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ELEMENTARY PROPERTIES OF CIRCLE MAP SEQUENCES

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Abstract. We study the combinatorial and structural properties of the circle map sequences. We introduce an embedding procedure which gives a map $\Phi : \Omega \to W := \{R, L\}^{\mathbf{N}}$ from the hull(closure of the set of translates) to the sequence of embedding operations through which we study the structure of Ω. We also study the set of admissible words and classify them in terms of their appearance.

1. Introduction

The circle map $v_0 \in \{0,1\}^{\mathbf{Z}}$ of rotation number $\alpha \in (0,1) \cap \mathbf{Q}^c$ is defined by

$$v_0(n) := 1_{[1-\alpha,1)}(n\alpha \bmod 1), \quad n \in \mathbf{Z}.$$

We first recall its basic properties [6]. Let

$$\alpha = [a_1, a_2, \ldots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}, \quad \alpha_n := [a_1, a_2, \ldots, a_n] = \frac{p_n}{q_n}$$

be the continued fraction expansion of α and its rational approximation ($a_n \in \mathbf{N}$ and p_n, q_n are relatively prime). p_n and q_n satisfy

$$\begin{cases} p_{n+1} = a_{n+1}p_n + p_{n-1} \\ q_{n+1} = a_{n+1}q_n + q_{n-1} \end{cases} \quad n \geq 0 \tag{1.1}$$

with $(p_{-1}, q_{-1}) = (1, 0)$, $(p_0, q_0) = (0, 1)$. Let $s_n \in \mathcal{A}^* := \bigcup_{n\geq 1}\{0,1\}^n$ be the word given recursively by

$$s_{-1} = 1, \ s_0 = 0, \ s_1 = s_0^{a_1-1}s_{-1}, \ s_{n+1} = s_n^{a_{n+1}}s_{n-1}, \ n \geq 1.$$

Then s_n has length q_n and coincides with $(v_0(1), v_0(2), \ldots, v_0(q_n))$ and also coincides with $(v_0(-q_n+1), v_0(-q_n+2), \ldots, v_0(-1), v_0(0))$ if n is even; in other words,

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$(v_0(n))_{n\geq 1}$ is the right limit of s_n and $(v_0(n))_{n\leq 0}$ is the left limit of s_{2n}. s_n $(n \geq 1)$ can be written as

$$s_n = \pi_n \begin{cases} (10) & (n : \text{even}) \\ (01) & (n : \text{odd}) \end{cases}$$

where π_n is a palindrome. If α is the reciprocal number of the golden number ($\alpha = \frac{1}{\tau} := \frac{\sqrt{5}-1}{2} = [1, 1, \ldots]$), then $s_1 = 1$, $s_2 = 10$, $s_3 = 101$, $s_4 = 10110$, $\ldots$ and v_0 is called the Fibonacci word which is thoroughly studied. We give the topology of pointwise convergence on $\{0, 1\}^{\mathbf{Z}}$ (the product topology of the discrete topology on $\{0, 1\}$) and let

$$\Omega := \text{ closure of } \{v_0(\cdot - m)\}_{m\in\mathbf{Z}}$$

which is called the hull of v_0 and has the following representation.

$$\begin{aligned} \Omega &= \{v_\theta\}_{\theta\in\mathbf{T}} \cup \{v_0'(\cdot - m)\}_{m\in\mathbf{Z}} \qquad (1.2) \\ v_\theta(n) &:= 1_{[1-\alpha,1)}(n\alpha + \theta \bmod 1), \quad \theta \in \mathbf{T}, \\ v_0'(n) &:= 1_{(1-\alpha,1]}(n\alpha \bmod 1). \end{aligned}$$

Circle map sequences have the property that (1) minimal complexity, and (2) aperiodic and balanced. Actually, these three conditions are mutually equivalent [7], and for that reason circle map sequences are sometimes called Sturmian sequences.

The purpose of this paper is to study some elementary properties of v_0. In Section 2, we consider Fibonacci word and introduce an "embedding procedure" to construct elements of Ω to study the the combinatorial properties of v_0. This is essentially a special case of the "desubstitution" [3, 6], which is studied well, though the formulation given here is slightly different. We review the relationship between this embedding and the two interval exchange dynamical system inheriting in the Fibonacci word, by which we study property of a measure on $\mathbf{T}$ induced by a random embedding.

In Section 3, we consider the set of admissible words of v_0 and study how they distribute in v_0. We classify them in terms of their occurrence in v_0 and compute their frequency. As is discussed (in more general context) in [1], this classification gives us an alternative proof of the three-distance theorem[10]. In Appendix 1, we collect some basic properties of the embedding procedure. In Appendix 2, we discuss a combinatorial property of the circle map sequence which follows easily from the embedding procedure.

In what follows, the definition of notation $|A|$ for a set A should be clear from the context: it means the number of its elements if $A \subset \mathbf{Z}$, while it means the Lebesgue measure if $A \subset \mathbf{R}$.

2. An embedding procedure

In this section, we consider the case of Fibonacci word: $a_n = 1$. We first define the "embedding procedure."

2.1. Definition

We first explain the motivation of considering this procedure. Since we have $s_{n+1} = s_n s_{n-1}$ in Fibonacci word, it is possible to embed s_k to a larger $s_{k'}$ by either of the following two operations.

(i) $R: \ s_n \mapsto s_{n+1} := s_n s_{n-1}$,

(ii) $L: \ s_n \mapsto s_{n+1} s_n =: s_{n+2}$.

After infinitely many operations, we will have an element of Ω. The converse will turn out to be true: every $v \in \Omega$ is obtained by this procedure. Utilizing this fact, we would like to consider an analogue of the "up-down generation" in the construction of the Penrose tiling. To define it properly, we first recall the results in [2] which applies to any circle map sequences. The $(n-1, n)$-partition is the non-overlapping covering of a sequence $\{v(n)\}_{n \in \mathbf{Z}}$ by two words s_{n-1}, s_n.

Lemma 2.1 *[2] For any $n \geq 0$, $v \in \Omega$ has unique $(n-1, n)$-partition.*

Corollary 2.1 *[2] In the $(n-1, n)$-partition of $v \in \Omega$,*

(i) *s_{n-1} does not appear consecutively (s_{n-1} is always isolated);*

(ii) *s_n always appears a_{n+1} or $(a_{n+1} + 1)$ times successively.*

Let

$$W := \{(O_1, O_2, \ldots) \mid O_j = R \text{ or } L\} = \{R, L\}^{\mathbf{N}}.$$

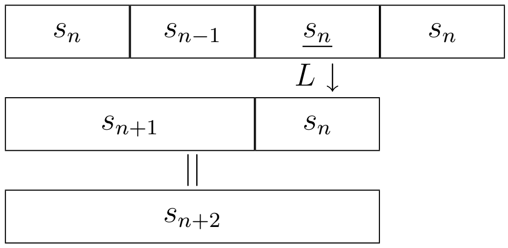
For given $v \in \Omega$, we construct the sequence $(O_1, O_2, \ldots) \in W$ of operations by the following procedure.

(i) When $v(0) = 1$, $v(0)$ is covered by s_1 in the $(0, 1)$-partition. Set $O_1 = R$. When $v(0) = 0$, $v(0)$ is covered by s_2 in the $(1, 2)$-partition, for we have $(v_0(-1), v_0(0), v_0(1)) = (1, 0, 1)$. Set $O_1 = L$.

(ii) Suppose $v(0)$ is covered by a block $\underline{s_n}$ in the $(n-1, n)$-partition after the k-th step. If we find s_{n-1} in the right to $\underline{s_n}$ in the $(n-1, n)$-partition, then $v(0)$ is covered by s_{n+1} in the $(n, n+1)$-partition. In this case we regard that the block $\underline{s_n}$ containing $v(0)$ grow up to s_{n+1} by putting s_{n-1} to its right end, so that we set $O_{k+1} = R$.

$$\begin{array}{l} \boxed{\quad \underline{s_n} \quad} \\ \ \ \downarrow R \\ \boxed{\quad s_n \quad}\boxed{s_{n-1}} \\ \qquad\quad \| \\ \boxed{\qquad s_{n+1} \qquad} \end{array}$$

If we find s_n in the right to $\underline{s_n}$, then $v(0)$ is still covered by s_n in the $(n, n+1)$-partition, and is then covered by s_{n+2} in the $(n+1, n+2)$-partition. In this case, we regard that the block $\underline{s_n}$ containing $v(0)$ grow up to s_{n+2} by putting s_{n+1} to its left end, so that we set $O_{k+1} = L$.



In other words, if we find s_{n-1} in the right to $\underline{s_n}$ in the $(n-1, n)$-partition, then we set $O_{k+1} = R$; otherwise we find s_{n+1} in the left to $\underline{s_n}$ in the $(n+1, n+2)$-partition, and we set $O_{k+1} = L$. Hence we have defined a map $\Phi : \Omega \to W$.

Remark 2.1 It is possible to define this map for any circle map sequences. In the n-th level, the embedding procedure is given by

$$\begin{aligned} &R_{(n,k)} : s_n \mapsto s_n^{a_{n+1}} s_{n-1}, \quad k = 1, 2, \ldots, a_{n+1} \\ &L_n : s_n \mapsto s_{n+1}^{a_{n+2}} s_n \end{aligned}$$

$R_{(n,k)}$ means to embed s_n to the k-th s_n in $s_{n+1} = s_n^{a_{n+1}} s_{n-1}$. This method also applies to the period-doubling sequence which is the fixed point of the substitution: $1 \mapsto 10$, $0 \mapsto 11$.

2.2. The inverse map

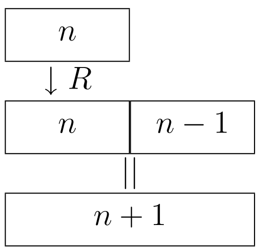
To see Φ is surjective and to find the subset of Ω on which Φ is one to one, we study how to reconstruct $v \in \Omega$ for given $(O_1, O_2, \ldots) \in W$ ($O_j = R$ or L).

$O_1 = R$: Set $v(0) = 1$. Then $v(0)$ is covered by s_1 in the $(0, 1)$-partition.

$O_1 = L$: Set $v(0) = 0$. Then we have $(v(-1), v(0), v(1)) = (1, 0, 1)$ so that $v(0)$ is covered by s_2 in the $(1, 2)$-partition.

After the k-th step, suppose that $v(0)$ is covered by $\underline{s_n}$ in the $(n-1, n)$-partition.

$O_{k+1} = R$: we put s_{n-1} to the right end of $\underline{s_n}$ in the $(n-1, n)$-partition.



Then $v(0)$ is covered by s_{n+1} in the $(n, n+1)$-partition.

$O_{k+1} = L$: we put s_{n+1} to the left end of $\underline{s_n}$ in the $(n, n+1)$-partition.

$$\begin{array}{cc} & \boxed{\quad n \quad} \\ & L\downarrow \\ \boxed{\; n+1 \;} & \boxed{\quad n \quad} \\ \multicolumn{2}{c}{\|} \\ \multicolumn{2}{c}{\boxed{\qquad\quad n+2 \qquad\quad}} \end{array}$$

Then $v(0)$ is covered by s_{n+2} in the $(n+1, n+2)$-partition. We remark that, when $v(0)$ is covered by $\underline{s_n}$ in the $(n-1, n)$-partition, a number of letters has been further determined to the right of that and thus, in most cases, repeating this procedure determines a bi-infinite sequence $(v(n))_{n\in\mathbf{Z}}$. In fact, we always find π_{n+1} to the next to $\underline{s_n}$, since we have either $\underline{s_n}s_{n-1}s_n$ $(O_{k+1} = R)$ or $\underline{s_n}s_n s_{n-1}$ $(O_{k+1} = L)$ in the $(n-1, n)$-partition. Because $s_{n-1}\pi_n = \pi_{n+1}$, they are equal to either $\underline{s_n}\pi_{n+1}(10)$ or $\underline{s_n}\pi_{n+1}(01)$. However if $O_j = R$ for large j, we have a semi-infinite sequence: $(v(n))_{n\geq -N}$ for some N, and $(v(n))_{n\leq -N-1}$ is not determined. In this case $(v(n))_{n\geq -N}$ is equal to a translation of $(v_0(n))_{n\geq 1}$: $v(-N+n-1) = v_0(n)$, $n \geq 1$. So by (1.2) we set either $(v(-N-2), v(-N-1)) = (1,0)$ or $(0,1)$ and further set $v(-N-n) = v_0(n-2)$ for $n \geq 3$ so that we obtain an element of

$$\Omega_R := \{v_0(\cdot + m),\ v_0'(\cdot + m) \mid m \geq 1\}.$$

Hence, Φ is two to one on Ω_R and one to one elsewhere. Under the topology of the pointwise convergence on Ω and W, Φ and $(\Phi : \Omega_R^c \to \Phi(\Omega_R^c))^{-1}$ are continuous. Φ has an unique fixed point $f := (L, R, R, L, R, L, R, R, L, \ldots)$ if we identify R, L with $1, 0$ respectively and $v(n)$ with O_{n+1}.

Remark 2.2 By this method we see the correlation (constraint condition) of letters between different sites. In fact, if n is even, both $(10)s_n\pi_{n+1}(10)$ and $(01)s_n\pi_{n+1}(10)$ are allowed while only $(01)s_n\pi_{n+1}(01)$ is possible (for odd n, exchange (10) with (01)).

2.3. Relation to the division of intervals in T

Let $\Psi : \mathbf{T} \to \Omega$ be the map $\theta \in \mathbf{T} \mapsto v_\theta \in \Omega$. We consider the inverse image of the cylinder set of Ω: e.g.,

$$\Psi^{-1}(\{v(0) = 1\}) = \left[\frac{1}{\tau^2}, 1\right), \quad \Psi^{-1}(\{v(0) = 0\}) = \left[0, \frac{1}{\tau^2}\right). \tag{2.1}$$

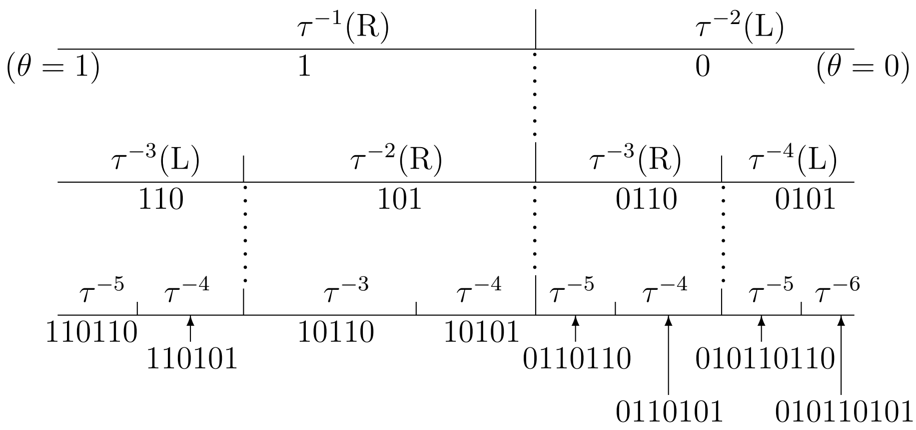
If we go further, each interval is divided into two intervals with ratio $\tau : 1$.

$$\begin{aligned}
\Psi^{-1}(\{(v(0), v(1), v(2)) = (1,1,0)\}) &= \left[1 - \frac{1}{\tau^3}, 1\right), \\
\Psi^{-1}(\{(v(0), v(1), v(2)) = (1,0,1)\}) &= \left[\frac{1}{\tau^2}, 1 - \frac{1}{\tau^3}\right), \\
\Psi^{-1}(\{(v(0), v(1), v(2), v(3)) = (0,1,1,0)\}) &= \left[\frac{1}{\tau^4}, \frac{1}{\tau^2}\right), \\
\Psi^{-1}(\{(v(0), v(1), v(2), v(3)) = (0,1,0,1)\}) &= \left[0, \frac{1}{\tau^4}\right).
\end{aligned}$$

Similarly, we consider $\Psi^{-1}(A_n)$ for $A_n = \{v \in \Omega \mid v(0) = a_0, v(1) = a_1, \ldots, v(n) = a_n\}$ which corresponds to the two interval exchange dynamical system given by (2.1). As n becomes large, we have many intervals whose endpoints belong to

$$D_- = \{x \mid x \equiv n\alpha \pmod 1),\ n = 0, -1, -2, \ldots\}$$

Since the induced system given by the first return map to each small interval is again the two interval exchange, each new interval is given by dividing each intervals into two ones with ratio $\tau : 1$, with the longer one has the previous dividing point as one of its endpoints.



The operations R, L correspond to those division of intervals in the following way [3].

Theorem 2.1 *The operation R (resp. L) corresponds to creating the longer (resp. smaller) interval.*

Proof. Since the division of intervals corresponds to the words $\underline{s_n}\pi_{n+1}(10)$ or $\underline{s_n}\pi_{n+1}$ (01), under the mapping $\Psi : \mathbf{T} \to \Omega$, it corresponds either to R or L. It then suffices to note that L creates the word with the same ending of the original one, while R

creates the word with the opposite ending: $\cdots(01) \xrightarrow{L} \cdots(01)$, $\cdots(01) \xrightarrow{R} \cdots(10)$.
$\square$

Remark 2.3 If $\alpha \neq \frac{1}{\tau}(=\frac{\sqrt{5}-1}{2})$, we do not have such a simple relation except for quadratic numbers. In fact, we have many types $R_{(n,k)}$'s of embedding operations for general α and the induced system given by the first return map is not the two interval exchange in general.

Remark 2.4 For given $w = (O_1, O_2, \ldots) \in W$, we can compute the corresponding $\theta = (\Phi \circ \Psi)^{-1}(w)$ as follows.

$$\begin{aligned} \theta &= \sum_{n=0}^{\infty} d_n, \\ d_0 &= 1,\ d_1 = -\frac{1}{\tau},\ d_{n+1} = (-1)^{a_n+1}\left(\frac{1}{\tau}\right)^{a_n+1}\left(\frac{1}{\tau^2}\right)^{b_n}, \quad n \geq 1 \end{aligned}$$

where $a_n := \sharp\{1 \leq k \leq n \ : \ O_k = R\}$, $b_n := \sharp\{1 \leq k \leq n \ : \ O_k = L\}$. This is equivalent to represent $\theta \in \mathbf{T}$ in terms of the sum of $\{\frac{1}{\tau^k}\}_{k \geq 1}$.

Remark 2.5 For $w = (w_1, w_2, \ldots, w_n) \in \mathcal{A}^*$, let $w^{-1} := (w_n, \ldots, w_2, w_1)$ be its mirror image. Then $t_n := s_n^{-1}$ satisfies

$$t_n = \begin{cases} (01)\pi_n & (n : \text{even}) \\ (10)\pi_n & (n : \text{odd}) \end{cases} \qquad \text{and} \qquad t_{n+1} = t_{n-1}t_n.$$

Since $v \in \Omega \iff v^{-1} \in \Omega$, $v \in \Omega$ always has $(n-1, n)$-partition by t_n so that we can define embedding R', L' by using t_n's in the same way as R, L (we set $v(-1)$ as the starting point). We have analogue of Theorem 2.1, and for n even both $(01)\pi_{n+1}t_n(10)$ and $(01)\pi_{n+1}t_n(01)$ are possible while only $(10)\pi_{n+1}t_n(10)$ is allowed.

2.4. A measure induced by the random embedding

Let m be a measure on $\{R, L\}$ with $m(\{R\}) = p \in (0,1)$, $m(\{L\}) = q := 1 - p$ and let $\mathbf{P} := \otimes_{\mathbf{N}} m$. In this section we study the measure μ on $\mathbf{T}$ induced by the mapping $\Phi \circ \Psi : \mathbf{T} \to W$. This may be regarded as an analogue of the Bernoulli convolution problem [8]. Since $\mathbf{P}(\{O_j = R \text{ for large } j\}) = \mathbf{P}(\{O_j = L \text{ for large } j\}) = 0$, μ is a probability measure. It is easy to see that μ does not have atoms.

Theorem 2.2 (i) *If $p = \frac{1}{\tau}$, $q = \frac{1}{\tau^2}$, μ is equal to the Lebesgue measure.*

(ii) *If $\frac{1}{\tau^2} < p < \frac{1}{2}$, μ has singular continuous component.*

Proof. (i) follows from Theorem 2.1. To prove (ii), we use the following fact [9]: set

$$D_\mu(x) := \limsup_{\delta\downarrow 0} \frac{\mu(x-\delta, x+\delta)}{\delta}, \quad A := \{x | D_\mu(x) = \infty\}.$$

Then $1_A d\mu$ is singular w.r.t. the Lebesgue measure. Take any $(O_1', O_2', \ldots, O_n') \in \{R, L\}^n$ and let $k_n = \sharp\{1 \le j \le n \mid O_j' = R\}$, $l_n = \sharp\{1 \le j \le n \mid O_j' = L\}$, $k_n + l_n = n$. Then $I_n = I_n(O_1', O_2', \ldots, O_n') := (\Phi \circ \Psi)^{-1}(\{w = (O_1, O_2, \ldots) \in W \mid O_1 = O_1', O_2 = O_2', \ldots, O_n = O_n'\})$ satisfies

$$|I_n| = \left(\frac{1}{\tau}\right)^{k_n} \left(\frac{1}{\tau^2}\right)^{l_n} = \frac{1}{\tau^{k_n+2l_n}}, \quad \mu(I_n) = p^{k_n} q^{l_n}$$

so that we have

$$r_n := \frac{\mu(I_n)}{|I_n|} = \frac{p^{k_n} q^{l_n}}{\left(\frac{1}{\tau}\right)^{k_n+2l_n}} = (p\tau)^n \left(\frac{(1-p)\tau}{p}\right)^{l_n}.$$

Define x and α by

$$p\tau =: x < 1, \quad \frac{\tau}{x}(\tau - x) = x^{-\alpha},$$

Then we have $\alpha > 1$ and

$$r_n = x^n x^{-\alpha l_n} = \left(\frac{1}{x}\right)^{\alpha l_n - n} \tag{2.2}$$

For $w = (O_1, O_2, \ldots) \in W$ let $k_n(w) = \sharp\{1 \le j \le n \mid O_j = R\}$, $l_n(w) = \sharp\{1 \le j \le n \mid O_j = L\}$, $k_n(w) + l_n(w) = n$. By (2.2) $\mu|_A$ is singular continuous, where

$$A := (\Phi \circ \Psi)^{-1}(\{w \in W \mid l_n(w) > \frac{n}{\alpha} \text{ for infinitely many } n\}).$$

Lemma 2.2 below shows $\mu(A) > 0$. $\square$

Let

$$B = (\Phi \circ \Psi)^{-1}(\{w \in W \mid l_n(w) \le \frac{n}{\alpha} \text{ for infinitely many } n\})$$

so that $\mathbf{T} = A \cup B = A \cup (B \setminus A)$.

Lemma 2.2 *If* $\frac{1}{\tau^2} < p < \frac{1}{2}$, $\mu(A) > 0$.

Proof. Suppose $\mu(A) = 0$, then $\mu(B \setminus A) > 0$. By definition,

$$\begin{aligned} A \setminus B &= (\Phi \circ \Psi)^{-1}(\{w \in W \mid n \gg 1, \ l_n(w) > \frac{n}{\alpha}\}) \\ B \setminus A &= \bigcup_{N \ge 1} \bigcap_{n \ge N} (\Phi \circ \Psi)^{-1}(\{x \mid \ k_n(w) \ge (1 - \frac{1}{\alpha})n\}) =: \bigcup_{N \ge 1} (B \setminus A)_N. \end{aligned}$$

Since $(B \setminus A)_N$ is monotone increasing, $\mu((B \setminus A)_N) > 0$ for some N. Let $(B \setminus A)'_N$ be the set with R and L being exchanged in $(B \setminus A)_N$:

$$(B \setminus A)'_N = \bigcap_{n \geq N} (\Phi \circ \Psi)^{-1}(\{w \in W \mid l_n(w) \geq (1 - \frac{1}{\alpha})n\}).$$

Since $\frac{1}{\tau^2} < p < \frac{1}{\tau}$, $l_n(w) \geq (1 - \frac{1}{\alpha})n$ implies $l_n(w) > \frac{n}{\alpha}$ so that $(B \setminus A)'_N \subset A \setminus B$. Hence

$$\mu((B \setminus A)'_N) \leq \mu(A \setminus B)(= 0).$$

It suffices to show

$$(0 <)\ \mu((B \setminus A)_N) \leq \mu((B \setminus A)'_N) \tag{2.3}$$

which leads us to a contradiction. To see (2.3), note that we may assume

$$l_{N-1}((\Phi \circ \Psi)(x)) \leq \frac{N-1}{2}$$

for $x \in (B \setminus A)_N$ by letting N large if necessary. Hence if we exchange R with L, $\sharp L$ increases in $(B \setminus A)_N$. Since $\mu(I_n) = \left(\frac{x}{\tau}\right)^n (\tau x^\alpha)^{-l_n}$ and since $\tau x^\alpha < 1$ for $p < \frac{1}{2}$, $\mu(I_n)$ is monotone increasing w.r.t. l_n which implies (2.3). □

3. Some combinatorial aspects of admissible words

In this section, we consider general circle map sequences except in subsection 3.2, and use the symbol A, B instead of 1, 0 respectively. Let P_n be the set of admissible words(factors of v_0) of length n. $|P_n| = n + 1$ is well known. We can find $t_n \in P_n$ uniquely such that t_nA, $t_nB \in P_{n+1}$ and for $a \in P_n \setminus \{t_n\}$ there exists unique $C = C(a) \in \{A, B\}$ with $aC(a) \in P_{n+1}$ [1]. For any k with $n \leq q_k - 2$ we have $t_n = (\pi_k(n), \pi_k(n-1), \ldots, \pi_k(1))$. In this section we study some combinatorial properties of admissible words.

3.1. Exhausting point

For $n \geq 2$, let $f(n) \in \mathbf{N}$ be the smallest number where we have seen all words in P_n in $(v_0(n))_{n \geq 1}$. For instance in the Fibonacci word,

$$ABA\underline{A}_2BA\underline{B}_3\underline{A}_4ABA\underline{A}_5\underline{B}_6AB\cdots$$

$\underline{*}_n$ corresponds to $f(n)$. Hence $f(2) = 4$, $f(3) = 7$, $f(4) = 8$ in this case.

Theorem 3.1 *Let $n \geq 2$ and take $k = 0, 1, \ldots$ such that*

$$q_k \leq n \leq q_{k+1} - 1.$$

Then writing $n = q_k + j$, we have

$$f(q_k + j) = q_{k+1} + q_k - 1 + j, \quad j = 0, 1, \ldots, q_{k+1} - q_k - 1. \tag{3.1}$$

[1] This is called the right special factor [6].

The corresponding exhausting points lies from the letter next to $s_{k+1}\pi_k$ to the last letter in $s_{k+1}\pi_{k+1}$. Therefore $(v_0(f(n)))_{n\geq 2}$ coincides with the original circle map sequence $(v_0(n))_{n\geq 1}$.

Corollary 3.1 *$v_0(f(n+1)) = v_0(n)$, for $n = 1, 2, 3, \ldots$.*

Let $g(n) \in \mathbf{N}$ be the smallest number where we have seen both $t_{n-1}A$ and $t_{n-1}B$. As a preparation, we prove the following lemma.

Lemma 3.1 *$f(n)$ is the smallest number satisfying following conditions.*

(i) $f(n-1) + 1 \leq f(n)$,

(ii) $g(n) \leq f(n)$.

Proof. We first show that $f(n)$ satisfies (i), (ii). (ii) is clear. We should have $f(n-1) < f(n)$, because cutting the rightmost letter in words in P_n yields all words in P_{n-1}. Hence $f(n)$ satisfies (i). Thus it suffices to show that if a number $f(n)$ satisfies (i), (ii), then we have already seen all words in P_n at $f(n)$. By the equation $P_n = \{aC(a)\}_{a\in P_{n-1}\setminus\{t_{n-1}\}} \cup \{t_{n-1}A, t_{n-1}B\}$, this is clear. □

Proof of Theorem 3.1. We prove (3.1) by induction on k. Let

$$k_0 = \begin{cases} 2 & (a_1 = 1, a_2 = 1) \\ 1 & (a_1 = 1, a_2 \geq 2, \quad \text{or} \quad a_1 = 2) \\ 0 & (a_1 \geq 3) \end{cases}$$

so that $q_{k_0} \leq 2 \leq q_{k_0+1} - 1$. For $n = 2, 3, \ldots, q_{k_0+1} - 1$, it is straightforward to see (3.1). We next suppose that (3.1) holds true for $k_0, k_0+1, \ldots, k-1$ and would like to prove it for $k(\geq k_0+1)$. Let $q_k \leq n \leq q_{k+1} - 1$. We note that t_{n-1} is a subword of π_{k+1} and is not a subword of π_k. Since $s_{k+1}\pi_k = s_k\pi_{k+1}$, both $\pi_{k+1}AB$ and $\pi_{k+1}BA$ are subwords of $s_{k+1}s_k$, which implies

$$g(n) \leq q_{k+1} + q_k - 1. \tag{3.2}$$

On the other hand, since we suppose (3.1) for $k-1$, $f(q_k - 1) = 2q_k - 2$. In $(v_0(1), v_0(2), \ldots, v_0(2q_k - 2))$, we have $(2q_k - 2) - (q_k - 1) + 1 = q_k$ of words of length $q_k - 1$. Since $|P_{q_k-1}| = q_k$, we find each element of P_{q_k-1} only once in $(v_0(1), v_0(2), \ldots, v_0(2q_k - 2))$. Hence t_{q_k-1} appears only once in $s_k\pi_k$. In what follows, we suppose that k is even. For k odd, we have only to exchange AB with BA in the argument below. Since t_{q_k-1} is the last subword of length $q_k - 1$ in $s_k\pi_{k-1}$, it appears a_{k+1} times in s_{k+1} and they have the form of $t_{q_k-1}BA$. The other one $t_{q_k-1}AB$ appears as the last subword of $s_{k+1}s_k$. Since t_{q_k-1} appears only once in $s_k\pi_k$, they exhaust all t_{q_k-1}'s in $s_{k+1}s_k$. Therefore we have $g(q_k) = q_{k+1} + q_k - 1$. Since $f(q_k - 1) < g(q_k)$,

$$f(q_k) = q_{k+1} + q_k - 1,$$

by Lemma 3.1. By the monotonicity of g, we have $g(n) \geq q_{k+1} + q_k - 1$ for $q_k \leq n \leq q_{k+1} - 1$ and together with (3.2), $g(n) = q_{k+1} + q_k - 1$ for such n. By Lemma 3.1 again,

$$f(q_k + j) = g(q_k + j) + j = q_{k+1} + q_k - 1 + j$$

for $j = 0, 1, \ldots, q_{k+1} - q_k - 1$ which proves (3.1) for k. $\square$

3.2. The classification of P_n and frequency: Fibonacci case

We consider the classification of words in P_n in terms of their frequency. We study Fibonacci case in this subsection. Let $\{F(n)\}$ be the Fibonacci sequence defined by

$$F_1 = 1, \quad F_2 = 2, \quad F_{n+1} = F_n + F_{n-1}.$$

By (1.1), $F_n = q_n = |s_n|$.

Theorem 3.2 *Let $a_n = 1$. We can decompose P_n into three disjoint subsets*

$$P_n = A_n \cup B_n \cup C_n$$

which are given explicitly as follows. Let $n = F_k + j$, $j = 0, 1, \ldots, F_{k-1} - 1$.

$$\begin{aligned}
A_n: \quad & (v_0(1), \ldots, v_0(F_k + j)) \\
& \quad \ddots \\
& \quad (v_0(1+m), \ldots, v_0(F_k + j + m)) \quad (m = 0, 1, \ldots, F_{k-1} - j - 2) \\
& \qquad \ddots \\
& \qquad (v_0(F_{k-1} - j - 1), \ldots, v_0(F_{k+1} - 2))
\end{aligned}$$

$$\begin{aligned}
B_n: \quad & (v_0(F_{k-1} - j), \ldots, v_0(F_{k+1} - 1)) \\
& \quad \ddots \\
& \quad (v_0(F_{k-1} - j + m), \ldots, v_0(F_{k+1} - 1 + m)) \quad (m = 0, 1, \ldots, F_{k-2} + j) \\
& \qquad \ddots \\
& \qquad (v_0(F_k), \ldots, v_0(2F_k + j - 1))
\end{aligned}$$

$$\begin{aligned}
A_n: \quad & (v_0(F_k + 1), \ldots, v_0(2F_k + j)) \\
& \quad \ddots \\
& \quad (v_0(F_k + 1 + m), \ldots, v_0(2F_k + j + m)) \quad (m = 0, 1, \ldots, F_{k-1} - j - 2) \\
& \qquad \ddots \\
& \qquad (v_0(F_{k+1} - j - 1), \ldots, v_0(F_{k+2} - 2))
\end{aligned}$$

$$\begin{aligned}
C_n: \quad & (v_0(F_{k+1} - j), \ldots, v_0(F_{k+2} - 1)) \\
& \quad \ddots \\
& \quad (v_0(F_{k+1} - j + m), \ldots, v_0(F_{k+2} - 1 + m)) \quad (m = 0, 1, \ldots, j) \\
& \qquad \ddots \\
& \qquad (v_0(F_{k+1}), \ldots, v_0(F_{k+2} + j - 1)).
\end{aligned}$$

They are characterized as follows. A_n consists of all words in P_n contained in $s_k\pi_{k-1}(=\pi_{k+1})$ and each ones in A_n appear twice before arriving at $f(n)$, while those in B_n, C_n appear only once. Starting from $v_0(1)$, we see words in A_n one after another. Words in B_n begin to appear after we have seen all words in A_n. Words in A_n appear for the second time after we have seen words in B_n. Words in C_n appear after we have seen all words in A_n for the second time. Moreover

(i) $|A_n| = (F_{k-1}-j-1)$ *and each word in A_n has overlaps of length j or $(F_{k-2}+j)$ with itself.*

(ii) $|B_n| = (F_{k-2}+j+1)$ *and each word in B_n has overlaps of length j or has distance $(F_{k-1}-j)$ with itself.*

(iii) $|C_n| = (j+1)$ *and each word in C_n has distance $(F_{k-1}-j)$ or $(F_{k+1}-j)$ with itself.*

Since each words in A_n has overlaps with itself, it can cover $v \in \Omega$ if overlap is allowed. Penrose tiling has analogous property [4, 5].

Proof. Before arriving at $f(n) = f(F_k + j) = F_{k+2} + j - 1$, we see F_{k+1} words of length n. Since $|P_n| = (F_k + j + 1)$, at least $F_{k+1} - (F_k + j + 1) = F_{k-1} - j - 1$ words should appear more than twice. On the other hand since $s_{k-1}\pi_k = \pi_{k+1}$, the words of length n contained in π_{k+1} (there are $F_{k-1} - j - 1$ of these) should appear at least twice before arriving at $f(n)$. Let A_n be the set of such words. Then the words in $P_n \setminus A_n$ appears only once. The properties of B_n, C_n follows from looking at words in $(v_0(1), \ldots, v_0(f(n)))$ explicitly, and the length of overlaps and distance follows from the $(k-1,k)$-partition of v_0. $\square$

We next compute the frequency of words in P_n.

Theorem 3.3 *Let $v \in \Omega$ and let $n = F_k + j$, $j = 0, 1, \ldots, F_{k-1} - 1$. For $a \in P_n$,*

$$\lim_{N\to\infty} \frac{\sharp\ a\text{'s in } (v(1), v(2), \ldots, v(N))}{N} = \begin{cases} \dfrac{1}{\tau^{k-1}} & (a \in A_n) \\ \dfrac{1}{\tau^{k}} & (a \in B_n) \\ \dfrac{1}{\tau^{k+1}} & (a \in C_n) \end{cases}$$

Proof. Due to the unique ergodicity of the dynamical system (Ω, T) ($(Tv)(n) := v(n+1)$, $v \in \Omega$ is the shift operator), we can work on some subsequence. In the $(k-1,k)$-partition, the frequency of s_k, s_{k-1} are equal to α, $1-\alpha$ respectively. Thus when $|s_k| + |s_{k-1}| = N$, $\sharp\{s_k\} = N\alpha(1+o(1))$, $\sharp\{s_{k-1}\} = N(1-\alpha)(1+o(1))$ so that the number of letters is equal to $\{\alpha F_k + (1-\alpha)F_{k-1}\} N(1+o(1))$. Each word in A_n is found in every s_k, s_{k-1} while each word in B_n (resp. C_n) is found in every s_k

(resp. s_{k-1}). Hence

$$\begin{aligned}
r_A &= \frac{N}{(\alpha F_k + (1-\alpha)F_{k-1})N} = \frac{1}{\tau^{k-1}} \\
r_B &= \frac{N\alpha}{(\alpha F_k + (1-\alpha)F_{k-1})N} = \frac{1}{\tau^{k}} \\
r_C &= \frac{N(1-\alpha)}{(\alpha F_k + (1-\alpha)F_{k-1})N} = \frac{1}{\tau^{k+1}}.
\end{aligned}$$

□

Theorem 3.4 *Let $n = F_k + j$, $j = 0, 1, \ldots, F_{k-1} - 1$ and $w \in P_n$. Then $I(w) := \Psi^{-1}(\{v \in \Omega \mid (v(0), v(1), \ldots, v(n-1)) = w\})$ satisfies*

$$|I(w)| = \begin{cases} \dfrac{1}{\tau^{k-1}} & (w \in A_n) \\ \dfrac{1}{\tau^{k}} & (w \in B_n) \\ \dfrac{1}{\tau^{k+1}} & (w \in C_n) \end{cases}$$

In other words, the width of intervals in $\mathbf{T}$ corresponding to words in A_n, B_n and C_n are $\frac{1}{\tau^{k-1}}$, $\frac{1}{\tau^k}$ and $\frac{1}{\tau^{k+1}}$ respectively.

Since the endpoints in these intervals are equal to the set $\{x \mid x \equiv -j\alpha \pmod 1),\ j = 1, 2, \ldots, n\}$, Theorem 3.4 implies the three-distance theorem [10, 1].

Proof. This follows directly from Theorem 3.3 and the ergodic theorem. It is also possible to prove Theorem 3.4 directly by using Theorem 2.1 and inductive argument. In doing so, we note that $t_n \in A_n$ for $n = F_k + j$, $j = 0, 1, \ldots, F_{k-1} - 2$ (resp. $t_n \in B_n$ for $n = F_k + F_{k-1} - 1 = F_{k+1} - 1$) and $t_n A$, $t_n B$ belong to B_{n+1}, C_{n+1}, while the corresponding intervals are divided into two intervals with ratio $\tau : 1$. □

Remark 3.1 Let us call $w_n \in P_n$ exhausting word if the endpoint of which is located at $f(n)$. By Theorem 3.2, $w_{F_k} = A\pi_k A$ (resp. $w_{F_k} = B\pi_k B$) if k is even (resp. odd). Then by the fact that $t_n \notin C_n$ and the embedding procedure we see that $\Psi^{-1}(w_n)$ is the interval which is closest to the endpoint of $\mathbf{T}$: if $n = F_k + j$, $j = 0, 1, \ldots, F_{k-1} - 1$,

$$\Psi^{-1}(w_n) = \begin{cases} [1 - \frac{1}{\tau^{k+1}}) & (k : \text{even}) \\ [0, \frac{1}{\tau^{k+1}}) & (k : \text{odd}) \end{cases}$$

3.3. Classification of admissible words and frequency: general case

The results in previous subsection is directly extended to the general case, though the statement becomes slightly complicated. We only state the results. For given n, take k such that $q_k \leq n \leq q_{k+1} - 1$.

(1) Classification

(i) $q_k \leq n \leq q_k + q_{k-1} - 1$:
writing $n = q_k + j$, $j = 0, 1, \ldots, q_{k-1} - 1$, we have

$$\begin{aligned}
A_n: &\quad (v_0(1+m+pq_k), \ldots, v_0(s_k + j + m + pq_k)) \\
&\quad m = 0, 1, \ldots, q_{k-1} - j - 2, \quad p = 0, 1, \ldots, a_{k+1} \\
B_n: &\quad (v_0(q_{k-1} - j + m + pq_k), \ldots, v_0(q_k + q_{k-1} - 1 + m + pq_k)) \\
&\quad m = 0, 1, \ldots, q_k - q_{k-1} + j, \quad p = 0, 1, \ldots, a_{k+1} - 1 \\
C_n: &\quad (v_0(q_{k+1} - j + m), \ldots, v_0(q_{k+1} + q_k - 1 + m)) \\
&\quad m = 0, 1, \ldots, j.
\end{aligned}$$

The order of their appearance is $\overbrace{(A_n, B_n), \ldots, (A_n, B_n)}^{a_{k+1}}, (A_n, C_n)$. They are characterized as follows.

- A_n: they are the words of length n in $s_k \pi_{k-1}$, $|A_n| = q_{k-1} - 1 - j$, and appear $(a_{k+1} + 1)$-times before arriving at $f(n)$. Each one has overlap of length j, $(q_k - q_{k-1} + j)$ with itself.
- B_n: $|B_n| = q_k - |A_n| = q_k - q_{k-1} + 1 + j$ and appear a_{k+1} before arriving at $f(n)$. Each one has overlap of length j or has distance of length $(q_{k-1} - j)$.
- C_n: $|C_n| = j + 1$ and appear only once before arriving at $f(n)$. Each one has distance of length $(q_{k+1} - j)$, $(q_{k+1} - q_k - j)$ from itself.

(ii) $lq_k + q_{k-1} \leq n \leq (l+1)q_k + q_{k-1} - 1$, $l = 1, 2, \ldots, (a_{k+1} - 1)$:
writing $n = lq_k + q_{k-1} + j$, $j = 0, 1, \ldots, (q_k - 1)$, we have

$$\begin{aligned}
A_n: &\quad (v_0(1+m+pq_k), \ldots, v_0(lq_k + q_{k-1} + j + m + pq_k)) \\
&\quad m = 0, 1, \ldots, q_k - j - 2, \quad p = 0, 1, \ldots, a_{k+1} - l \\
B_n: &\quad (v_0(q_k - j + m + pq_k), \ldots, v_0((l+1)q_k + q_{k-1} - 1 + m + pq_k)) \\
&\quad m = 0, 1, \ldots, j, \quad p = 0, 1, \ldots, a_{k+1} - l - 1 \\
C_n: &\quad (v_0(q_{k+1} - (l-1)q_k - q_{k-1} - j + m), \ldots, v_0(q_{k+1} + q_k - 1 + m)) \\
&\quad m = 0, 1, \ldots, (l-1)q_k + q_{k-1} + j.
\end{aligned}$$

The order of their appearance is $\overbrace{(A_n, B_n), \ldots, (A_n, B_n)}^{a_{k+1} - l}, (A_n, C_n)$. They are characterized as follows.

- A_n: they are the words of length n in $s_k^{l+1} \pi_{k-1}$, $|A_n| = q_k - 1 - j$, and appear $(a_{k+1} - l + 1)$-times before arriving at $f(n)$. Each one has overlaps with itself of length $(l-1)q_k + q_{k-1} + j$, $(l-2)q_k + q_{k-1} + j$, $\ldots$ $[(2l - a_{k+1} - 1)]_+ q_k + q_{k-1} + j$ and j.
- B_n: $|B_n| = j + 1$, and appear $(a_{k+1} - l)$-times before arriving at $f(n)$. Each one has overlaps with itself of length $(l-1)q_k + q_{k-1} + j$, $(l-2)q_k + q_{k-1} + j$, $\ldots$ $[(2l - a_{k+1})]_+ q_k + q_{k-1} + j$ or has distance of length $q_k - j$.

- C_n: $|C_n| = (l-1)q_k + q_{k-1} + 1 + j$, and appear only once before arriving at $f(n)$. Each one has distance from itself of length $(a_{k+1} - l)q_k - j$, $(a_{k+1} + 1 - l)q_k - j$.

(2) Frequency

To compute the frequency, set

$$\beta_k := [1, a_{k+2}, a_{k+3}, \ldots].$$

Let r_{A_n} (resp. r_{B_n}, r_{C_n}) be the frequency of the words in A_n (resp. B_n, C_n).
(i) $q_k \leq n \leq q_k + q_{k-1} - 1$:

$$\begin{aligned} r_{A_n} &= \frac{\beta_k(a_{k+1}+1) + (1-\beta_k)}{\beta_k q_{k+1} + (1-\beta_k)q_k} \\ r_{B_n} &= \frac{\beta_k a_{k+1} + (1-\beta_k)}{\beta_k q_{k+1} + (1-\beta_k)q_k} \\ r_{C_n} &= \frac{\beta_k}{\beta_k q_{k+1} + (1-\beta_k)q_k}. \end{aligned}$$

(ii) $lq_k + q_{k-1} \leq n \leq (l+1)q_k + q_{k-1} - 1$, $l = 1, 2, \ldots, (a_{k+1} - 1)$:

$$\begin{aligned} r_{A_n} &= \frac{\beta_k(a_{k+1} - l + 1) + (1-\beta_k)}{\beta_k q_{k+1} + (1-\beta_k)q_k} \\ r_{B_n} &= \frac{\beta_k(a_{k+1} - l) + (1-\beta_k)}{\beta_k q_{k+1} + (1-\beta_k)q_k} \\ r_{C_n} &= \frac{\beta_k}{\beta_k q_{k+1} + (1-\beta_k)q_k}. \end{aligned}$$

4. Appendix 1: Basic properties of embedding procedure

4.1. Fixed point of Φ

In this subsection, we would like to represent the fixed point

$$f = (L, R, R, L, R, L, R, R, L, \ldots) \in W$$

of $\Phi : \Omega \to W$ in terms of the recursion relation of the sequence of words $\{u_n\}_{n=0}^{\infty}$ such that f is the right limit of that: $f = \lim_{n\to\infty} u_n$. In whalt follows, we identify $1 \leftrightarrow R$, $0 \leftrightarrow L$. Let

$$\begin{aligned} u_0 &= s_0 \\ v_0 &= s_1 \\ k(0) &= 1 \end{aligned}$$

$k(0)$ stands for the suffix of $s_\sharp$ in v_0. To go further, we prepare some notations. For $s \in \mathcal{S} := \{s_{l_1} s_{l_2} \cdots s_{l_N} : l_1 < l_2 < \cdots < l_N,\ N \in \mathbf{N}\}$, we define an operation $\mathcal{O}(v)$ as follows. Arrange its elements like $(R, L, R, R, L, \ldots)$, partition it in terms of R and RL like $((RL), R, (RL), \ldots)$, and replace R(resp. RL) by A_R (resp. A_{RL}).

$$\begin{array}{rcl} v &=& s_{l_1} s_{l_2} \ldots s_{l_r} \\ &=& (O_1, O_2, O_3, \ldots, O_N), \quad O_j = R \text{ or } RL \\ && \qquad\qquad \downarrow \\ \mathcal{O}(v) &=& (A_1, A_2, A_3, \ldots, A_N), \quad A_j = \mathcal{O}(O_j) = A_R \text{ or } A_{RL} \end{array}$$

where $\mathcal{O}(R) = A_R$, $\mathcal{O}(RL) = A_{RL}$, whose operation on $\mathcal{S}_0 := \{s_l : l \in \mathbf{N}\}$, are defined by

$$A_R s_k := s_{k+1}, \quad A_{RL} s_k := s_{k+3},$$

and the action of $\mathcal{O}(v)$ on s_k is defined by

$$(A_1, A_2, A_3, \ldots, A_N)s_k := (A_1 s_k)(A_2 A_1 s_k) \cdots (A_N A_{N-1} \cdots A_2 A_1 s_k).$$

By using notations above, the recursion relation between $(u_n, v_n, k(n))$ and $(u_{n+1}, v_{n+1}, k(n+1))$ is given by

$$\begin{array}{rcl} u_{n+1} &=& u_n v_n \\ v_{n+1} &=& \mathcal{O}(v_n) s_{k(n)} \\ k(n+1) &=& k(n) + (3\sharp\{(RL)\text{'s in } v_n\} + \sharp\{R\text{'s in } v_n\}). \end{array}$$

$k(n)$ is equal to the suffix of the rightmost word in $v_n \in \mathcal{S}$. The followings are computations of a few of them.

$$\left\{\begin{array}{l} u_1 = u_0 v_0 = s_0 s_1 \\ v_1 = \mathcal{O}(v_0) s_{k(0)} = \mathcal{O}(s_1) s_1 = s_2 \\ k(1) = k(0) + (3\sharp\{(RL)\text{'s in } v_0\} + \sharp\{R\text{'s in } v_0\}) = 1 + 1 = 2, \end{array}\right.$$

$$\left\{\begin{array}{l} u_2 = u_1 v_1 = u_1 s_2 \\ v_2 = \mathcal{O}(v_1) s_{k(1)} = \mathcal{O}(s_2) s_2 = \mathcal{O}(RL) s_2 = s_5 \\ k(2) = k(1) + (3\sharp\{(RL)\text{'s in } v_1\} + \sharp\{R\text{'s in } v_1\}) = 5, \end{array}\right.$$

$$\left\{\begin{array}{l} u_3 = u_2 v_2 = u_2 s_5 \\ v_3 = \mathcal{O}(v_2) s_{k(2)} = \mathcal{O}(s_5) s_5 = \mathcal{O}((RL)R(RL)(RL)R) s_5 \\ \quad = (A_{RL}, A_R, A_{RL}, A_{RL}, A_R) s_5 \\ \quad = (A_{RL} s_5)(A_R A_{RL} s_5)(A_{RL} A_R A_{RL} s_5) \\ \qquad\qquad (A_{RL} A_{RL} A_R A_{RL} s_5)(A_R A_{RL} A_{RL} A_R A_{RL} s_5) \\ \quad = s_8 s_9 s_{12} s_{15} s_{16} \\ k(3) = k(2) + (3\sharp\{(RL)\text{'s in } v_2\} + \sharp\{R\text{'s in } v_2\}) = 16 \end{array}\right.$$

4.2. Concrete examples

From the discussion in subsection 2.2, $\Phi(v)$ is seen to reflect some combinatorial aspects of $v \in \Omega$, and $\Phi(v)$ in turn can be derived by Theorem 2.1. In this subsection we explicitly give $\Phi(v)$ for some examples of v: $v_0(\cdot - m)$, $v_0'(\cdot - m)$ and v_{AA}, v_A, v_B defined later.

(1) $v_0(\cdot - m)$, $v_0'(\cdot - m)$: it is easy to see

$$\Phi(v_0) = (L, L, \ldots), \quad \Phi(v_0') = (R, L, L, \ldots). \tag{4.1}$$

Moreover

$$\Omega_L := \{v_0(\cdot - m),\ v_0'(\cdot - m) \mid m \geq 0\}, \quad \Omega_R := \{v_0(\cdot + m),\ v_0'(\cdot + m) \mid m \geq 1\}$$

satisfy

$$\Phi(\Omega_L) = \{(O_1, O_2, \ldots) \mid O_j = L \text{ for large } j\} \tag{4.2}$$
$$\Phi(\Omega_R) = \{(O_1, O_2, \ldots) \mid O_j = R \text{ for large } j\} \tag{4.3}$$

In fact, to see (4.2) we note that $\Psi^{-1}(v_0(\cdot - m)) \in D_-$ $(m \geq 0)$ by definition. Therefore, by a successive application of R or L, say after the k-th step we reach the interval with $\Psi^{-1}(v_0(\cdot - m))$ its left endpoint, and then we set $O_{k+1} = R$, $O_{k+2} = O_{k+3} = \cdots = L$. For $\Phi(v_0'(\cdot - m))$, we approach $\Psi^{-1}(v_0(\cdot - m))$ from the opposite direction. Conversely, if $w \in \{(O_1, O_2, \ldots) \mid O_j = L \text{ for large } j\}$, we have $(\Phi \circ \Psi)^{-1}(w) \in D_-$ by Theorem 2.1.

To see (4.3), we recall that $m \in \mathbf{N}$ has the following unique representation

$$m = F_{k_1} + F_{k_2} + \cdots + F_{k_N}, \quad l_j := k_j - k_{j-1} \geq 2,\ j = 2, 3, \ldots, N,$$

by which $\Phi(v_0(\cdot + m))$, $\Phi(v_0'(\cdot + m))$ are given explicitly below.
(i) k_1: odd

$$\begin{aligned} & \Phi(v_0(\cdot + m)) = \Phi(v_0'(\cdot + m)) \\ = & (R, \overbrace{L, \ldots, L}^{\frac{k_1-1}{2}}, \overbrace{R, \ldots, R}^{l_2-1}, L, \overbrace{R, \ldots, R}^{l_3-2}, L, \ldots, \overbrace{R, \ldots, R}^{l_N-2}, L, R, R, \ldots) \end{aligned}$$

(ii) k_1: even

$$\begin{aligned} & \Phi(v_0(\cdot + m)) = \Phi(v_0'(\cdot + m)) \\ = & (\overbrace{L, \ldots, L}^{\frac{k_1}{2}}, \overbrace{R, \ldots, R}^{l_2-1}, L, \overbrace{R, \ldots, R}^{l_3-2}, L, \ldots, \overbrace{R, \ldots, R}^{l_N-2}, L, R, R, \ldots) \end{aligned}$$

The converse is clear.

For the general case, given $\theta \in \mathbf{T}$ we take a sequence $\{N_k\}_{k=1}^{\infty}$ with $N_k\alpha \downarrow \theta$ in $\mathbf{T}$, and then the above argument tells us how to obtain $\Phi(v_\theta)$.

(2) symmetric sequences: Ω contains words with mirror symmetry

$$\begin{aligned} v_{AA} &:= \cdots 110101|101011\cdots =: h_{AA}^{-1}h_{AA} \\ v_A &:= \cdots 10110\underline{1}01101\cdots =: h_A^{-1}Ah_A \\ v_B &:= \cdots 101\underline{1}0\underline{1}101\cdots =: h_B^{-1}Bh_B \end{aligned}$$

which do not belong to $\Omega_R \cup \Omega_L$.

(i) v_{AA} : setting $v_{AA}(-1) = v_{AA}(0) = 1$ gives $\theta_{AA} := \Psi^{-1}(v_{AA}) = \frac{1}{2}$ and

$$\Phi(v_{AA}) = (R, R, L, R, L, \ldots).$$

(ii) v_A: setting $v_A(-1) = 1$ gives $\theta_A := \Psi^{-1}(v_A) = \frac{\alpha}{2}$ and

$$\Phi(v_A) = (L, R, L, R, \ldots).$$

(iii) v_B: setting $v_B(1) = 0$ gives $\theta_B := \Psi^{-1}(v_B) = \frac{1}{2} - \frac{3}{2}\alpha$ and

$$\Phi(v_B) = (R, L, R, L, \ldots).$$

4.3. Symmetric words

In this subsection, we further study some combinatorial properties of v_{AA}, v_A and v_B. When n is odd, $s_{n+3} = s_{n+1}s_ns_{n+1} = \pi_{n+1}(AB)\pi_n(BA)\pi_{n+1}(AB)$ from which we have

$$\pi_{n+3} = \pi_{n+1}(AB)\pi_n(BA)\pi_{n+1} \tag{4.4}$$

For even n we exchange AB with BA. Hence π_n and π_{n+3} have the same symmetry and v_A, v_B and v_{AA} can be derived by using this equation for $n = 3k$, $n = 3k+1$ and $n = 3k+2$ respectively. In fact, define h_n by the following equation.

$$s_n =: \begin{cases} h_n^{-1}\cdot A\cdot h_n, & (n = 3k = 3, 6, 9, \ldots) \\ h_n^{-1}\cdot B\cdot h_n, & (n = 3k+1 = 4, 7, 10, \ldots) \\ h_n^{-1}\cdot h_n, & (n = 3k+2 = 5, 8, 11, \ldots) \end{cases}$$

By (4.4) we have

$$h_{n+3} = \begin{cases} h_n(BA)\pi_{n+1} & (n : \text{odd}) \\ h_n(AB)\pi_{n+1} & (n : \text{even}) \end{cases}$$

whose right limits coincide with h_A, h_B and h_{AA} respectively.

We next study some substitutive properties of v_{AA}. Recall $h_{AA} \in \{0,1\}^{\mathbf{N}}$ is defined by the equation $v_{AA} = h_{AA}^{-1}h_{AA}$.

Proposition 4.1 (i) *h_{AA} is the fixed point of the following substitution rule:*

$$\sigma: \qquad \begin{aligned} &A \mapsto AB', A' \mapsto BA', \\ &B \mapsto A, B' \mapsto A' \end{aligned}$$

under the identification of A, B with A', B'.

(ii) *Define the sequence of words* $\{t_n\}_{n\geq 0}$ *by*

$$t_{n+1} = t_n\overline{t_{n-1}},\ n \geq 1, \quad t_0 = B, \quad t_1 = A.$$

where $\overline{s}$ *is obtained by exchanging* A, B *with* A', B' *in* s^{-1}. *Let* t *be the right limit of* t_n *(we identify* A, B *with* A', B' *in* t*). Then* $t = h_{AA}$.

Proof. We can show $t_n = \sigma^n(A)$ by the inductive argument and the equation $\sigma(\overline{s}) = \overline{\sigma(s)}$. $t = h_{AA}$ then follows from Lemma 4.1 given below. □

Let t'_n be the word obtained by identifying A, B with A', B' in t_n.

Lemma 4.1 *For* $n \geq 1$ *odd, we have*

$$\begin{aligned}
t'_{3n+2} &= h_{3n+2}(BA)h^{-1}_{3n+2} \\
t'_{3n+3} &= h_{3n+2}(BA)\pi_{3n+1}(AB)h^{-1}_{3n+2} \\
t'_{3n+4} &= h_{3n+2}(BA)h^{-1}_{3n+5} = h_{3n+5}(AB)h^{-1}_{3n+2}
\end{aligned}$$

(for even n*, we exchange* AB *with* BA*)*

5. Appenix 2: Robustness against local move

In the Fibonacci case ($\alpha = \frac{1}{\tau}$), we can exchange 10 with 01 in $v \in \Omega$ at some site. A natural question is whether it remains in the hull after this exchange. Let $\mathcal{E}^{(i,i+1)}$ be this exchange operation at site $i, i+1$ (we always assume $v(i) \neq v(i+1)$). We can see $\mathcal{E}^{(-1,0)}v_0 = v'_0$ which is, however, essentially the only case where this exchange is possible.

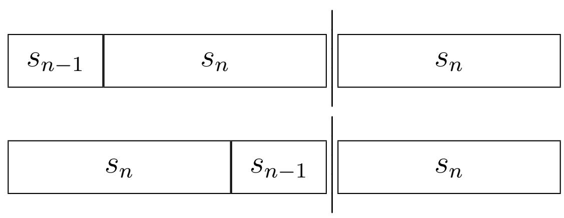
Theorem 5.1 *Let* $\alpha \in \mathbf{Q}^c \cap (0,1)$. *If* $v \in \Omega \setminus (\Omega_R \cup \Omega_L)$, *then* $\mathcal{E}^{(i,i+1)}v \notin \Omega$ *for any* i.

As a preparation, we prove

Lemma 5.1 *Let* $v \in \Omega$. *If* $\mathcal{E}^{(m-1,m)}v \in \Omega$ *for some* m, *then for any* $n \geq 0$ *the* $(n-1,n)$*-partition of* v *has one of the following form:*

(a)

(b)



where m *is the site left to* $|$. *Furthermore, if the* $(n-1,n)$*-partition satisfies* (a) *(resp.* (b)*), then the* $(n,n+1)$*-partition satisfies* (b) *(resp.* (a)*), where* n *is replaced by* $n+1$.

Lemma 5.1 is proved by induction. Then Theorem 5.1 follows from the fact that v_0 (resp. v_0') is the right limit of s_n and the left limit of s_{2n} (resp. s_{2n+1}). We can also consider exchanging s_k with s_{k-1} somewhere in the $(k-1,k)$-partition of v and the same result as Theorem 5.1 holds.

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