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LINEAR ISOMETRIES ON SPACES OF CONTINUOUSLY DIFFERENTIABLE AND LIPSCHITZ CONTINUOUS FUNCTIONS

HIRONAO KOSHIMIZU

ABSTRACT. We characterize the surjective linear isometries on $C^{(n)}[0,1]$ and $\mathrm{Lip}[0,1]$. Here $C^{(n)}[0,1]$ denotes the Banach space of n-times continuously differentiable functions on $[0,1]$ equipped with the norm

$$\|f\| = \sum_{k=0}^{n-1} |f^{(k)}(0)| + \sup_{x\in[0,1]} |f^{(n)}(x)| \quad (f \in C^{(n)}[0,1]),$$

and $\mathrm{Lip}[0,1]$ denotes the Banach space of Lipschitz continuous functions on $[0,1]$ equipped with the norm

$$\|f\| = |f(0)| + \operatorname*{ess\,sup}_{x\in[0,1]} |f'(x)| \quad (f \in \mathrm{Lip}[0,1]).$$

1. Introduction

The linear isometries on various function spaces have been studied by many mathematicians (see [5]). The source of this subject is the classical Banach-Stone theorem, which characterizes the surjective linear isometries on $C(X)$, the Banach space of all complex-valued continuous functions on a compact Hausdorff space X with the supremum norm $\|\cdot\|_\infty$. It states that every surjective linear isometry T from $C(X)$ onto itself has the canonical form: $Tf = \omega(f \circ \varphi)$ for all $f \in C(X)$, where φ is a homeomorphism of X onto itself and ω is a unimodular continuous function on X. In this paper, we investigate the surjective linear isometries on two types of the spaces $C^{(n)}[0,1]$ and $\mathrm{Lip}[0,1]$.

We denote by $C^{(n)}[0,1]$ for a positive integer n the $\mathbb{K}$-linear space of $\mathbb{K}$-valued n-times continuously differentiable functions on the closed unit interval $[0,1]$, where $\mathbb{K}$ is the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. With each of the following five equivalent

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norms the space $C^{(n)}[0,1]$ is a Banach space respectively:

$$\|f\|_C = \max\left\{\sum_{k=0}^{n} \frac{|f^{(k)}(x)|}{k!} : x \in [0,1]\right\},$$
$$\|f\|_\Sigma = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_\infty}{k!},$$
$$\|f\|_M = \max\{\|f\|_\infty, \|f'\|_\infty, \ldots, \|f^{(n)}\|_\infty\},$$
$$\|f\|_m = \max\{|f(0)|, |f'(0)|, \ldots, |f^{(n-1)}(0)|, \|f^{(n)}\|_\infty\},$$
$$\|f\|_\sigma = \sum_{k=0}^{n-1} |f^{(k)}(0)| + \|f^{(n)}\|_\infty,$$

for $f \in C^{(n)}[0,1]$. Among them, $(C^{(n)}[0,1], \|\cdot\|_C)$ and $(C^{(n)}[0,1], \|\cdot\|_\Sigma)$ are unital semisimple commutative Banach algebras.

In [2], Cambern characterized the surjective linear isometries on $(C^{(1)}[0,1], \|\cdot\|_C)$. Later, Pathak [12] extended this result to $(C^{(n)}[0,1], \|\cdot\|_C)$. The other extensions may be found in [3] and [11]. On the other hand, Rao and Roy [13] and Jarosz and Pathak [7] characterized the surjective linear isometries on $(C^{(1)}[0,1], \|\cdot\|_\Sigma)$ and $(C^{(1)}[0,1], \|\cdot\|_M)$, respectively. Those results say that every surjective linear isometry has the canonical form. However, the author [10] proved that the surjective linear isometries on $(C^{(n)}[0,1], \|\cdot\|_m)$ have the different form. In this paper, we show a similar result for the space $(C^{(n)}[0,1], \|\cdot\|_\sigma)$.

To state our theorem, we introduce an integral operator S: for any $f \in C([0,1])$, we put $(Sf)(x) = \int_0^x f(t)dt$ for all $x \in [0,1]$. Then S is a linear operator of $C([0,1])$ onto $\{f \in C^{(1)}[0,1] : f(0) = 0\}$, and S^n maps $C([0,1])$ onto $\{f \in C^{(n)}[0,1] : f^{(k)}(0) = 0 \text{ for } k = 0, 1, \ldots, n-1\}$. Hence $\{f^{(n)} : f \in C^{(n)}[0,1]\} = C([0,1])$. Moreover we have

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + (S^n f^{(n)})(x) \qquad (x \in [0,1],\ f \in C^{(n)}[0,1]).$$

The following is a characterization of the surjective linear isometries on $(C^{(n)}[0,1], \|\cdot\|_\sigma)$.

Theorem 1.1. *Let T be a linear operator from $(C^{(n)}[0,1], \|\cdot\|_\sigma)$ onto itself. Then T is an isometry if and only if there exist a homeomorphism φ of $[0,1]$ onto itself, a unimodular continuous function ω on $[0,1]$, a permutation $\{\tau(0), \tau(1), \ldots, \tau(n-1)\}$ of $\{0, 1, \ldots, n-1\}$ and unimodular constants $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ such that*

$$(Tf)(x) = \sum_{k=0}^{n-1} \frac{\lambda_k f^{(\tau(k))}(0)}{k!} x^k + \left(S^n(\omega(f^{(n)} \circ \varphi))\right)(x) \tag{1.1}$$

for all $x \in [0,1]$ *and* $f \in (C^{(n)}[0,1], \|\cdot\|_\sigma)$.

We denote the $\mathbb{K}$-linear space of $\mathbb{K}$-valued Lipschitz continuous functions on $[0,1]$ by $\mathrm{Lip}[0,1]$. Every $f \in \mathrm{Lip}[0,1]$ has the derivative $f'(x)$ for almost all $x \in [0,1]$, and the set $\{f' : f \in \mathrm{Lip}[0,1]\}$ coincides with $L^\infty[0,1]$; the Banach algebra of $\mathbb{K}$-valued essentially bounded functions on $[0,1]$ with the essential supremum norm $\|\cdot\|_{L^\infty}$. With each of the following four equivalent norms the space $\mathrm{Lip}[0,1]$ is a Banach space respectively:

$$\begin{aligned}
\|f\|_\Sigma &= \|f\|_\infty + \|f'\|_{L^\infty}, \\
\|f\|_M &= \max\{\|f\|_\infty, \|f'\|_{L^\infty}\}, \\
\|f\|_m &= \max\{|f(0)|, \|f'\|_{L^\infty}\}, \\
\|f\|_\sigma &= |f(0)| + \|f'\|_{L^\infty},
\end{aligned}$$

for $f \in \mathrm{Lip}[0,1]$. Among them, $(\mathrm{Lip}[0,1], \|\cdot\|_\Sigma)$ is a unital semisimple commutative Banach algebra. It is known that every surjective linear isometry on $(\mathrm{Lip}[0,1], \|\cdot\|_\Sigma)$ or $(\mathrm{Lip}[0,1], \|\cdot\|_M)$ has the canonical form ([7, 8, 13]). In [10], the author proved that the surjective linear isometries on $(\mathrm{Lip}[0,1], \|\cdot\|_m)$ have the different form. The following is a characterization of the surjective linear isometries on $(\mathrm{Lip}[0,1], \|\cdot\|_\sigma)$.

Theorem 1.2. *Let* T *be a linear operator from* $(\mathrm{Lip}[0,1], \|\cdot\|_\sigma)$ *onto itself. Then* T *is an isometry if and only if there exist an algebra automorphism* Φ *of* $L^\infty[0,1]$*, a unimodular function* $\omega \in L^\infty[0,1]$ *and a unimodular constant* λ *such that*

$$(Tf)(x) = \lambda f(0) + \int_0^x \omega(t)(\Phi f')(t)dt \tag{1.2}$$

for all $x \in [0,1]$ *and* $f \in (\mathrm{Lip}[0,1], \|\cdot\|_\sigma)$.

It is known that every algebra automorphism Φ of $L^\infty[0,1]$ has the form: $\Phi f = f \circ \varphi$ for all $f \in L^\infty[0,1]$, where $\varphi \in L^\infty[0,1]$ and $\varphi(x) \in [0,1]$ for almost all $x \in [0,1]$. This fact is proved by the method of the proof of [6, Theorem 1].

Remark. Theorems 1.1 and 1.2 are the same results as the cases $(C^{(n)}[0,1], \|\cdot\|_m)$ and $(\mathrm{Lip}[0,1], \|\cdot\|_m)$, respectively (see [10]). However we need a different consideration for their proofs.

Throughout this paper, we use the notations below: Put $\mathbb{T} = \{z \in \mathbb{K} : |z| = 1\}$. If $\mathbb{K} = \mathbb{R}$, then $\mathbb{T} = \{1, -1\}$. If $\mathbb{K} = \mathbb{C}$, then $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$. For any nonnegative integer ℓ, we define $i^\ell(x) = x^\ell$ for $x \in [0,1]$. In particular, we write $i^0 = 1$ and $i^1 = i$. Let $f \in C^{(n)}[0,1]$ and $\ell = 1, 2, \ldots, n$. Then $f = i^\ell$ if and only if $f(0) = f'(0) = \cdots = f^{(\ell-1)}(0) = 0$ and $f^{(\ell)}(x) = \ell!$ for $x \in [0,1]$. For a normed linear space $\mathcal{B}$, we put $\mathrm{ball}\,\mathcal{B} = \{\xi \in \mathcal{B} : \|\xi\|_{\mathcal{B}} \le 1\}$ and denote its dual space by $\mathcal{B}^*$.

2. Lemmas

Before proving the theorem we state useful lemmas.

Lemma 2.1. *Let $\mathcal{S}_1, \ldots, \mathcal{S}_\ell$ be normed linear spaces, and let $\mathcal{B} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_\ell$ be the product space equipped with the norm*

$$\|(s_1, \ldots, s_\ell)\|_{\mathcal{B}} = \max\{\|s_1\|_{\mathcal{S}_1}, \ldots, \|s_\ell\|_{\mathcal{S}_\ell}\} \quad ((s_1, \ldots, s_\ell) \in \mathcal{B}).$$

Then $(s_1, \ldots, s_\ell)$ is an extreme point of $\operatorname{ball} \mathcal{B}$ *if and only if s_k is an extreme point of* $\operatorname{ball} \mathcal{S}_k$ *for all $k = 1, \ldots, \ell$.*

Proof. Suppose s_k is an extreme point of $\operatorname{ball} \mathcal{S}_k$ for all k. To prove that $(s_1, \ldots, s_\ell)$ is an extreme point of $\operatorname{ball} \mathcal{B}$, write $(s_1, \ldots, s_\ell) = ((s'_1, \ldots, s'_\ell) + (s''_1, \ldots, s''_\ell))/2$, where $(s'_1, \ldots, s'_\ell), (s''_1, \ldots, s''_\ell) \in \operatorname{ball} \mathcal{B}$. Then for each $k = 1, \ldots, \ell$ we have

$$s_k = \frac{1}{2} s'_k + \frac{1}{2} s''_k.$$

Also, $\|s'_k\|_{\mathcal{S}_k} \leq \max\{\|s'_1\|_{\mathcal{S}_1}, \ldots, \|s'_\ell\|_{\mathcal{S}_\ell}\} = \|(s'_1, \ldots, s'_\ell)\|_{\mathcal{B}} \leq 1$. Similarly, $\|s''_k\|_{\mathcal{S}_k} \leq 1$. By hypothesis, $s_k = s'_k = s''_k$. Hence $(s_1, \ldots, s_\ell) = (s'_1, \ldots, s'_\ell) = (s''_1, \ldots, s''_\ell)$. Thus $(s_1, \ldots, s_\ell)$ is an extreme point of $\operatorname{ball} \mathcal{B}$.

The converse can be proved in a similar manner. □

Lemma 2.2. *Suppose that ψ_1 and ψ_2 are injective continuous mappings from $[0, 1]$ into $[0, 1]$. Let $\alpha \in \mathbb{C}$. If $\alpha(g \circ \psi_1) + (g \circ \psi_2)$ is constant on $[0, 1]$ for all real-valued continuous functions g on $[0, 1]$, then $\psi_1 = \psi_2$.*

Proof. Assume $\psi_1 \neq \psi_2$. Then $\psi_1(p) \neq \psi_2(p)$ for some $p \in [0, 1]$. Since ψ_1 is continuous there exists $q \in [0, 1]$ such that $q \neq p$ and $\psi_1(q) \neq \psi_2(p)$. Since ψ_2 is injective, $\psi_2(q) \neq \psi_2(p)$. Applying the Urysohn's lemma there exists a real-valued continuous function g_0 on $[0, 1]$ so that $g_0(\psi_2(p)) = 1$ and $g_0(\psi_1(p)) = g_0(\psi_1(q)) = g_0(\psi_2(q)) = 0$. Then we have $\alpha g_0(\psi_1(p)) + g_0(\psi_2(p)) = 1$ and $\alpha g_0(\psi_1(q)) + g_0(\psi_2(q)) = 0$. This contradicts the fact that $\alpha(g_0 \circ \psi_1) + (g_0 \circ \psi_2)$ is constant. Hence $\psi_1 = \psi_2$. □

3. Proof of Theorem 1.1

From now on, we write simply $C^{(n)}$ and C for the Banach spaces $(C^{(n)}[0, 1], \| \cdot \|_\sigma)$ and $(C([0, 1]), \| \cdot \|_\infty)$, respectively.

We first give a proof for the elementary part:

Proof of the "if" part. Suppose T has the form (1.1). It is clear that T is linear. Let $f \in C^{(n)}$. For each $\ell = 0, 1, \ldots, n-1$ we have

$$(Tf)^{(\ell)}(x) = \sum_{k=\ell}^{n-1} \frac{\lambda_k f^{(\tau(k))}(0)}{(k-\ell)!} x^{k-\ell} + \left(S^{n-\ell}(\omega(f^{(n)} \circ \varphi)\right)(x) \quad (x \in [0, 1]).$$

Thus $(Tf)^{(\ell)}(0) = \lambda_\ell f^{(\tau(\ell))}(0)$ since $(Sg)(0) = 0$ for all $g \in C$. Moreover $(Tf)^{(n)} = \omega(f^{(n)} \circ \varphi)$. Therefore

$$\|Tf\|_\sigma = \sum_{\ell=0}^{n-1} |\lambda_\ell f^{(\tau(\ell))}(0)| + \|\omega(f^{(n)} \circ \varphi)\|_\infty = \sum_{k=0}^{n-1} |f^{(k)}(0)| + \|f^{(n)}\|_\infty = \|f\|_\sigma.$$

Hence T is an isometry.

To prove that T is surjective let $g \in C^{(n)}$. Put

$$f(x) = \sum_{k=0}^{n-1} \frac{\overline{\lambda_{\tau^{-1}(k)}} g^{(\tau^{-1}(k))}(0)}{k!} x^k + \left(S^n \left(\frac{g^{(n)} \circ \varphi^{-1}}{\omega \circ \varphi^{-1}}\right)\right)(x) \quad (x \in [0,1]).$$

Then $f^{(\ell)}(0) = \overline{\lambda_{\tau^{-1}(\ell)}} g^{(\tau^{-1}(\ell))}(0)$ for $\ell = 0, 1, \ldots, n-1$ and $f^{(n)} = (g^{(n)} \circ \varphi^{-1})/(\omega \circ \varphi^{-1})$. Hence

$$\begin{aligned}(Tf)(x) &= \sum_{k=0}^{n-1} \frac{\lambda_k \overline{\lambda_k} g^{(k)}(0)}{k!} x^k + \left(S^n \left(\omega\left(\frac{g^{(n)} \circ \varphi^{-1}}{\omega \circ \varphi^{-1}} \circ \varphi\right)\right)\right)(x) \\ &= \sum_{k=0}^{n-1} \frac{g^{(k)}(0)}{k!} x^k + \left(S^n g^{(n)}\right)(x) = g(x)\end{aligned}$$

for all $x \in [0,1]$. □

The rest of this section is devoted to the proof of the "only if" part. Let T be a linear isometry of $C^{(n)}$ onto itself. Let $\mathbb{K}^n$ denote the product space of n copies of $\mathbb{K}$. The points of $\mathbb{K}^n$ are thus ordered n-tuples $\boldsymbol{a} = (a_0, a_1, \ldots, a_{n-1})$, where $a_0, a_1, \ldots, a_{n-1} \in \mathbb{K}$. For instance, we write $\boldsymbol{b} = (b_0, b_1, \ldots, b_{n-1})$, $\mathbf{1} = (1, 1, \ldots, 1)$ and so on.

Definition 3.1. For each $(\boldsymbol{a}, c, x) \in \mathbb{T}^n \times \mathbb{T} \times [0,1]$ we define a functional $\Lambda_{(\boldsymbol{a},c,x)}$ on $C^{(n)}$ by

$$\Lambda_{(\boldsymbol{a},c,x)}(f) = \sum_{k=0}^{n-1} a_k f^{(k)}(0) + c f^{(n)}(x) \quad (f \in C^{(n)}).$$

It is clear that $\Lambda_{(\boldsymbol{a},c,x)} \in \mathrm{ball}(C^{(n)})^*$.

Proposition 3.2. *Let $\xi \in (C^{(n)})^*$. Then ξ is an extreme point of* $\mathrm{ball}(C^{(n)})^*$ *if and only if there exists $(\boldsymbol{a}, c, x) \in \mathbb{T}^n \times \mathbb{T} \times [0,1]$ such that $\xi = \Lambda_{(\boldsymbol{a},c,x)}$.*

Proof. If the product spaces $\mathbb{K}^n \times C$ and $\mathbb{K}^n \times C^*$ are equipped with the norms

$$\begin{aligned}\|(\boldsymbol{b}, g)\| &= \sum_{k=0}^{n-1} |b_k| + \|g\|_\infty && ((\boldsymbol{b}, g) \in \mathbb{K}^n \times C), \\ \|(\boldsymbol{a}, \eta)\| &= \max\{|a_0|, |a_1|, \ldots, |a_{n-1}|, \|\eta\|\} && ((\boldsymbol{a}, \eta) \in \mathbb{K}^n \times C^*),\end{aligned}$$

then $(\mathbb{K}^n \times C)^*$ is linearly isometric to $\mathbb{K}^n \times C^*$. In fact, the linear isometry Q of $\mathbb{K}^n \times C^*$ onto $(\mathbb{K}^n \times C)^*$ is given by

$$(Q(\boldsymbol{a},\eta))(\boldsymbol{b},g) = \sum_{k=0}^{n-1} a_k b_k + \eta(g) \quad ((\boldsymbol{a},\eta) \in \mathbb{K}^n \times C^*,\ (\boldsymbol{b},g) \in \mathbb{K}^n \times C).$$

Now, define a mapping P of $C^{(n)}$ into $\mathbb{K}^n \times C$ by

$$Pf = \big((f(0), f'(0), \ldots, f^{(n-1)}(0)), f^{(n)}\big) \quad (f \in C^{(n)}).$$

Clearly P is a linear isometry of $C^{(n)}$ onto $\mathbb{K}^n \times C$. Then the conjugate operator P^* of P is a linear isometry of $(\mathbb{K}^n \times C)^*$ onto $(C^{(n)})^*$. Hence P^*Q is a linear isometry of $\mathbb{K}^n \times C^*$ onto $(C^{(n)})^*$. Thus $\xi \in (C^{(n)})^*$ is an extreme point of $\mathrm{ball}(C^{(n)})^*$ if and only if $\xi = P^*Q(\boldsymbol{a},\eta)$, where $(\boldsymbol{a},\eta)$ is an extreme point of $\mathrm{ball}(\mathbb{K}^n \times C^*)$. Note that the set of all extreme points of $\mathrm{ball}\,\mathbb{K}$ is $\mathbb{T}$. Also it is known that the set of all extreme points of $\mathrm{ball}\,C^*$ is $\{ce_x : c \in \mathbb{T}, x \in [0,1]\}$, where e_x is the evaluation functional at x: $e_x(g) = g(x)$ for $g \in C$ (see [4, Theorem V.8.4]). By Lemma 2.1, $(\boldsymbol{a},\eta)$ is an extreme point of $\mathrm{ball}(\mathbb{K}^n \times C^*)$ if and only if $\boldsymbol{a} \in \mathbb{T}^n$ and $\eta = ce_x$, where $c \in \mathbb{T}$, $x \in [0,1]$. Thus the conclusion follows from

$$\begin{aligned}(P^*Q(\boldsymbol{a},ce_x))(f) &= (Q(\boldsymbol{a},ce_x))\big((f(0), f'(0), \ldots, f^{(n-1)}(0)), f^{(n)}\big)\\ &= \sum_{k=0}^{n-1} a_k f^{(k)}(0) + cf^{(n)}(x) = \Lambda_{(\boldsymbol{a},c,x)}(f)\end{aligned}$$

for $f \in C^{(n)}$. □

Claim 3.3. *For any* $(\boldsymbol{a},c,x) \in \mathbb{T}^n \times \mathbb{T} \times [0,1]$ *there exists a unique* $(\boldsymbol{b},d,y) \in \mathbb{T}^n \times \mathbb{T} \times [0,1]$ *such that* $T^*\Lambda_{(\boldsymbol{a},c,x)} = \Lambda_{(\boldsymbol{b},d,y)}$.

Proof. Let $(\boldsymbol{a},c,x) \in \mathbb{T}^n \times \mathbb{T} \times [0,1]$. By Proposition 3.2, $\Lambda_{(\boldsymbol{a},c,x)}$ is an extreme point of $\mathrm{ball}(C^{(n)})^*$. Since T^* is a linear isometry of $(C^{(n)})^*$ onto itself, $T^*\Lambda_{(\boldsymbol{a},c,x)}$ is an extreme point of $\mathrm{ball}(C^{(n)})^*$. By Proposition 3.2 there exists $(\boldsymbol{b},d,y) \in \mathbb{T}^n \times \mathbb{T} \times [0,1]$ such that $T^*\Lambda_{(\boldsymbol{a},c,x)} = \Lambda_{(\boldsymbol{b},d,y)}$.

For the uniqueness of $(\boldsymbol{b},d,y)$ suppose $T^*\Lambda_{(\boldsymbol{a},c,x)} = \Lambda_{(\boldsymbol{b}',d',y')}$ for some $(\boldsymbol{b}',d',y') \in \mathbb{T}^n \times \mathbb{T} \times [0,1]$, where $\boldsymbol{b}' = (b'_0, b'_1, \ldots, b'_{n-1})$. Then $\Lambda_{(\boldsymbol{b},d,y)} = \Lambda_{(\boldsymbol{b}',d',y')}$ and so

$$\sum_{k=0}^{n-1} b_k f^{(k)}(0) + df^{(n)}(y) = \sum_{k=0}^{n-1} b'_k f^{(k)}(0) + d'f^{(n)}(y') \quad (f \in C^{(n)}). \tag{3.1}$$

For each $\ell = 0, 1, \ldots, n-1$ put $f = i^\ell$ in (3.1). Then $b_\ell = b'_\ell$ holds hence $\boldsymbol{b} = \boldsymbol{b}'$. Substituting $f = i^n$ and $f = i^{n+1}$ respectively in (3.1) we obtain $d = d'$ and $y = y'$. □

Definition 3.4. Let $(\boldsymbol{a}, x) \in \mathbb{T}^n \times [0,1]$. Applying Claim 3.3 there exists a unique $(\boldsymbol{b}, d, y) \in \mathbb{T}^n \times \mathbb{T} \times [0,1]$ such that $T^*\Lambda_{(\boldsymbol{a},1,x)} = \Lambda_{(\boldsymbol{b},d,y)}$. Since $\boldsymbol{b} = (b_0, \ldots, b_{n-1})$, d and y depend on $(\boldsymbol{a}, x)$ we write

$$b_k = u_k(\boldsymbol{a}, x) \quad (k = 0, 1, \ldots, n-1), \quad d = v(\boldsymbol{a}, x) \quad \text{and} \quad y = \psi(\boldsymbol{a}, x).$$

Thus u_k and v are unimodular functions on $\mathbb{T}^n \times [0,1]$ and ψ is a mapping of $\mathbb{T}^n \times [0,1]$ into $[0,1]$. Moreover we have

$$\Lambda_{(\boldsymbol{a},1,x)}(Tf) = (T^*\Lambda_{(\boldsymbol{a},1,x)})(f) = \Lambda_{((u_0(\boldsymbol{a},x),\ldots,u_{n-1}(\boldsymbol{a},x)),v(\boldsymbol{a},x),\psi(\boldsymbol{a},x))}(f)$$

for $f \in C^{(n)}$ and so

$$\sum_{k=0}^{n-1} a_k (Tf)^{(k)}(0) + (Tf)^{(n)}(x) = \sum_{\ell=0}^{n-1} u_\ell(\boldsymbol{a}, x) f^{(\ell)}(0) + v(\boldsymbol{a}, x) f^{(n)}(\psi(\boldsymbol{a}, x)). \tag{3.2}$$

Substituting $f = i^m$ for $m = 0, 1, \ldots, n-1$ respectively in (3.2) we have

$$\sum_{k=0}^{n-1} a_k (Ti^m)^{(k)}(0) + (Ti^m)^{(n)}(x) = m! u_m(\boldsymbol{a}, x). \tag{3.3}$$

Substituting i^n and i^{n+1} for f in (3.2) we have

$$\sum_{k=0}^{n-1} a_k (Ti^n)^{(k)}(0) + (Ti^n)^{(n)}(x) = n! v(\boldsymbol{a}, x), \tag{3.4}$$

$$\sum_{k=0}^{n-1} a_k (Ti^{n+1})^{(k)}(0) + (Ti^{n+1})^{(n)}(x) = (n+1)! v(\boldsymbol{a}, x) \psi(\boldsymbol{a}, x). \tag{3.5}$$

Claim 3.5. *For $k = 0, 1, \ldots, n-1$, u_k and v are unimodular continuous functions on $\mathbb{T}^n \times [0,1]$. Also, ψ is a continuous mapping of $\mathbb{T}^n \times [0,1]$ onto $[0,1]$.*

Proof. Note that the left hand sides of (3.3), (3.4) and (3.5) are continuous in $(\boldsymbol{a}, x) \in \mathbb{T}^n \times [0,1]$. The first two equations show that u_k and v are continuous. Since v is unimodular, (3.5) implies that ψ is also continuous.

To prove that $\psi : \mathbb{T}^n \times [0,1] \to [0,1]$ is surjective let $y \in [0,1]$. Since T^* is a linear isometry of $(C^{(n)})^*$ onto itself, Proposition 3.2 gives $(\boldsymbol{a}, c, x) \in \mathbb{T}^n \times \mathbb{T} \times [0,1]$ such that $T^*\Lambda_{(\boldsymbol{a},c,x)} = \Lambda_{(\mathbf{1},1,y)}$. Then we have

$$\begin{aligned}(T^*\Lambda_{(\bar{c}\boldsymbol{a},1,x)})(f) &= \bar{c}\left(\sum_{k=0}^{n-1} a_k (Tf)^{(k)}(0) + c(Tf)^{(n)}(x)\right) = \bar{c}(T^*\Lambda_{(\boldsymbol{a},c,x)})(f) \\ &= \bar{c}(\Lambda_{(\mathbf{1},1,y)})(f) = \left(\sum_{k=0}^{n-1} \bar{c} f^{(k)}(0) + \bar{c} f^{(n)}(y)\right) = \Lambda_{(\bar{c}\mathbf{1},\bar{c},y)}(f)\end{aligned}$$

for $f \in C^{(n)}$. By the definition of ψ we get $\psi(\bar{c}\boldsymbol{a}, x) = y$. Hence ψ is surjective. □

Claim 3.6. *For any fixed $x \in [0,1]$, $\psi(\mathbb{T}^n \times \{x\})$ is a singleton.*

Proof in case $\mathbb{K} = \mathbb{R}$. Fix $a_1, \ldots, a_{n-1} \in \mathbb{T} = \{1, -1\}$. For $t \in \{1, -1\}$ put $\boldsymbol{a}_t = (t, a_1, \ldots, a_{n-1})$. By Claim 3.5 functions $u_k(\boldsymbol{a}_t, x)$ and $v(\boldsymbol{a}_t, x)$ are continuous and take values within -1 and 1, so that they are constant functions as the interval $[0,1]$ is connected. Let

$$u_k(\boldsymbol{a}_t, x) = \alpha_{t,k} \quad \text{and} \quad v(\boldsymbol{a}_t, x) = \beta_t \qquad (x \in [0,1]),$$

where $\alpha_{t,k}$ and β_t are 1 or -1. Define $\psi_t(x) = \psi(\boldsymbol{a}_t, x)$ for all $t \in \{1, -1\}$ and $x \in [0,1]$. Putting $\boldsymbol{a} = \boldsymbol{a}_t$ in (3.2) we have

$$t(Tf)(0) + \sum_{k=1}^{n-1} a_k (Tf)^{(k)}(0) + (Tf)^{(n)}(x) = \sum_{\ell=0}^{n-1} \alpha_{t,\ell} f^{(\ell)}(0) + \beta_t f^{(n)}(\psi_t(x)) \tag{3.6}$$

for all $x \in [0,1]$ and $f \in C^{(n)}$.

By Claim 3.5 ψ_t is continuous. We show that ψ_t is injective. Since T is surjective we can choose $f_0 \in C^{(n)}$ so that $Tf_0 = i^{n+1}/(n+1)!$. Putting $f = f_0$ in (3.6) we have

$$x = \sum_{\ell=0}^{n-1} \alpha_{t,\ell} f_0^{(\ell)}(0) + \beta_t f_0^{(n)}(\psi_t(x)).$$

Since the left hand side is injective in $x \in [0,1]$, ψ_t must be injective.

Now the difference of (3.6) with $t = 1$ and (3.6) with $t = -1$ is

$$2(Tf)(0) = \sum_{\ell=0}^{n-1} (\alpha_{1,\ell} - \alpha_{-1,\ell}) f^{(\ell)}(0) + \beta_1 f^{(n)}(\psi_1(x)) - \beta_{-1} f^{(n)}(\psi_{-1}(x))$$

for all $x \in [0,1]$ and $f \in C^{(n)}$. If $\gamma = -\beta_1/\beta_{-1}$, then the above equation implies that $\gamma(f^{(n)} \circ \psi_1) + (f^{(n)} \circ \psi_{-1})$ is constant on $[0,1]$ for all $f \in C^{(n)}$. In other words, $\gamma(g \circ \psi_1) + (g \circ \psi_{-1})$ is constant for all $g \in C$. Hence Lemma 2.2 yields $\psi_1 = \psi_{-1}$, that is,

$$\psi(1, a_1, \ldots, a_{n-1}, x) = \psi_1(x) = \psi_{-1}(x) = \psi(-1, a_1, \ldots, a_{n-1}, x) \quad (x \in [0,1]).$$

If we fix $x \in [0,1]$, then the set $\psi(\mathbb{T} \times \{a_1\} \times \cdots \times \{a_{n-1}\} \times \{x\})$ is a singleton.

By the similar argument we can show that for each $\ell = 0, 1, \ldots, n-1$ and for fixed $a_0, \ldots, a_{\ell-1}, a_{\ell+1}, \ldots, a_{n-1} \in \mathbb{T}$ and $x \in [0,1]$ the set

$$\psi(\{a_0\} \times \cdots \times \{a_{\ell-1}\} \times \mathbb{T} \times \{a_{\ell+1}\} \times \cdots \times \{a_{n-1}\} \times \{x\})$$

is a singleton. Since ℓ is arbitrary we see that $\psi(\mathbb{T}^n \times \{x\})$ is also a singleton. □

Proof in case $\mathbb{K} = \mathbb{C}$. Fix $a_1, \ldots, a_{n-1} \in \mathbb{T}$ and $x \in [0,1]$. Since $\mathbb{T} \times \{a_1\} \times \cdots \times \{a_{n-1}\} \times \{x\}$ is connected and compact the continuity of ψ implies that $\psi(\mathbb{T} \times \{a_1\} \times \cdots \times \{a_{n-1}\} \times \{x\})$ is connected and compact in $[0,1]$. Hence we can write $\psi(\mathbb{T} \times \{a_1\} \times \cdots \times \{a_{n-1}\} \times \{x\}) = [s,t]$, where $s, t \in [0,1]$ and $s \leq t$. To show that $s = t$ assume $s < t$. Then we easily find three distinct points $p, q, r \in [s,t]$ and a function $f_0 \in C^{(n)}$ such that $f_0(0) = f_0'(0) = \cdots = f_0^{(n-1)}(0) = f_0^{(n)}(p) = f_0^{(n)}(q) = 0$ and $f_0^{(n)}(r) = 1$. Since $p, q, r \in \psi(\mathbb{T} \times \{a_1\} \times \cdots \times \{a_{n-1}\} \times \{x\})$ there exist three distinct points $b, c, d \in \mathbb{T}$ such that $\psi(b, a_1, \ldots, a_{n-1}, x) = p$, $\psi(c, a_1, \ldots, a_{n-1}, x) = q$ and $\psi(d, a_1, \ldots, a_{n-1}, x) = r$. Putting $f = f_0$ in (3.2) we have

$$b(Tf_0)(0) + \sum_{k=1}^{n-1} a_k (Tf_0)^{(k)}(0) + (Tf_0)^{(n)}(x) = 0, \tag{3.7}$$

$$c(Tf_0)(0) + \sum_{k=1}^{n-1} a_k (Tf_0)^{(k)}(0) + (Tf_0)^{(n)}(x) = 0, \tag{3.8}$$

$$d(Tf_0)(0) + \sum_{k=1}^{n-1} a_k (Tf_0)^{(k)}(0) + (Tf_0)^{(n)}(x) = v(d, a_1, \ldots, a_{n-1}, x). \tag{3.9}$$

By (3.7) and (3.8) we have $(Tf_0)(0) = 0$ and $\sum_{k=1}^{n-1} a_k (Tf_0)^{(k)}(0) + (Tf_0)^{(n)}(x) = 0$ because $b \neq c$. It follows by (3.9) that $0 = v(d, a_1 \ldots, a_{n-1}, x)$. This contradicts the fact that v is unimodular. Thus we obtain $s = t$, and $\psi(\mathbb{T} \times \{a_1\} \times \cdots \times \{a_{n-1}\} \times \{x\})$ is a singleton $\{s\}$.

A similar argument shows that for each $\ell = 0, 1, \ldots, n-1$ and for fixed $a_0, \ldots, a_{\ell-1}$, $a_{\ell+1}, \ldots, a_{n-1} \in \mathbb{T}$ the set

$$\psi(\{a_0\} \times \cdots \times \{a_{\ell-1}\} \times \mathbb{T} \times \{a_{\ell+1}\} \times \cdots \times \{a_{n-1}\} \times \{x\})$$

is a singleton. Hence we see that $\psi(\mathbb{T}^n \times \{x\})$ is also a singleton. This concludes the claim. □

Definition 3.7. Define a mapping φ of $[0,1]$ into $[0,1]$ by

$$\varphi(x) = \psi(\mathbf{1}, x) \quad (x \in [0,1]).$$

Since ψ is a continuous mapping of $\mathbb{T}^n \times [0,1]$ onto $[0,1]$, φ is a continuous mapping of $[0,1]$ onto $[0,1]$. By Claim 3.6 we have $\varphi(x) = \psi(\mathbf{1}, x) = \psi(\boldsymbol{a}, x)$ for $(\boldsymbol{a}, x) \in \mathbb{T}^n \times [0,1]$. Moreover for any $(\boldsymbol{a}, x) \in \mathbb{T}^n \times [0,1]$ and $f \in C^{(n)}$, (3.2) is written as

$$\sum_{k=0}^{n-1} a_k (Tf)^{(k)}(0) + (Tf)^{(n)}(x) = \sum_{\ell=0}^{n-1} u_\ell(\boldsymbol{a}, x) f^{(\ell)}(0) + v(\boldsymbol{a}, x) f^{(n)}(\varphi(x)).$$

Applying (3.3) and (3.4) we have by removing u_ℓ and v the equation

$$\begin{aligned}\sum_{k=0}^{n-1} a_k (Tf)^{(k)}(0) + (Tf)^{(n)}(x) \\ = \sum_{k=0}^{n-1} a_k \left(\sum_{\ell=0}^{n-1} \frac{(Ti^\ell)^{(k)}(0)}{\ell!} f^{(\ell)}(0) + \frac{(Ti^n)^{(k)}(0)}{n!} f^{(n)}(\varphi(x)) \right) \\ + \sum_{\ell=0}^{n-1} \frac{(Ti^\ell)^{(n)}(x)}{\ell!} f^{(\ell)}(0) + \frac{(Ti^n)^{(n)}(x)}{n!} f^{(n)}(\varphi(x)).\end{aligned}$$

Since this holds for all $\boldsymbol{a} = (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{T}^n$ we have

$$(Tf)^{(k)}(0) = \sum_{\ell=0}^{n-1} \frac{(Ti^\ell)^{(k)}(0)}{\ell!} f^{(\ell)}(0) + \frac{(Ti^n)^{(k)}(0)}{n!} f^{(n)}(\varphi(x)), \tag{3.10}$$

$$(Tf)^{(n)}(x) = \sum_{\ell=0}^{n-1} \frac{(Ti^\ell)^{(n)}(x)}{\ell!} f^{(\ell)}(0) + \frac{(Ti^n)^{(n)}(x)}{n!} f^{(n)}(\varphi(x)). \tag{3.11}$$

Claim 3.8. *For each $k = 0, 1, \ldots, n-1$, $(Ti^n)^{(k)}(0) = 0$ and*

$$(Tf)^{(k)}(0) = \sum_{\ell=0}^{n-1} \frac{(Ti^\ell)^{(k)}(0)}{\ell!} f^{(\ell)}(0) \quad (f \in C^{(n)}) \tag{3.12}$$

Proof. Fix $k = 0, 1, \ldots, n-1$. Putting $f = i^{n+1}$ in (3.10) we have

$$(Ti^{n+1})^{(k)}(0) = (Ti^n)^{(k)}(0)(n+1)\varphi(x) \quad (x \in [0,1]).$$

Note that the left hand side is constant while φ maps $[0,1]$ onto $[0,1]$. We must have $(Ti^n)^{(k)}(0) = 0$. Substituting this into (3.10) we obtain (3.12). □

Definition 3.9. Define $\omega(x) = (Ti^n)^{(n)}(x)/n!$ for all $x \in [0,1]$. Clearly ω is a continuous function on $[0,1]$.

Claim 3.10. *The function ω is a unimodular continuous function on $[0,1]$.*

Proof. By Claim 3.8 and Equation (3.4) we have

$$|(Ti^n)^{(n)}(x)| = \left| \sum_{k=0}^{n-1} (Ti^n)^{(k)}(0) + (Ti^n)^{(n)}(x) \right| = |n! v(\mathbf{1}, x)| = n!$$

for all $x \in [0,1]$. Hence $|\omega(x)| = 1$ for $x \in [0,1]$. □

Claim 3.11. *For each $k \in \{0, 1, \ldots, n-1\}$ there exist a unique $m \in \{0, 1, \ldots, n-1\}$ and a unique $\alpha \in \mathbb{C}$ such that $Ti^m = \alpha i^k$ and $|\alpha| = m!/k!$.*

Proof. Let $k \in \{0, 1, \dots, n-1\}$. Assume $(Ti^{\ell})^{(k)}(0) = 0$ for all $\ell \in \{0, 1, \dots, n-1\}$. Then (3.12) shows that $(Tf)^{(k)}(0) = 0$ for all $f \in C^{(n)}$, which is a contradiction if we choose f so that $Tf = i^k$ because T is surjective. Therefore there exists $m \in \{0, 1, \dots, n-1\}$ such that $(Ti^m)^{(k)}(0) \neq 0$. By (3.3) we have

$$\begin{aligned} m! = |m! u_m(\boldsymbol{a}, x)| &= \left| \sum_{\ell=0}^{n-1} a_\ell (Ti^m)^{(\ell)}(0) + (Ti^m)^{(n)}(x) \right| \\ &\leq \sum_{\ell=0}^{n-1} |(Ti^m)^{(\ell)}(0)| + |(Ti^m)^{(n)}(x)| \leq \|Ti^m\|_\sigma = \|i^m\|_\sigma = m! \end{aligned}$$

for all $(\boldsymbol{a}, x) \in \mathbb{T}^n \times [0,1]$. Since the equality holds in the first inequality for all $\boldsymbol{a} = (a_0, a_1, \dots, a_{n-1}) \in \mathbb{T}^n$ and since $(Ti^m)^{(k)}(0) \neq 0$, we must have $(Ti^m)^{(\ell)}(0) = 0$ for all $\ell \in \{0, 1, \dots, n-1\} \setminus \{k\}$ and $(Ti^m)^{(n)}(x) = 0$ for all $x \in [0,1]$. Moreover $|(Ti^m)^{(k)}(0)| = m!$. Put $\alpha = (Ti^m)^{(k)}(0)/k!$. Then $|\alpha| = m!/k!$ and

$$(Ti^m)(x) = \sum_{\ell=0}^{n-1} \frac{(Ti^m)^{(\ell)}(0)}{\ell!} x^\ell + (Ti^m)^{(n)}(x) = \frac{(Ti^m)^{(k)}(0)}{k!} x^k = \alpha i^k(x) \quad (x \in [0,1]).$$

For the uniqueness assume $Ti^{m'} = \alpha' i^k$, where $m' \in \{0, 1, \dots, n-1\}$, $\alpha' \in \mathbb{C}$ and $|\alpha'| = m!/k!$. Then $T(i^m/\alpha) = i^k = T(i^{m'}/\alpha')$. Since T is injective we have $i^m/\alpha = i^{m'}/\alpha'$. This yields $\alpha = \alpha'$ and $m = m'$. □

Definition 3.12. According to Claim 3.11, with each $k \in \{0, 1, \dots, n-1\}$ we associate $m \in \{0, 1, \dots, n-1\}$ and $\alpha \in \mathbb{C}$ such that $Ti^m = \alpha i^k$ and $|\alpha| = m!/k!$. Since m and α depend on k we write

$$m = \tau(k) \quad \text{and} \quad \alpha = \frac{m!}{k!} \lambda_k.$$

Then we have

$$Ti^{\tau(k)} = \frac{\tau(k)!}{k!} \lambda_k i^k \quad \text{and} \quad |\lambda_k| = 1.$$

To complete the proof it remains to show the following claim:

Claim 3.13. (a) *φ is a homeomorphism of* $[0,1]$ *onto* $[0,1]$.
(b) $\{\tau(0), \tau(1), \dots, \tau(n-1)\}$ *is a permutation of* $\{0, 1, \dots, n-1\}$.
(c) *T has the form* (1.1).

Proof. We first show (b). For (b), it suffices to show that τ is injective. Suppose $\tau(k) = \tau(k')$, where $k, k' \in \{0, 1, \dots, n-1\}$. Then

$$\frac{\tau(k)!}{k!} \lambda_k i^k = Ti^{\tau(k)} = Ti^{\tau(k')} = \frac{\tau(k')}{k'!} \lambda_{k'} i^{k'}.$$

This implies $k = k'$. So τ is injective.

For (c), let $x \in [0,1]$ and $f \in C^{(n)}$. Since we have established (b), (3.12) implies

$$(Tf)^{(k)}(0) = \sum_{\ell=0}^{n-1} \frac{(Ti^{\tau(\ell)})^{(k)}(0)}{\tau(\ell)!} f^{(\tau(\ell))}(0) = \sum_{\ell=0}^{n-1} \frac{1}{\tau(\ell)!} \left(\frac{\tau(\ell)!}{\ell!}\lambda_\ell i^\ell\right)^{(k)}(0)\, f^{(\tau(\ell))}(0)$$
$$= \sum_{\ell=0}^{n-1} \frac{\lambda_\ell (i^\ell)^{(k)}(0)}{\ell!} f^{(\tau(\ell))}(0) = \lambda_k f^{(\tau(k))}(0).$$

On the other hand, by (b) for any $\ell \in \{0,1,\ldots,n-1\}$ there is $k \in \{0,1,\ldots,n-1\}$ such that $\tau(k) = \ell$. Then

$$(Ti^\ell)^{(n)}(x) = (Ti^{\tau(k)})^{(n)}(x) = \left(\frac{\tau(k)!}{k!}\lambda_k i^k\right)^{(n)}(x) = 0$$

because $k < n$. Hence (3.11) shows

$$(Tf)^{(n)}(x) = \omega(x) f^{(n)}(\varphi(x)). \tag{3.13}$$

Thus it follows that

$$(Tf)(x) = \sum_{k=0}^{n-1} \frac{(Tf)^{(k)}(0)}{k!} x^k + \left(S^n (Tf)^{(n)}\right)(x)$$
$$= \sum_{k=0}^{n-1} \frac{\lambda_k f^{(\tau(k))}(0)}{k!} x^k + \left(S^n(\omega(f^{(n)} \circ \varphi))\right)(x).$$

Finally we show (a). Since φ is continuous and surjective it suffices to show that φ is injective. Choose $f_0 \in C^{(n)}$ so that $Tf_0 = i^{n+1}/(n+1)!$ because T is surjective. Using Claim 3.10 and Equation (3.13) we have

$$|f_0^{(n)}(\varphi(x))| = |\omega(x) f_0^{(n)}(\varphi(x))| = |(Tf_0)^{(n)}(x)| = |i(x)| = |x| = x \quad (x \in [0,1]).$$

Hence if $\varphi(x_1) = \varphi(x_2)$, then $x_1 = |f_0^{(n)}(\varphi(x_1))| = |f_0^{(n)}(\varphi(x_2))| = x_2$. Therefore φ is injective, as desired. Thus we finish the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

Throughout the rest of this paper, we write simply Lip and L^∞ for the Banach space $(\mathrm{Lip}[0,1], \|\cdot\|_\sigma)$ and the Banach algebra $(L^\infty[0,1], \|\cdot\|_{L^\infty})$, respectively. If we indicate the scalar field $\mathbb{K}$, we write $L^\infty_{\mathbb{K}}$ instead of L^∞.

Let $\mathfrak{M}$ be the maximal ideal space of $L^\infty_{\mathbb{C}}$. Then $\mathfrak{M}$ is a compact Hausdorff space. We know that $\mathfrak{M}$ is totally disconnected, that is, every component of $\mathfrak{M}$ consists of one point ([1, Theorem 1.3.4]) and that $\mathfrak{M}$ has no isolated points ([14, Exercise 11.18]).

We write $C_{\mathbb{K}}(\mathfrak{M})$ or simply $C(\mathfrak{M})$ for the Banach algebra of all $\mathbb{K}$-valued continuous functions on $\mathfrak{M}$ with the supremum norm $\|\cdot\|_\infty$. For any $g \in L^\infty_{\mathbb{C}}$, $\widehat{g}$ denotes the

Gelfand representation of g. The Gelfand-Naimark theorem says that the Gelfand transformation $\Gamma : g \mapsto \widehat{g}$ is an algebra *-isomorphism of $L^\infty_{\mathbb{C}}$ onto $C_{\mathbb{C}}(\mathfrak{M})$ and $\|g\|_{L^\infty} = \|\widehat{g}\|_\infty$. Also Γ maps $L^\infty_{\mathbb{R}}$ onto $C_{\mathbb{R}}(\mathfrak{M})$, and $\{\widehat{f'} : f \in \mathrm{Lip}\} = C(\mathfrak{M})$.

We first give a proof of the "if" part:

Proof of the "if" part. Suppose T has the form (1.2). It is clear that T is linear. Define $\Psi = \Gamma\Phi\Gamma^{-1}$. Then Ψ is an algebra automorphism of $C(\mathfrak{M})$. By [9, Theorem 3.4.3], Ψ has the form $\Psi h = h \circ \varphi$ for some homeomorphism φ of $\mathfrak{M}$ onto itself. Hence Ψ is an isometry of $C(\mathfrak{M})$ onto itself and so Φ is an isometry of L^∞ onto itself. Also we have $(Tf)(0) = \lambda f(0)$ and $(Tf)' = \omega(\Phi f')$ for $f \in \mathrm{Lip}$. Therefore

$$\|Tf\|_\sigma = |\lambda f(0)| + \|\omega(\Phi f')\|_{L^\infty} = |f(0)| + \|\Phi f'\|_{L^\infty} = |f(0)| + \|f'\|_{L^\infty} = \|f\|_\sigma.$$

Hence T is an isometry.

To prove that T is surjective let $g \in \mathrm{Lip}$. Put

$$f(x) = \overline{\lambda} g(0) + \int_0^x (\Phi^{-1}(\overline{\omega} g'))(t)dt \quad (x \in [0,1]).$$

Then $f(0) = \overline{\lambda} g(0)$ and $f' = \Phi^{-1}(\overline{\omega} g')$, and so

$$(Tf)(x) = \lambda\overline{\lambda} g(0) + \int_0^x \omega(t)(\Phi\Phi^{-1}(\overline{\omega} g'))(t)dt = g(0) + \int_0^x g'(t)dt = g(x)$$

for all $x \in [0,1]$. □

The rest of the paper is devoted to the proof of the "only if " part. Let T be a linear isometry of Lip onto itself.

Definition 4.1. For each $(a,c,m) \in \mathbb{T} \times \mathbb{T} \times \mathfrak{M}$ we define a functional $\Lambda_{(a,c,m)}$ on Lip by

$$\Lambda_{(a,c,m)}(f) = af(0) + c\widehat{f'}(m) \quad (f \in \mathrm{Lip}).$$

It is clear that $\Lambda_{(a,c,m)} \in \mathrm{ball}(\mathrm{Lip})^*$.

Proposition 4.2. *Let $\xi \in (\mathrm{Lip})^*$. Then ξ is an extreme point of* ball(Lip)* *if and only if there exists $(a,c,m) \in \mathbb{T} \times \mathbb{T} \times \mathfrak{M}$ such that $\xi = \Lambda_{(a,c,m)}$.*

Proof. If the product spaces $\mathbb{K} \times L^\infty$ and $\mathbb{K} \times C(\mathfrak{M})^*$ are equipped with the norms

$$\|(b,g)\| = |b| + \|g\|_{L^\infty} \qquad ((b,g) \in \mathbb{K} \times L^\infty),$$
$$\|(a,\eta)\| = \max\{|a|, \|\eta\|\} \quad ((a,\eta) \in \mathbb{K} \times C(\mathfrak{M})^*),$$

then the next operator Q is a linear isometry of $\mathbb{K} \times C(\mathfrak{M})^*$ onto $(\mathbb{K} \times L^\infty)^*$:

$$(Q(a,\eta))(b,g) = ab + \eta(\widehat{g}) \quad ((a,\eta) \in \mathbb{K} \times C(\mathfrak{M})^*, (b,g) \in \mathbb{K} \times L^\infty).$$

Define a linear isometry P of Lip onto $\mathbb{K} \times L^\infty$ by

$$Pf = (f(0), f') \quad (f \in \mathrm{Lip}).$$

Then P^*Q is a linear isometry of $\mathbb{K} \times C(\mathfrak{M})^*$ onto $(\mathrm{Lip})^*$. Hence $\xi \in (\mathrm{Lip})^*$ is an extreme point of $\mathrm{ball}(\mathrm{Lip})^*$ if and only if $\xi = P^*Q(a, \eta)$, where (a, η) is an extreme point of $\mathrm{ball}(\mathbb{K} \times C(\mathfrak{M})^*)$. By Lemma 2.1 this condition on (a, η) is equivalent to the following: $a \in \mathbb{T}$ and there exist $c \in \mathbb{T}$ and $m \in \mathfrak{M}$ such that $\eta(g) = ce_m(g) = cg(m)$ for $g \in C(\mathfrak{M})$. Thus the conclusion follows from

$$P^*(Q(a, ce_m))(f) = (Q(a, ce_m))(f(0), f') = af(0) + c\widehat{f'}(m) = \Lambda_{(a,c,m)}(f)$$

for $f \in \mathrm{Lip}$. □

Claim 4.3. *For any $(a, c, m) \in \mathbb{T} \times \mathbb{T} \times \mathfrak{M}$ there exists a unique $(b, d, n) \in \mathbb{T} \times \mathbb{T} \times \mathfrak{M}$ such that $T^*\Lambda_{(a,c,m)} = \Lambda_{(b,d,n)}$.*

Proof. Let $(a, c, m) \in \mathbb{T} \times \mathbb{T} \times \mathfrak{M}$. Since T^* is a linear isometry of $(\mathrm{Lip})^*$ onto itself, Proposition 4.2 shows the existence of $(b, d, n) \in \mathbb{T} \times \mathbb{T} \times \mathfrak{M}$ such that $T^*\Lambda_{(a,c,m)} = \Lambda_{(b,d,n)}$.

For the uniqueness of (b, d, n) suppose $T^*\Lambda_{(a,c,m)} = \Lambda_{(b',d',n')}$ for some $(b', d', n') \in \mathbb{T} \times \mathbb{T} \times \mathfrak{M}$. Then $\Lambda_{(b,d,n)} = \Lambda_{(b',d',n')}$, that is,

$$bf(0) + d\widehat{f'}(n) = b'f(0) + d'\widehat{f'}(n') \quad (f \in \mathrm{Lip}). \tag{4.1}$$

Substituting 1 and i for f in (4.1) we get $b = b'$ and $d = d'$, respectively. Hence (4.1) shows $\widehat{f'}(n) = \widehat{f'}(n')$ for all $f \in \mathrm{Lip}$. In other words, $h(n) = h(n')$ for all $h \in C(\mathfrak{M})$. This implies $n = n'$. □

Definition 4.4. By Claim 4.3 for each $(a, m) \in \mathbb{T} \times \mathfrak{M}$ there exists a unique $(b, d, n) \in \mathbb{T} \times \mathbb{T} \times \mathfrak{M}$ such that $T^*\Lambda_{(a,1,m)} = \Lambda_{(b,d,n)}$. Since b, d and y depend on (a, m) we write

$$b = u(a, m), \quad d = v(a, m) \quad \text{and} \quad n = \psi(a, m).$$

Thus u and v are unimodular functions on $\mathbb{T} \times \mathfrak{M}$ and ψ is a mapping of $\mathbb{T} \times \mathfrak{M}$ into $\mathfrak{M}$. Moreover we have

$$\Lambda_{(a,1,m)}(Tf) = (T^*\Lambda_{(a,1,m)})(f) = \Lambda_{(u(a,m),v(a,m),\psi(a,m))}(f)$$

for $f \in \mathrm{Lip}$ and so

$$a(Tf)(0) + \widehat{(Tf)'}(m) = u(a, m)f(0) + v(a, m)\widehat{f'}(\psi(a, m)). \tag{4.2}$$

Substituting 1 and i for f we have

$$a(T1)(0) + \widehat{(T1)'}(m) = u(a, m), \tag{4.3}$$

$$a(Ti)(0) + \widehat{(Ti)'}(m) = v(a, m). \tag{4.4}$$

Claim 4.5. *The mapping ψ is a continuous mapping of $\mathbb{T} \times \mathfrak{M}$ onto $\mathfrak{M}$.*

Proof. By (4.3) and (4.4) we see that u and v are continuous on $\mathbb{T} \times \mathfrak{M}$. Since v is unimodular, (4.2) implies that $\widehat{f'} \circ \psi$ is continuous on $\mathbb{T} \times \mathfrak{M}$ for all $f \in \mathrm{Lip}$. In other words, $h \circ \psi$ is continuous on $\mathbb{T} \times \mathfrak{M}$ for all $h \in C(\mathfrak{M})$. To prove that $\psi : \mathbb{T} \times \mathfrak{M} \to \mathfrak{M}$ is continuous let $(a_0, m_0) \in \mathbb{T} \times \mathfrak{M}$ and let V be an open neighborhood of $\psi(a_0, m_0)$ in $\mathfrak{M}$. By the Urysohn's lemma there exists $h_0 \in C(\mathfrak{M})$ such that $h_0(\psi(a_0, m_0)) = 1$ and $h_0(n) = 0$ for all $n \in \mathfrak{M} \setminus V$. Put $U = \{(a, m) \in \mathbb{T} \times \mathfrak{M} : |(h_0 \circ \psi)(a, m)| > 0\}$. Since $h_0 \circ \psi$ is continuous, U is an open neighborhood of (a_0, m_0). Moreover we can easily see that $\psi(U) \subset V$. Thus ψ is continuous.

To prove that ψ is surjective let $n \in \mathfrak{M}$. Since T^* is a linear isometry of $(\mathrm{Lip})^*$ onto itself, Proposition 4.2 gives $(a, c, m) \in \mathbb{T} \times \mathbb{T} \times \mathfrak{M}$ such that $T^* \Lambda_{(a,c,m)} = \Lambda_{(1,1,n)}$. Then

$$\begin{aligned}(T^* \Lambda_{(\bar{c}a,1,m)})(f) &= \bar{c}(a(Tf)(0) + c\widehat{(Tf)'}(m)) = \bar{c}(T^* \Lambda_{(a,c,m)})(f) \\ &= \bar{c}(\Lambda_{(1,1,n)})(f) = \bar{c}f(0) + \bar{c}\widehat{f'}(n) = \Lambda_{(\bar{c},\bar{c},n)}(f)\end{aligned}$$

for $f \in \mathrm{Lip}$. By the definition of ψ we get $\psi(\bar{c}a, m) = n$. Hence ψ is surjective. □

Claim 4.6. *For any fixed $m \in \mathfrak{M}$, $\psi(\mathbb{T} \times \{m\})$ is a singleton.*

Proof in case $\mathbb{K} = \mathbb{R}$. For $t \in \mathbb{T} = \{1, -1\}$ put $\psi_t(m) = \psi(t, m)$ for all $m \in \mathfrak{M}$. The difference of (4.3) with $a = 1$ and (4.3) with $a = -1$ is $2(T1)(0) = u(1, m) - u(-1, m)$. Hence the difference of (4.2) with $a = 1$ and (4.2) with $a = -1$ shows that

$$2(Tf)(0) = 2(T1)(0) + v(1, m)\widehat{f'}(\psi_1(m)) - v(-1, m)\widehat{f'}(\psi_{-1}(m)) \qquad (4.5)$$

for $m \in \mathfrak{M}$ and $f \in \mathrm{Lip}$.

Assume that $\psi_1(m_0) \neq \psi_{-1}(m_0)$ for some $m_0 \in \mathfrak{M}$. Then we find disjoint open sets V_1 and V_2 in $\mathfrak{M}$ such that $\psi_1(m_0) \in V_1$ and $\psi_{-1}(m_0) \in V_2$. Since $\mathfrak{M}$ has no isolated points there exists $n \in V_1 \setminus \{\psi_1(m_0)\}$. Since $\psi : \mathbb{T} \times \mathfrak{M} \to \mathfrak{M}$ is surjective there exists $(t, m_1) \in \mathbb{T} \times \mathfrak{M}$ such that $\psi(t, m_1) = n$. Clearly $\psi_t(m_1) \neq \psi_1(m_0)$. We also have $\psi_t(m_1) \neq \psi_{-1}(m_0)$ because $n \notin V_2$.

Here we consider the case when $\psi_{-t}(m_1) = \psi_{-1}(m_0)$. In this case, we can choose $f_0 \in \mathrm{Lip}$ so that $\widehat{f_0'}(\psi_1(m_0)) = 1$ and $\widehat{f_0'}(\psi_{-1}(m_0)) = \widehat{f_0'}(\psi_t(m_1)) = \widehat{f_0'}(\psi_{-t}(m_1)) = 0$ because of $\{\widehat{f'} : f \in \mathrm{Lip}\} = C(\mathfrak{M})$ and the Urysohn's lemma. Put $f = f_0$ in (4.5) and evaluate it at m_0 and m_1. Then we get

$$2(Tf_0)(0) = 2(T1)(0) + v(1, m_0) \quad \text{and} \quad 2(Tf_0)(0) = 2(T1)(0).$$

Hence $v(1, m_0) = 0$, which is a contradiction because v is unimodular.

On the other hand if $\psi_{-t}(m_1) \neq \psi_{-1}(m_0)$, then we choose $f_0 \in \mathrm{Lip}$ so that $\widehat{f_0'}(\psi_{-1}(m_0)) = 1$ and $\widehat{f_0'}(\psi_1(m_0)) = \widehat{f_0'}(\psi_t(m_1)) = \widehat{f'}(\psi_{-t}(m_1)) = 0$. A similar argument shows that $v(-1, m_0) = 0$, which is a contradiction.

In any case, we reach a contradiction. Hence $\psi_1(m) = \psi_{-1}(m)$, that is, $\psi(1, m) = \psi(-1, m)$ for all $m \in \mathfrak{M}$. If we fix $m \in \mathfrak{M}$, then the set $\psi(\mathbb{T} \times \{m\})$ is a singleton. □

Proof in case $\mathbb{K} = \mathbb{C}$. Fix $m \in \mathfrak{M}$. Since $\mathbb{T} \times \{m\}$ is connected the continuity of ψ implies that $\psi(\mathbb{T} \times \{m\})$ is connected in $\mathfrak{M}$. Since $\mathfrak{M}$ is totally disconnected, $\psi(\mathbb{T} \times \{m\})$ is a singleton. □

Definition 4.7. Define a mapping φ of $\mathfrak{M}$ into $\mathfrak{M}$ by

$$\varphi(m) = \psi(1, m) \quad (m \in \mathfrak{M}).$$

Since ψ is a continuous mapping of $\mathbb{T} \times \mathfrak{M}$ onto $\mathfrak{M}$, φ is a continuous mapping of $\mathfrak{M}$ onto itself. By Claim 4.6 we have $\varphi(x) = \psi(1, m) = \psi(a, m)$ for $(a, m) \in \mathbb{T} \times \mathfrak{M}$. Moreover for any $(a, m) \in \mathbb{T} \times \mathfrak{M}$ and $f \in \mathrm{Lip}$, (4.2) is written as

$$a(Tf)(0) + \widehat{(Tf)'}(m) = u(a, m)f(0) + v(a, m)\widehat{f'}(\varphi(m)).$$

Applying (4.3) and (4.4) we have by removing u and v the equation

$$\begin{aligned} &a(Tf)(0) + \widehat{(Tf)'}(m) \\ &\quad = a\Big((T1)(0)f(0) + (Ti)(0)\widehat{f'}(\varphi(m))\Big) + \Big(\widehat{(T1)'}(m)f(0) + \widehat{(Ti)'}(m)\widehat{f'}(\varphi(m))\Big). \end{aligned}$$

Since this holds for all $a \in \mathbb{T}$ we have

$$(Tf)(0) = (T1)(0)f(0) + (Ti)(0)\widehat{f'}(\varphi(m)), \tag{4.6}$$

$$\widehat{(Tf)'}(m) = \widehat{(T1)'}(m)f(0) + \widehat{(Ti)'}(m)\widehat{f'}(\varphi(m)). \tag{4.7}$$

Definition 4.8. Define a constant λ and a function $\omega \in L^\infty$ by

$$\lambda = (T1)(0) \quad \text{and} \quad \omega = (Ti)'.$$

Claim 4.9. (a) $|\lambda| = 1$.
(b) $(Tf)(0) = \lambda f(0)$ *for all* $f \in \mathrm{Lip}$.
(c) ω *is unimodular.*
(d) $\widehat{(Tf)'}(m) = \widehat{\omega}(m)\widehat{f'}(\varphi(m))$ *for all* $m \in \mathfrak{M}$ *and* $f \in \mathrm{Lip}$.

Proof. We first show (b) and $\lambda \neq 0$. Equation (4.6) says that $(Ti)(0)(\widehat{f'} \circ \varphi)$ is constant on $\mathfrak{M}$ for all $f \in \mathrm{Lip}$. In other words, $(Ti)(0)(h \circ \varphi)$ is constant for $h \in C(\mathfrak{M})$. Since φ is surjective and $C(\mathfrak{M})$ separates the points of $\mathfrak{M}$ we must have $(Ti)(0) = 0$. Thus (b) follows from (4.6). Moreover if $\lambda = 0$, (b) yields $(Tf)(0) = 0$ for all $f \in \mathrm{Lip}$, which is a contradiction because T is surjective. Hence $\lambda \neq 0$.

For (c), we use $(Ti)(0) = 0$ and (4.4) to get

$$|\widehat{\omega}(m)| = |\widehat{(Ti)'}(m)| = |(Ti)(0) + \widehat{(Ti)'}(m)| = |v(1, m)| = 1 \quad (m \in \mathfrak{M}).$$

This implies that $\widehat{\omega}\overline{\widehat{\omega}}$ is an identity of $C(\mathfrak{M})$. Since the transformation $\Gamma : g \mapsto \widehat{g}$ is a *-isomorphism of L^∞ onto $C(\mathfrak{M})$, $\omega\overline{\omega}$ is an identity of L^∞. This implies (c).

For (a) and (d), we use (4.3) and compute as follows:

$$\begin{aligned} 1 &= |u(a,m)| = |a(T1)(0) + \widehat{(T1)'}(m)| = |a\lambda + \widehat{(T1)'}(m)| \le |\lambda| + |\widehat{(T1)'}(m)| \\ &\le |\lambda| + \|\widehat{(T1)'}\|_\infty = |(T1)(0)| + \|(T1)'\|_{L^\infty} = \|T1\|_\sigma = \|1\|_\sigma = 1 \end{aligned} \tag{4.8}$$

for all $(a,m) \in \mathbb{T} \times \mathfrak{M}$. Since the equality holds in the first inequality for all $a \in \mathbb{T}$ and since $\lambda \neq 0$ we must have $\widehat{(T1)'}(m) = 0$. Hence (4.7) implies (d). At the same time, we obtain $|\lambda| = 1$ because the equalities hold in (4.8). □

Claim 4.10. *The mapping φ is a homeomorphism of $\mathfrak{M}$ onto itself.*

Proof. Since $\mathfrak{M}$ is a compact Hausdorff space and φ is continuous and surjective it suffices to show that φ is injective. Assume $m_1 \neq m_2$ and $\varphi(m_1) = \varphi(m_2)$, where $m_1, m_2 \in \mathfrak{M}$. Then we can choose $f_1 \in \mathrm{Lip}$ such that $\widehat{f_1'}(m_1) = 1$ and $\widehat{f_1'}(m_2) = 0$ because of $\{\widehat{f'} : f \in \mathrm{Lip}\} = C(\mathfrak{M})$ and the Urysohn's lemma. Since T is surjective there exists $f_0 \in \mathrm{Lip}$ such that $Tf_0 = f_1$. By (c) and (d) of Claim 4.9 we have

$$|\widehat{f_0'}(\varphi(m))| = |\widehat{\omega}(m)\widehat{f_0'}(\varphi(m))| = |\widehat{(Tf_0)'}(m)| = |\widehat{f_1'}(m)| \quad (m \in \mathfrak{M}).$$

Hence $1 = |\widehat{f_1'}(m_1)| = |\widehat{f_0'}(\varphi(m_1))| = |\widehat{f_0'}(\varphi(m_2))| = |\widehat{f_1'}(m_2)| = 0$, which is a contradiction. Therefore φ is injective. □

Definition 4.11. For each $h \in C(\mathfrak{M})$ we define a function Ψh on $\mathfrak{M}$ by

$$(\Psi h)(m) = h(\varphi(m)) \quad (m \in \mathfrak{M}).$$

Since φ is a homeomorphism of $\mathfrak{M}$ onto itself, Ψ is an algebra automorphism of $C(\mathfrak{M})$. Put $\Phi = \Gamma^{-1}\Psi\Gamma$. Since the Gelfand transformation Γ is an algebra isomorphism of L^∞ onto $C(\mathfrak{M})$, Φ is an algebra automorphism of L^∞.

Claim 4.12. *The operator T has the form* (1.2).

Proof. Let $f \in \mathrm{Lip}$. By Claim 4.9 (d) we have

$$\begin{aligned} \widehat{(Tf)'}(m) &= \widehat{\omega}(m)\widehat{f'}(\varphi(m)) = \widehat{\omega}(m)(\Psi\widehat{f'})(m) = \widehat{\omega}(m)(\Psi\Gamma f')(m) \\ &= \widehat{\omega}(m)(\Gamma\Phi f')(m) = \widehat{\omega}(m)\widehat{\Phi f'}(m) = \widehat{\omega \cdot (\Phi f')}(m). \end{aligned}$$

for any $m \in \mathfrak{M}$. Hence $(Tf)' = \omega \cdot (\Phi f')$. Together with Claim 4.9 (b) we obtain

$$(Tf)(x) = (Tf)(0) + \int_0^x (Tf)'(t)dt = \lambda f(0) + \int_0^x \omega(t)(\Phi f')(t)dt$$

for $x \in [0,1]$. This completes the proof of Theorem 1.2. □

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(Hironao Koshimizu) Department of Mathematics and System Development, Interdisciplinary Graduate School of Science and Technology, Shinshu University, Matsumoto 390–8621, Japan
E-mail address: `s09t201@shinshu-u.ac.jp`

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