j(s)) \\ &< 2R + d(p_i(s), p_j(s)), \end{aligned}$$

for $i \neq j$. Since $d(p, p_i(s)) = d(p, p_j(s)) = s$, we see that

$$\frac{2s}{d(p_i(s), p_j(s))} \longrightarrow 1$$

as $s \longrightarrow \infty$. The convergence is uniform for $i \neq j$ as $s \longrightarrow \infty$. A minimum spanning tree $\mathrm{MST}(P(s))$ consists of $n-1$ minimal geodesic segments interconnecting $P(s)$ which are in the set of the minimal geodesic segments $T(p_i(s), p_j(s))$ for $i \neq j$. Thus, we have that

$$\frac{2(n-1)s}{L(\mathrm{MST}(P(s)))} \longrightarrow 1$$

as $s \longrightarrow \infty$. On one hand, we have that

$$L(\mathrm{SMT}(P(s))) \leq \sum_{i=1}^{n} d(p, p_i(s)) = ns.$$

Therefore, we have that

$$\frac{L(\mathrm{SMT}(P(s)))}{L(\mathrm{MST}(P(s)))} \leq \frac{2(n-1)s}{L(\mathrm{MST}(P(s)))} \frac{ns}{2(n-1)s},$$

and the right hand side converges to $n/2(n-1)$ as $s \longrightarrow \infty$, and hence,

$$\rho(M) \leq \frac{1}{2} \frac{n}{n-1}.$$

This completes the proof of Theorem 1.1. $\square$

Proof of Corollary 1.1. From the assumption there exists a domain U surrounded by n straight lines. The domain U has n ends. Since the boundary ∂U of U consists of straight lines, the domain U is convex, and hence, a complete locally compact length space. Theorem 1.1 and Lemma 2.3 prove that $\rho(M) \leq \rho(U) \leq n/2(n-1)$. This completes the proof of Corollary 1.1. $\square$

Proof of Corollary 1.2. If M has a point satisfying the assumption in Corollary 1.2, then for any positive integer n we can have n straight lines in M which surround a domain with n ends. This completes the proof of Corollary 1.2. $\square$

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(Shinetsu Tamura) Graduate School of Science and Technology, Niigata University, Niigata 950–2181, Japan

(Nobuhiro Innami) Department of Mathematics, Faculty of Science, Niigata University, Niigata 950–2181, Japan
E-mail address: `innami@math.sc.niigata-u.ac.jp`

Received April 1, 2008
Revised September 17, 2008