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A GENERALIZATION OF THE BANACH-STONE THEOREM FOR COMMUTATIVE BANACH ALGEBRAS

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Abstract. Let I be an index set, not necessarily a subset of any Banach algebra. Let $\mathcal{A}$ and $\mathcal{B}$ be unital semisimple commutative Banach algebras with maximal ideal spaces $M_{\mathcal{A}}$ and $M_{\mathcal{B}}$, respectively. If surjective mappings $S_1, S_2 \colon I \to \mathcal{A}$ and $T_1, T_2 \colon I \to \mathcal{B}$ satisfy $\mathrm{r}(T_1(\lambda) - T_2(\mu)) = \mathrm{r}(S_1(\lambda) - S_2(\mu))$ for all $\lambda, \mu \in I$, where $\mathrm{r}(a)$ is the spectral radius of a, then there exist $p, w \in \mathcal{B}$, a homeomorphism $\varphi \colon M_{\mathcal{B}} \to M_{\mathcal{A}}$ and a closed and open subset K of $M_{\mathcal{B}}$ such that $|\hat{w}| = 1$ on $M_{\mathcal{B}}$ and that

$$\widehat{T_k(\lambda)}(y) - \hat{p}(y) = \begin{cases} \hat{w}(y)\widehat{S_k(\lambda)}(\varphi(y)) & y \in K \\ \hat{w}(y)\overline{\widehat{S_k(\lambda)}(\varphi(y))} & y \in M_{\mathcal{B}} \setminus K \end{cases}$$

for all $\lambda \in I$ ($k = 1, 2$). In particular, if $\mathcal{A}$ and $\mathcal{B}$ are uniform algebras, and if $S_1, S_2 \colon I \to \mathcal{A}$ and $T_1, T_2 \colon I \to \mathcal{B}$ satisfy

$$\sigma_\pi\left(T_1(\lambda) - T_2(\mu)\right) \cap \sigma_\pi\left(S_1(\lambda) - S_2(\mu)\right) \neq \emptyset \qquad (\forall \lambda, \mu \in I),$$

where $\sigma_\pi(f)$ is the peripheral spectrum of f, then $\widehat{T_k(\lambda)}(y) = \hat{p}(y) + \widehat{S_k(\lambda)}(\varphi(y))$ for all $\lambda \in I$ and $y \in M_{\mathcal{B}}$ ($k = 1, 2$).

1. Introduction

The study of spectrum preserving surjections between Banach algebras is one of the most active areas in Banach algebra theory. The theorem of Kowalski and Słodkowski [8, Theorem 1.2] states that if a complex-valued mapping $T \colon \mathcal{A} \to \mathbb{C}$ defined on a Banach algebra $\mathcal{A}$ satisfies $T(a) - T(b) = \sigma(a - b)$ for all $a, b \in \mathcal{A}$, then T is linear and multiplicative (cf. [5, Theorem 3.1]), where $\sigma(\cdot)$ denotes the spectrum of the algebra element. This result is a generalization of the theorem of Gleason, Kahane and Żelazko [3, 7, 23].

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It seems interesting to consider a part of the spectrum instead of the spectrum. In fact, Rao, Tonev and Toneva [18] considered the peripheral spectrum $\sigma_\pi(f)$ of f defined by $\sigma_\pi(f) = \{z \in \sigma(f) : |z| = \max_{\lambda\in\sigma(f)} |\lambda|\}$, and proved that a surjective mapping $T\colon A \to B$ between uniform algebras is an isometric isomorphism whenever T satisfies $\sigma_\pi(T(f)+T(g)) = \sigma_\pi(f+g)$ for all $f, g \in A$ under some additional assumption on the maximum moduli of functions. Tonev and Yates introduced *norm-linear* condition $\|\alpha T(f) + \beta T(g)\|_\infty = \|\alpha f + \beta g\|_\infty$ for $f, g \in A$ and $\alpha, \beta \in \mathbb{C}$, where $\|f\|_\infty$ is the supremum norm of f. Among other things, they [21, Theorem 20] proved that if $T\colon A \to B$ is a norm-linear surjective mapping between uniform algebras with $T(1) = 1$ and $T(i) = i$, then T is an isometric algebra isomorphism. Tonev [20, Corollary 6] proved that if a surjective mapping $T\colon \mathcal{A} \to \mathcal{B}$ between unital semisimple commutative Banach algebras satisfies $\mathrm{r}(\alpha T(a) + \beta T(b)) = \mathrm{r}(\alpha a + \beta b)$ for all $a, b \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$, then T is an algebra isomorphism. Here, $\mathrm{r}(a)$ denotes the spectral radius of a. Let $M_{\mathcal{A}}$ and $M_{\mathcal{B}}$ be maximal ideal spaces of $\mathcal{A}$ and $\mathcal{B}$, respectively. It was shown in [6] that if a surjective mapping $T\colon \mathcal{A} \to \mathcal{B}$ satisfies $\mathrm{r}(T(a)+T(b)) = \mathrm{r}(a+b)$ for all $a, b \in \mathcal{A}$, then there exit a homeomorphism $\varphi\colon M_{\mathcal{B}} \to M_{\mathcal{A}}$ and a closed and open subset K of $M_{\mathcal{B}}$ such that

$$\widehat{T(a)}(y) = \begin{cases} \widehat{T(e)}(y)\hat{a}(\varphi(y)) & y \in K \\ \widehat{T(e)}(y)\overline{\hat{a}(\varphi(y))} & y \in M_{\mathcal{B}} \setminus K \end{cases}$$

for all $a \in \mathcal{A}$, where e is unit of $\mathcal{A}$ and $\hat{a}$ is the Gelfand transform of a.

In this paper we investigate surjective mappings $S_1, S_2\colon I \to \mathcal{A}$ and $T_1, T_2\colon I \to \mathcal{B}$ satisfying $\mathrm{r}(T_1(\lambda)-T_2(\mu)) = \mathrm{r}(S_1(\lambda)-S_2(\mu))$ for all $\lambda, \mu \in I$, where I is an index set. We will prove that (S_k, T_k) are represented by a homeomorphism between maximal ideal spaces of $\mathcal{A}$ and $\mathcal{B}$ ($k = 1, 2$). We also investigate surjective mappings that satisfy $\sigma_\pi(T_1(\lambda) - T_2(\mu)) \subset \sigma_\pi(S_1(\lambda) - S_2(\mu))$ for all $\lambda, \mu \in I$.

2. The main theorem

Let $\mathcal{A}$ be a unital semisimple commutative Banach algebra with maximal ideal space $M_{\mathcal{A}}$ and Shilov boundary $\partial\mathcal{A}$, the smallest closed boundary of $\mathcal{A}$. Denote by $\sigma(a)$ the spectrum of $a \in \mathcal{A}$, namely, $\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda e \notin \mathcal{A}^{-1}\} = \{\hat{a}(x) \in \mathbb{C} : x \in M_{\mathcal{A}}\}$, where $\hat{a}$ is the Gelfand transform of $a \in \mathcal{A}$. Recall that the spectral radius $\mathrm{r}(a)$ of an element $a \in \mathcal{A}$ is the maximum modulus of λ in the spectrum of a, that is $\mathrm{r}(a) = \sup_{x\in M_{\mathcal{A}}} |\hat{a}(x)| = \sup_{x\in\partial\mathcal{A}} |\hat{a}(x)|$. The *peripheral spectrum* $\sigma_\pi(a)$ of $a \in \mathcal{A}$ is defined by $\sigma_\pi(a) = \{\lambda \in \mathbb{C} : |\lambda| = \mathrm{r}(a)\}$. The norm $\|a\|$ of $a \in \mathcal{A}$ and the spectral radius $\mathrm{r}(a)$ are not equal in general. However, if $\mathcal{A}$ is a uniform algebra, then the supremum norm $\|a\|_\infty$ and the spectral radius $\mathrm{r}(a)$ of $a \in \mathcal{A}$ coincide.

Let A be a uniform algebra on a compact Hausdorff space X with the supremum norm $\|\cdot\|$. Recall that $h \in A$ is a *peaking function* of A if and only if $\|h\| = 1$ and $h(x) \neq 1$ implies $|h(x)| < 1$ for $x \in X$. Equivalently, $h \in A$ is a peaking function of A if and only if $\sigma_\pi(h) = \{1\}$. We say that $K \subset X$ is a peak set of A if $K = h^{-1}(1) = \{x \in X : h(x) = 1\}$ for some peaking function h of A. If the intersection of a family of peak sets of A is a singleton, then the unique element in the intersection is called a *weak peak point*, or a *p-point* of A. The set of all weak peak points of A is the *Choquet boundary* of A, denoted by $\mathrm{Ch}(A)$, which is a boundary of A, and dense in the Shilov boundary ∂A of A.

Theorem 2.1. *Let I be an index set, not necessarily a subset of any Banach algebras. Let $\mathcal{A}$ and $\mathcal{B}$ be unital semisimple commutative Banach algebras with maximal ideal spaces $M_{\mathcal{A}}$ and $M_{\mathcal{B}}$, and Shilov boundaries $\partial\mathcal{A}$ and $\partial\mathcal{B}$, respectively. If $S_1, S_2\colon I \to \mathcal{A}$ and $T_1, T_2\colon I \to \mathcal{B}$ are surjective mappings satisfying*

$$\mathrm{r}(T_1(\lambda) - T_2(\mu)) = \mathrm{r}(S_1(\lambda) - S_2(\mu)) \qquad (\forall \lambda, \mu \in I), \tag{2.1}$$

then there exist $p, w \in \mathcal{B}$, a homeomorphism $\varphi\colon M_{\mathcal{B}} \to M_{\mathcal{A}}$ and a closed and open set K of $M_{\mathcal{B}}$ such that $|\hat{w}| = 1$ on $M_{\mathcal{B}}$ and that

$$\widehat{T_k(\lambda)}(y) - \hat{p}(y) = \begin{cases} \hat{w}(y)\widehat{S_k(\lambda)}(\varphi(y)) & y \in K \\ \hat{w}(y)\overline{\widehat{S_k(\lambda)}(\varphi(y))} & y \in M_{\mathcal{B}} \setminus K \end{cases}$$

for all $\lambda \in I$ $(k = 1, 2)$.

Proof. Let $\mu \in I$. Since T_2 is surjective, there exists $\mu' \in I$ such that $T_2(\mu') = T_1(\mu)$. According to (2.1), $\mathrm{r}(S_1(\mu) - S_2(\mu')) = \mathrm{r}(T_1(\mu) - T_2(\mu')) = \mathrm{r}(T_1(\mu) - T_1(\mu)) = 0$. Since $\mathcal{A}$ is semisimple, $S_2(\mu') = S_1(\mu)$. Therefore, for each $\lambda \in I$, $\mathrm{r}(T_1(\lambda) - T_1(\mu)) = \mathrm{r}(T_1(\lambda) - T_2(\mu')) = \mathrm{r}(S_1(\lambda) - S_2(\mu')) = \mathrm{r}(S_1(\lambda) - S_1(\mu))$. Consequently,

$$\mathrm{r}(T_1(\lambda) - T_1(\mu)) = \mathrm{r}(S_1(\lambda) - S_1(\mu)) \qquad (\forall \lambda, \mu \in I). \tag{2.2}$$

Define $\hat{T}_1\colon \hat{\mathcal{A}} \to \hat{\mathcal{B}}$ by $\hat{T}_1(\hat{a}) = \widehat{T_1(\lambda_a)}$ for $\hat{a} \in \hat{\mathcal{A}}$, where λ_a is any element of $\{\lambda \in I : S_1(\lambda) = a\}$. Here, $T_1(\lambda_a)$ does not depend on a choice of an element in $\{\lambda \in I : S_1(\lambda) = a\}$: for if $S_1(\lambda) = S_1(\lambda')$, then $T_1(\lambda) = T_1(\lambda')$ by (2.2). Since $\mathcal{A}$ is semisimple, $\hat{T}_1$ is well-defined. Let $\hat{a}_i \in \hat{\mathcal{A}}$ and take $\lambda_i \in I$ so that $S_1(\lambda_i) = a_i$ for $i = 1, 2$. According to (2.2),

$$\begin{aligned} \sup_{y \in M_{\mathcal{B}}} |\hat{T}_1(\hat{a}_1)(y) - \hat{T}_1(\hat{a}_2)(y)| &= \sup_{y \in M_{\mathcal{B}}} |\widehat{T_1(\lambda_1)}(y) - \widehat{T_1(\lambda_2)}(y)| \\ &= \mathrm{r}(T_1(\lambda_1) - T_1(\lambda_2)) = \mathrm{r}(S_1(\lambda_1) - S_1(\lambda_2)) \\ &= \mathrm{r}(a_1 - a_2) = \sup_{x \in M_{\mathcal{A}}} |\hat{a}_1(x) - \hat{a}_2(x)|. \end{aligned}$$

Thus, $\hat{T}_1\colon \hat{\mathcal{A}} \to \hat{\mathcal{B}}$ is an isometry with respect to the supremum norms. Since T_1 is surjective, for each $\hat{b} \in \hat{\mathcal{B}}$, there exists $\lambda \in I$ such that $T_1(\lambda) = b$, and thus $\hat{T}_1(\widehat{S_1(\lambda)}) = \widehat{T_1(\lambda)} = \hat{b}$. Consequently, $\hat{T}_1$ is a surjective isometry between normed linear spaces $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$. By the Mazur-Ulam theorem [12], $\hat{T}_1$ is affine (cf. [22]). Set $\mathcal{T} = \hat{T}_1 - \hat{T}_1(0)$. Then $\mathcal{T}\colon \hat{\mathcal{A}} \to \hat{\mathcal{B}}$ is a real-linear surjective isometry with respect to the supremum norms.

Next, we show that there exists a real-linear surjective isometry $\widetilde{\mathcal{T}}\colon A \to B$ such that $\widetilde{\mathcal{T}}|_{\hat{\mathcal{A}}} = \mathcal{T}$, where A and B are uniform closures of $\hat{\mathcal{A}} \subset C(M_{\mathcal{A}})$ and $\hat{\mathcal{B}} \subset C(M_{\mathcal{B}})$, respectively. It should be mentioned that the following arguments are used in [6]. Just for the sake of convenience, here we give a proof. It is well-known that A and B are uniform algebras on $M_{\mathcal{A}}$ and $M_{\mathcal{B}}$, respectively, so that $M_A = M_{\mathcal{A}}$, $M_B = M_{\mathcal{B}}$, $\partial A = \partial\mathcal{A}$ and $\partial B = \partial\mathcal{B}$. Since A is the uniform closure of $\hat{\mathcal{A}}$, for each $f \in A$, there exists $\{\hat{a}_n\} \subset \hat{\mathcal{A}}$ such that $\sup_{x\in M_{\mathcal{A}}} |\hat{a}_n(x) - f(x)| \to 0$ as $n \to \infty$. Thus, $\{\mathcal{T}(\hat{a}_n)\} \subset \hat{\mathcal{B}}$ is a Cauchy sequence since $\mathcal{T}$ is a real-linear isometry with respect to the supremum norms. Therefore, there exists an element $\widetilde{\mathcal{T}}(f) \in B$ such that $\sup_{y\in M_{\mathcal{B}}} |\mathcal{T}(\hat{a}_n)(y) - \widetilde{\mathcal{T}}(f)(y)| \to 0$ as $n \to \infty$. We see that $\widetilde{\mathcal{T}}(f)$ does not depend on a choice of $\{\hat{a}_n\} \subset \hat{\mathcal{A}}$ so that $\sup_{x\in M_{\mathcal{A}}} |\hat{a}_n(x) - f(x)| \to 0$ as $n \to \infty$. Thus, there arises a well-defined, real-linear surjective isometry $\widetilde{\mathcal{T}}\colon A \to B$ such that $\widetilde{\mathcal{T}}|_{\hat{\mathcal{A}}} = \mathcal{T}$ as claimed.

Let $\gamma_A\colon A \to A|_{\partial\mathcal{A}}$ and $\gamma_B\colon B \to B|_{\partial\mathcal{B}}$ are restriction mappings. Then γ_A and γ_B are isometric algebra isomorphisms. Set $\widetilde{\mathcal{T}}_\gamma = \gamma_B \circ \widetilde{\mathcal{T}} \circ {\gamma_A}^{-1}$, and thus the mapping $\widetilde{\mathcal{T}}_\gamma\colon A|_{\partial\mathcal{A}} \to B|_{\partial\mathcal{B}}$ is a real-linear surjective isometry.

$$
\begin{array}{ccc}
A & \xrightarrow{\widetilde{\mathcal{T}}} & B \\
{\scriptstyle\gamma_A}\big\downarrow & & \big\downarrow{\scriptstyle\gamma_B} \\
A|_{\partial\mathcal{A}} & \xrightarrow[\widetilde{\mathcal{T}}_\gamma]{} & B|_{\partial\mathcal{B}}
\end{array}
$$

By a result of Ellis [2, Theorem], $\widetilde{\mathcal{T}}_\gamma(1)$ satisfies $|\widetilde{\mathcal{T}}_\gamma(1)| = 1$ on $\partial\mathcal{B}$, and there exist a homeomorphism $\phi\colon \partial\mathcal{B} \to \partial\mathcal{A}$ and a closed and open subset E of $\partial\mathcal{B}$ such that

$$
\widetilde{\mathcal{T}}_\gamma(h)(y) = \begin{cases} \widetilde{\mathcal{T}}_\gamma(1)(y)h(\phi(y)) & y \in E \\ \widetilde{\mathcal{T}}_\gamma(1)(y)\overline{h(\phi(y))} & y \in \partial\mathcal{B} \setminus E \end{cases}
$$

for all $h \in A|_{\partial\mathcal{A}}$. Since $\widetilde{\mathcal{T}}_\gamma(1) = \widetilde{\mathcal{T}}(1)|_{\partial\mathcal{B}} = \mathcal{T}(\hat{e})|_{\partial\mathcal{B}} \in \hat{\mathcal{B}}|_{\partial\mathcal{B}}$, there exists $w \in \mathcal{B}$ such that $\hat{w} = \widetilde{\mathcal{T}}(1) \in B$, where e is unit of $\mathcal{A}$, and thus, $|\hat{w}| = 1$ on $\partial\mathcal{B}$, and

$$
\widetilde{\mathcal{T}}(f)(y) = \begin{cases} \hat{w}(y)f(\phi(y)) & y \in E \\ \hat{w}(y)\overline{f(\phi(y))} & y \in \partial\mathcal{B} \setminus E \end{cases} \tag{2.3}
$$

for all $f \in A$.

Choose an element $f_0 \in A$ such that $\widetilde{\mathcal{T}}(f_0) = 1$. Such $f_0 \in A$ exists since $\widetilde{\mathcal{T}}$ is surjective. By (2.3), $f_0 \circ \phi = 1/\hat{w}$ on E and $\overline{f_0 \circ \phi} = 1/\hat{w}$ on $\partial\mathcal{B} \setminus E$. Set $f_1 = f_0^2 \in A$. Then (2.3) shows that $\widetilde{\mathcal{T}}(f_1) = 1/\hat{w}$ on $\partial\mathcal{B}$ and therefore, $\hat{w}\widetilde{\mathcal{T}}(f_1) = 1$ on $\partial\mathcal{B}$. It follows that $|\hat{w}| = 1$ on $M_{\mathcal{B}}$. Define $\mathcal{U}\colon A \to B$ by $\mathcal{U}(f) = \widetilde{\mathcal{T}}(f_1)\widetilde{\mathcal{T}}(f)$ for $f \in A$. Then $\mathcal{U}$ is surjective since so is $\widetilde{\mathcal{T}}$. By (2.3)

$$\mathcal{U}(f)(y) = \begin{cases} f(\phi(y)) & y \in E \\ \overline{f(\phi(y))} & y \in \partial\mathcal{B} \setminus E \end{cases} \tag{2.4}$$

for all $f \in A$, where we have used $\hat{w}\widetilde{\mathcal{T}}(f_1) = 1$ on $\partial\mathcal{B}$. Here, we notice that $\phi(\mathrm{Ch}(B)) = \mathrm{Ch}(A)$ by (2.4). If we define $\epsilon = (1 - i\mathcal{U}(i))/2 \in B$, then (2.4) yields $\epsilon = 1$ on E and $\epsilon = 0$ on $\partial\mathcal{B} \setminus E$, and thus $\epsilon^2 = \epsilon$ on $\partial\mathcal{B}$. Since $\partial\mathcal{B}$ is a boundary of B, ϵ is an idempotent. Now we use the idea of A. Luttman and S. Lambert in [10, Theorem 2.1]. Set $\epsilon' = 1 - \epsilon \in B$. Then ϵ' is an idempotent such that $\epsilon\epsilon' = 0$. Then $B' = B\epsilon \oplus \overline{B}\epsilon'$ is a uniform algebra on $M_{\mathcal{B}}$ with $M_{B'} = M_{\mathcal{B}}$. We define $\widetilde{\mathcal{U}}\colon A \to B'$ by

$$\widetilde{\mathcal{U}}(f) = \mathcal{U}(f)\epsilon + \overline{\mathcal{U}(f)}\epsilon' \qquad (\forall f \in A). \tag{2.5}$$

By (2.4), with $\epsilon = 1$ on E and $\epsilon = 0$ on $\partial\mathcal{B} \setminus E$, $\widetilde{\mathcal{U}}(f) = f \circ \phi$ on $\partial\mathcal{B}$ for all $f \in A$. Since $\partial\mathcal{B}$ is a boundary of B', $\widetilde{\mathcal{U}}$ is an algebra homomorphism. We show that $\widetilde{\mathcal{U}}$ is a bijection. In fact, if $\widetilde{\mathcal{U}}(f) = \widetilde{\mathcal{U}}(g)$ for $f, g \in A$, then $f \circ \phi = g \circ \phi$ on $\partial\mathcal{B}$, and thus $f = g$ since $\phi(\partial\mathcal{B}) = \partial\mathcal{A}$. Hence $\widetilde{\mathcal{U}}$ is injective. For each $k \in B'$ choose $k_1, k_2 \in B$ such that $k = k_1\epsilon + \overline{k_2}\epsilon'$. There exists $g_0 \in A$ such that $\mathcal{U}(g_0) = k_1\epsilon + k_2\epsilon' \in B$. Since $\epsilon^2 = \epsilon$ and $\epsilon\epsilon' = 0$, (2.5) shows that $\widetilde{\mathcal{U}}(g_0)\epsilon = \mathcal{U}(g_0)\epsilon = k_1\epsilon$ and

$$\widetilde{\mathcal{U}}(g_0)(1 - \epsilon) = \widetilde{\mathcal{U}}(g_0)\epsilon' = \overline{\mathcal{U}(g_0)}\epsilon' = \overline{k_2}\epsilon'.$$

Consequently, $\widetilde{\mathcal{U}}(g_0) = k_1\epsilon + \overline{k_2}\epsilon' = k$, and therefore, $\widetilde{\mathcal{U}}$ is surjective. Hence $\widetilde{\mathcal{U}}$ is a bijection, as claimed. Since $\widetilde{\mathcal{U}}$ is an algebra isomorphism between uniform algebras, there exists a homeomorphism $\varphi\colon M_{\mathcal{B}} \to M_{\mathcal{A}}$ such that $\widetilde{\mathcal{U}}(f) = f \circ \varphi$ for all $f \in A$. According to (2.5), using $\epsilon^2 = \epsilon$ and $\epsilon\epsilon' = 0$, $\widetilde{\mathcal{U}}(f)\epsilon = \mathcal{U}(f)\epsilon$ and $\overline{\widetilde{\mathcal{U}}(f)}\epsilon' = \mathcal{U}(f)\epsilon' = \mathcal{U}(f)(1 - \epsilon)$. Therefore,

$$\mathcal{U}(f) = \widetilde{\mathcal{U}}(f)\epsilon + \overline{\widetilde{\mathcal{U}}(f)}\epsilon'$$

for all $f \in A$. Set $K = \{y \in M_{\mathcal{B}} : \epsilon(y) = 1\}$. Since $\epsilon^2 = \epsilon$, K is a (possibly empty) closed and open subset of $M_{\mathcal{B}}$. Using $\widetilde{\mathcal{U}}(f) = f \circ \varphi$,

$$\mathcal{U}(f)(y) = \begin{cases} f(\varphi(y)) & y \in K \\ \overline{f(\varphi(y))} & y \in M_{\mathcal{B}} \setminus K \end{cases}$$

for all $f \in A$. Since $f \circ \varphi = \widetilde{\mathcal{U}}(f) = f \circ \phi$ on $\partial\mathcal{B}$, we have $\varphi = \phi$ on $\partial\mathcal{B}$. Thus, $\varphi(\partial\mathcal{B}) = \phi(\partial\mathcal{B}) = \partial\mathcal{A}$ and $\varphi(\mathrm{Ch}(B)) = \phi(\mathrm{Ch}(B)) = \mathrm{Ch}(A)$.

Finally, by the definition of $\mathcal{U}$, $\widetilde{\mathcal{T}}(f) = \hat{w}\mathcal{U}(f)$ for all $f \in A$. In addition, recall that

$$\widetilde{\mathcal{T}}(\widehat{S_1(\lambda)}) = \mathcal{T}(\widehat{S_1(\lambda)}) = \hat{T}_1(\widehat{S_1(\lambda)}) - \hat{T}_1(0) = \widehat{T_1(\lambda)} - \hat{T}_1(0)$$

for all $\lambda \in I$. Consequently,

$$\widehat{T_1(\lambda)}(y) - \hat{p}(y) = \begin{cases} \hat{w}(y)\widehat{S_1(\lambda)}(\varphi(y)) & y \in K \\ \hat{w}(y)\overline{\widehat{S_1(\lambda)}(\varphi(y))} & y \in M_{\mathcal{B}} \setminus K \end{cases}$$

for all $\lambda \in I$, where $p \in \mathcal{B}$ with $\hat{p} = \hat{T}_1(0)$. Since T_1 is surjective, for each $\lambda \in I$ there exists $\lambda' \in I$ such that $T_1(\lambda') = T_2(\lambda)$ and $S_1(\lambda') = S_2(\lambda)$, where we have used (2.1). Thus, for each $y \in K$,

$$\begin{aligned} \widehat{T_2(\lambda)}(y) - \hat{p}(y) &= \widehat{T_1(\lambda')}(y) - \hat{p}(y) = \hat{w}(y)\widehat{S_1(\lambda')}(\varphi(y)) \\ &= \hat{w}(y)\widehat{S_2(\lambda)}(\varphi(y)). \end{aligned}$$

By the same argument, we have $\widehat{T_2(\lambda)}(y) - \hat{p}(y) = \hat{w}(y)\overline{\widehat{S_2(\lambda)}(\varphi(y))}$ for all $y \in M_{\mathcal{B}} \setminus K$. □

Remark 2.1. In Theorem 2.1 if we consider the case when $I = \mathcal{A} = C(X)$, $\mathcal{B} = C(Y)$, $S_1 = S_2 = \mathrm{Id}$, the identity mapping and $T_1 = T_2$ are complex-linear, then we have the Banach-Stone theorem [1, 19]. Theorem 2.1 is also a generalization of Nagasawa theorem stating that any unital, surjective, complex-linear isometry between uniform algebras is an algebra isomorphism.

Corollary 2.2. *Let I be an index set, not necessarily a subset of any Banach algebras. Let $\mathcal{A}$ and $\mathcal{B}$ be unital semisimple commutative Banach algebras with maximal ideal spaces $M_{\mathcal{A}}$ and $M_{\mathcal{B}}$, respectively. If $S_1, S_2 \colon I \to \mathcal{A}$ and $T_1, T_2 \colon I \to \mathcal{B}$ are surjective mappings satisfying*

$$\sigma_\pi\left(T_1(\lambda) - T_2(\mu)\right) \subset \sigma_\pi\left(S_1(\lambda) - S_2(\mu)\right) \qquad (\forall \lambda, \mu \in I), \tag{2.6}$$

then there exist $p \in \mathcal{B}$ and a homeomorphism $\varphi \colon M_{\mathcal{B}} \to M_{\mathcal{A}}$ such that

$$\widehat{T_k(\lambda)}(y) - \hat{p}(y) = \widehat{S_k(\lambda)}(\varphi(y))$$

for all $\lambda \in I$ and $y \in M_{\mathcal{B}}$ ($k = 1, 2$).

Proof. By (2.6), $\mathrm{r}(T_1(\lambda) - T_2(\mu)) = \mathrm{r}(S_1(\lambda) - S_2(\mu))$ holds for all $\lambda, \mu \in I$. According to Theorem 2.1, there exist $p, w \in \mathcal{B}$, a homeomorphism $\varphi \colon M_{\mathcal{B}} \to M_{\mathcal{A}}$ and a closed and open subset K of $M_{\mathcal{B}}$ such that $|\hat{w}| = 1$ on $M_{\mathcal{B}}$ and that

$$\widehat{T_k(\lambda)}(y) - \hat{p}(y) = \begin{cases} \hat{w}(y)\widehat{S_k(\lambda)}(\varphi(y)) & y \in K \\ \hat{w}(y)\overline{\widehat{S_k(\lambda)}(\varphi(y))} & y \in M_{\mathcal{B}} \setminus K \end{cases} \tag{2.7}$$

for all $\lambda \in I$ $(k = 1, 2)$. Since S_2 is surjective, there exists $\mu_0 \in I$ such that $S_2(\mu_0) = 0$. Thus $\widehat{T_2(\mu_0)} = \hat{p}$ on M_B, and consequently $T_2(\mu_0) = p$. By (2.6),

$$\sigma_\pi \left(T_1(\lambda) - p\right) \subset \sigma_\pi \left(S_1(\lambda)\right) \tag{2.8}$$

for all $\lambda \in I$. First, we will prove that $\hat{w} = 1$ on $M_{\mathcal{B}}$. Let $\lambda_0 \in I$ with $S_1(\lambda_0) = e$, unit element of $\mathcal{A}$. On one hand,

$$\sigma_\pi \left(T_1(\lambda_0) - p\right) \subset \sigma_\pi \left(S_1(\lambda_0)\right) = \{1\},$$

and thus $\sigma_\pi \left(T_1(\lambda_0) - p\right) = \{1\}$. On the other hand, $\widehat{T_1(\lambda_0)} - \hat{p} = \hat{w}$ on $M_{\mathcal{B}}$ by (2.7), and hence $|\widehat{T_1(\lambda_0)} - \hat{p}| = |\hat{w}| = 1$ on $M_{\mathcal{B}}$. It follows that $\widehat{T_1(\lambda_0)} - \hat{p} = 1$ on $M_{\mathcal{B}}$, and consequently, $\hat{w} = 1$ on $M_{\mathcal{B}}$ as claimed. Finally, we show that $K = M_{\mathcal{B}}$. Suppose, on the contrary, that there exists $y_1 \in M_{\mathcal{B}} \setminus K$. Choose $\lambda_1 \in I$ so that $S_1(\lambda_1) = ie$. According to (2.7), $\widehat{T_1(\lambda_1)}(y_1) - \hat{p}(y_1) = -i$ and $\widehat{T_1(\lambda_1)}(y) - \hat{p}(y) = \pm i$ for $y \in M_{\mathcal{B}}$. Hence $-i \in \sigma_\pi \left(T_1(\lambda_1) - p\right)$. By (2.8), $\sigma_\pi \left(T_1(\lambda_1) - p\right) \subset \sigma_\pi \left(S_1(\lambda_1)\right) = \{i\}$, which is a contradiction. This implies that $K = M_{\mathcal{B}}$. Thus, $\widehat{T_k(\lambda)}(y) - \hat{p}(y) = \widehat{S_k(\lambda)}(\varphi(y))$ for all $\lambda \in I$ and $y \in M_{\mathcal{B}}$ $(k = 1, 2)$. $\square$

Corollary 2.3. *Let I be an index set, not necessarily a subset of any Banach algebras. Let A and B be uniform algebras with maximal ideal spaces M_A and M_B, respectively. If $S_1, S_2 \colon I \to A$ and $T_1, T_2 \colon I \to B$ are surjective mappings satisfying*

$$\sigma_\pi \left(T_1(\lambda) - T_2(\mu)\right) \cap \sigma_\pi \left(S_1(\lambda) - S_2(\mu)\right) \neq \emptyset \qquad (\forall \lambda, \mu \in I), \tag{2.9}$$

then there exist $p \in B$ and a homeomorphism $\varphi \colon M_B \to M_A$ such that

$$\widehat{T_k(\lambda)}(y) - \hat{p}(y) = \widehat{S_k(\lambda)}(\varphi(y))$$

for all $\lambda \in I$ and $y \in M_B$ $(k = 1, 2)$.

Proof. According to Theorem 2.1, there exist $p, w \in B$, a homeomorphism $\varphi \colon M_B \to M_A$ and a closed and open subset K of M_B such that $|w| = 1$ on $M_{\mathcal{B}}$ and that (2.7) holds for all $\lambda \in I$ $(k = 1, 2)$. By the same argument to the Proof of Corollary 2.2, we see that

$$\sigma_\pi \left(T_1(\lambda) - p\right) \cap \sigma_\pi \left(S_1(\lambda)\right) \neq \emptyset \tag{2.10}$$

for all $\lambda \in I$. As mentioned in the proof of Theorem 2.1, we also have that $\varphi(\mathrm{Ch}(B)) = \mathrm{Ch}(A)$.

First, we will prove that $w = 1$. Let $\lambda_0 \in I$ with $S_1(\lambda_0) = 1$, the unit element of A. By (2.7) and (2.10), $1 \in \sigma_\pi \left(T_1(\lambda_0) - p\right) = \sigma_\pi (w)$. Thus, $F = \{y \in \partial B : w(y) = 1\}$ is a non-empty closed subset of the Shilov boundary ∂B of B. Suppose, on the contrary, that there exists $y_0 \in \partial B \setminus F$. Since $\mathrm{Ch}(B)$ is dense in ∂B, we may and do assume that $y_0 \in \mathrm{Ch}(B) \setminus F$. Set $x_0 = \varphi(y_0) \in \mathrm{Ch}(A)$. Then $\varphi(F)$ is a closed subset of $\varphi(\partial B) = \partial A$ with $x_0 \notin \varphi(F)$. Hence, there exists $\lambda_1 \in I$ such that $\sigma_\pi(S_1(\lambda_1)) = \{1\}$, $S_1(\lambda_1)(x_0) = 1$ and $|S_1(\lambda_1)| < 1$ on $\varphi(F)$. On one hand, $1 \in \sigma_\pi(T_1(\lambda_1) - p)$ by (2.9). Thus $T_1(\lambda_1)(y_1) - p(y_1) = 1$ for some $y_1 \in \mathrm{Ch}(B)$. Then $y_1 \notin F$ because $|T_1(\lambda_1)(y) - p(y)| = |S_1(\lambda_1)(\varphi(y))| < 1$ for all $y \in F$. Hence, $w(y_1) \neq 1$ by the definition of F. On the other hand, since $\sigma_\pi(S_1(\lambda_1)) = \{1\}$, $1 = |T_1(\lambda_1)(y_1) - p(y_1)| = |S_1(\lambda_1)(\varphi(y_1))|$ implies that $S_1(\lambda_1)(\varphi(y_1)) = 1$. According to (2.7), $T_1(\lambda_1)(y_1) - p(y_1) = w(y_1) \neq 1$, which is a contradiction. Consequently, $\partial B = F$, that is $w = 1$ on ∂B. Since ∂B is a boundary of B, we obtain that $w = 1$ as claimed.

Next, we show that $K \neq \emptyset$. In fact, choose $\lambda_2 \in I$ so that $S_1(\lambda_2) = i$. Then by (2.7) with $w = 1$, $\widehat{T_1(\lambda_2)} - \hat{p} = i$ on K and $\widehat{T_1(\lambda_2)} - \hat{p} = -i$ on $M_{\mathcal{B}} \setminus K$. Since $\sigma_\pi(S_1(\lambda_2)) = \{i\}$, (2.10) yields $K \neq \emptyset$. Finally, we prove that $K = M_B$. Suppose that there exists $y_2 \in \mathrm{Ch}(B) \setminus K$. Since K is a non-empty closed subset of $M_{\mathcal{B}}$ with $y_2 \notin K$, there exists $\lambda_3 \in I$ so that $\sigma_\pi(S_1(\lambda_3)) = \{i\}$, $S_1(\lambda_3)(y_2) = i$ and $|S_1(\lambda_3)| < 1$ on K. By (2.7), $\sigma_\pi(T_1(\lambda_3) - p) = \{-i\}$, which contradicts (2.10). This shows that $\mathrm{Ch}(B) \setminus K = \emptyset$, and thus $\mathrm{Ch}(B) \subset K$. By the choice of $\lambda_2 \in I$, $T_1(\lambda_2) - p = i$ on $\mathrm{Ch}(B)$. Since $\mathrm{Ch}(B)$ is a boundary of B, $T_1(\lambda_2) - p = i$, and thus $M_B \setminus K$ is empty. Consequently, $K = M_B$ as claimed. According to (2.7), we conclude that $\widehat{T_k(\lambda)}(y) - \hat{p}(y) = \widehat{S_k(\lambda)}(\varphi(y))$ for all $\lambda \in I$ and $y \in M_B$ $(k = 1, 2)$. $\square$

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