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A POLYNOMIALLY SPECTRUM PRESERVING MAP BETWEEN UNIFORM ALGEBRAS

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ABSTRACT. We determine the general forms of polynomially spectrum preserving maps between uniform algebras for polynomials of the type $p(z,w) = zw + az + bw + c$.

1. Introduction

Let X be a compact Hausdorff space. The algebra of all complex valued continuous functions on X is denoted by $C(X)$. A uniform algebra on X is a closed subalgebra of $C(X)$ which contains constants and separates the points of X. A uniform algebra is a unital semisimple commutative Banach algebra with respect to the supremum norm which is denoted by $\|\cdot\|$ in this paper. For simplicity the Gelfand transform of a uniform algebra A and $f \in A$ are also denoted by A and f respectively, throughout the paper. The spectrum $\sigma(f)$ of $f \in A$ is the usual set of all complex numbers λ such that $f - \lambda$ is not invertible in A, and the subset $\{z \in \sigma(f) : |z| = \|f\|\}$ of $\sigma(f)$ is called the peripheral spectrum and is denoted by $\sigma_\pi(f)$.

Molnár [6] initiated the study of multiplicatively spectrum preserving maps on certain Banach algebras. Luttman and Tonev [5] introduced the peripherally multiplicatively spectrum preserving maps and show a generalization of a theorem of Molnár.

Polynomially spectrum preserving maps are first considered by Hatori, Miura and Takagi in [3]. In particular, they considered the surjective map T between uniform algebras A and B such that $\sigma_\pi(p(T(f), T(g))) = \sigma_\pi(f, g)$ holds for every pair $f, g \in A$ with respect to the polynomial of the type $p(z,w) = zw + az + bw + ab$. They asked a question for the case of $p(z,w) = zw + az + bw + c$ without assuming that $c = ab$. In this paper we give an answer for the question by showing a similar result in [3].

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2. Main Result

Theorem 2.1. *Let A and B be uniform algebras on compact Hausdorff spaces X and Y with maximal ideal spaces M_A and M_B, respectively. Let $p(z,w) = zw + az + bw + c$ be a two-variable polynomial with coefficients a, b and c of complex numbers. Suppose that $T : A \to B$ is a surjective map such that the peripheral spectrum inclusion*

$$\sigma_\pi(p(T(f), T(g))) \subset \sigma_\pi(p(f,g)) \tag{2.1}$$

holds for every pair f and g in A. Then we have the following:

(1) *if $a \neq b$, then T is an algebra isomorphism. Thus there exists a homeomorphism Φ from M_B onto M_A such that the equality*

$$T(f)(y) = f(\Phi(y)), \quad y \in M_B$$

holds for every $f \in A$;

(2) *if $a = b$, then there exist a continuous map $\eta : M_B \to \{-1, 1\}$ and a homeomorphism Φ from M_B onto M_A such that the equality*

$$T(f)(y) = \eta(y) f(\Phi(y)) + a(\eta(y) - 1), \quad y \in M_B$$

holds for every $f \in A$.

Proof. We note that every map of the forms (1) for the case of $a \neq b$ and (2) for the case of $a = b$ satisfies the spectral equation $\sigma(p(T(f), T(g)) = \sigma(p(f,g))$. The content of the theorem is that the reverse statement with a weaker assumption is also true.

To begin with the proof, we define two surjections $S_1 : A \to B$ and $S_2 : A \to B$ as $S_1(h) = T(h - a) + a$ and $S_2(h) = T(h - b) + b$ for $h \in A$. Since $p(z,w) = (z + b)(w + a) + c - ab$ we see by a simple calculation that

$$\sigma_\pi(S_1(h_1)S_2(h_2) + c - ab) \subset \sigma_\pi(h_1 h_2 + c - ab) \tag{2.2}$$

holds for every pair h_1 and h_2 in A.

Firstly, we consider the case where $c = ab$. In this case we have

$$\sigma_\pi(S_1(h_1)S_2(h_2)) \subset \sigma_\pi(h_1 h_2) \tag{2.3}$$

holds for every pair h_1 and h_2 in A. Then by [4, Corollary 1] there exists a homeomorphism $\phi : \mathrm{Ch}(B) \to \mathrm{Ch}(A)$ such that

$$\frac{S_1(h)(y)}{S_1(1)(y)} = \frac{S_2(h)(y)}{S_2(1)(y)} = h(\phi(y)), \quad \forall y \in \mathrm{Ch}(B) \tag{2.4}$$

holds for every $h \in A$, where $\mathrm{Ch}(\cdot)$ is the Choquet boundary. We consider both cases of $a \neq b$ and $a = b$. Now we suppose that the first case : $a \neq b$. Then

$$T(-b) = S_1(-b + a) - a = -S_1(1)b + (S_1(1) - 1)a$$

and

$$T(-b) = S_2(-b + b) - b = -b$$

on $\mathrm{Ch}(B)$. Then we have $S_1(1) = 1$ for $a - b \neq 0$. It follows that

$$T(h)(y) = S_1(h + a)(y) - a = h(\phi(y)), \quad \forall y \in \mathrm{Ch}(B)$$

holds for every $h \in A$; T is an algebra isomorphism from A onto B. Applying general theory for commutative Banach algebra we see that there exists a homeomorphism Φ from M_B onto M_A with

$$T(h)(y) = h(\Phi(y)), \quad \forall y \in M_B$$

for every $h \in A$. Next we suppose that the second case : $a = b$. Then $S_1 = S_2$ in this case. Applying the first part of the proof of [4, Corollary 1] we see that $(S_1(1))^2 = 1$ holds. Then by (2.4) the map $\frac{S_1}{S_1(1)}$ defines an algebra isomorphism from A onto B. Hence there exists a homeomorphism $\Phi : M_B \to M_A$ such that

$$S_1(h)(y) = \eta(y)h(\Phi(y)), \quad \forall y \in M_B$$

holds for every $h \in A$, where η is (precisely the Gelfand transform of) $S_1(1)$. It follows by a calculation that

$$T(h)(y) = \eta(y)h(\Phi(y)) + a(\eta(y) - 1), \quad \forall y \in M_B$$

holds for every $h \in A$.

Secondly, we consider the case where $c \neq ab$. Put $d = c - ab \neq 0$ and rewrite (2.2) we have

$$\sigma_\pi(S_1(h_1)S_2(h_2) + d) \subset \sigma_\pi(h_1h_2 + d)$$

for every pair h_1 and h_2 in A. Henceforce

$$\|S_1(h_1)S_2(h_2) + d\| = \|h_1h_2 + d\|$$

holds for every pair h_1 and h_2 in A. Then by [4, Theorem 3] there exists a homeomorphism $\Phi : M_B \to M_A$ and a clopen subset K of M_B such that

$$S_1(1)(y)S_2(1)(y) = \begin{cases} 1, \ y \in K, \\ d/\bar{d}, \ y \in M_B \setminus K \end{cases} \tag{2.5}$$

and

$$\frac{S_1(h)(y)}{S_1(1)(y)} = \frac{S_2(h)(y)}{S_2(1)(y)} = \begin{cases} h(\Phi(y)), \ y \in K, \\ \overline{h(\Phi(y))}, \ y \in M_B \setminus K \end{cases} \tag{2.6}$$

hold for every $h \in A$. We intend to prove $K = M_B$. Suppose that $K \neq M_B$. We will lead a contradiction. Put a complex number α such that $|d| < |\bar{\alpha}^2 d/\bar{d} + d|$,

$|d| < |\alpha^2 + d|$, and $\bar{\alpha}^2 d/\bar{d} + d \neq \alpha^2 + d$. By the Šilov idempotent theorem there exists an $h_0 \in A$ with

$$h_0(y) = \begin{cases} 0, \ y \in \Phi(K), \\ \alpha, \ y \in M_A \setminus \Phi(K). \end{cases}$$

Then by (2.5) and (2.6) we have

$$S_1(h_0)(y)S_2(h_0)(y) = \begin{cases} 0, \ y \in K, \\ \bar{\alpha}^2 d/\bar{d}, \ y \in M_B \setminus K. \end{cases}$$

Hence we see that

$$\sigma_\pi(S_1(h_0)S_2(h_0) + d) = \{\bar{\alpha}^2 d/\bar{d} + d\}$$

since $|d| < |\bar{\alpha}^2 d/\bar{d} + d|$. On the other hand since

$$(h_0)^2 + d = \begin{cases} d \text{ on } \Phi(K), \\ \alpha^2 + d \text{ on } M_A \setminus \Phi(K) \end{cases}$$

holds we have

$$\sigma_\pi((h_0)^2 + d) = \{\alpha^2 + d\}$$

since $|d| < |\alpha^2 + d|$, which contradicts to the inclusion

$$\sigma_\pi(S_1(h_0)S_2(h_0) + d) \subset \sigma_\pi((h_0)^2 + d).$$

We have just concluded that $K = M_B$. Henceforce

$$S_1(1)S_2(1) = 1$$

and

$$S_1(h) = S_1(1)h \circ \Phi, \ S_2(h) = S_2(1)h \circ \Phi$$

hold on M_B. The rest of the proof is similar to the proof for the case of $c = ab$ and we see that

$$T(h) = h \circ \Phi$$

on M_B for every $h \in A$ if $a = b$, and if $a \neq b$, then there is a continuous function $\eta : M_B \to \{-1, 1\}$ such that

$$T(h) = \eta h \circ \Phi + a(\eta - 1)$$

on M_B holds for every $h \in A$. □

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