Nihonkai Math. J.
Vol.21(2010), 91–103

CONE-SEMICONTINUITY OF SET-VALUED MAPS BY ANALOGY WITH REAL-VALUED SEMICONTINUITY

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Abstract. In the paper, we propose how we can treat several kinds of semicontinuity with respect to cone for set-valued maps by analogy with semicontinuity for real-valued functions and investigate the inheritance properties on cone-(semi)continuity of parent set-valued maps via scalarization.

1. Introduction

In general, it is well known that the composite function of two continuous functions is also continuous. Göpfert, Riahi, Tammer and Zălinescu [2] show several continuity properties of the composition of two set-valued maps or of a function with a set-valued map. Kuwano, Tanaka and Yamada [6] prove inheritance properties on continuity of set-valued maps via scalarization. These studies are concerned with several types of inheritance property on continuity of parent functions for composite functions. If we obtain some scalarizing function ϕ which preserves some kinds of continuity of a parent vector-valued or set-valued function f, then we can get a clue to confirm the continuity of its parent function by checking the continuity of its composite function $\phi \circ f$.

On the other hand, it is well known that there are various definitions of semicontinuity for real-valued functions. Let X be a topological space, then a real-valued function $f : X \to \mathbb{R}$ is lower semicontinuous at $\bar{x} \in X$ if $\liminf_{x\to\bar{x}} f(x) \geq f(\bar{x})$, which is equivalent to the following condition: for any $a \in \mathbb{R}$ with $f(\bar{x}) > a$, there exists an open neighborhood V of $\bar{x}$ such that $f(x) > a$ for all $x \in V$. In other words, f is lower semicontinuous at $\bar{x}$ if for any interval $(a, b) \subset \mathbb{R}$ with $f(\bar{x}) \in (a, b)$, there exists an open neighborhood V of $\bar{x}$ such that $f(x) \in (a, b) + \mathbb{R}_+$ for all $x \in V$ where

2000 *Mathematics Subject Classification.* Primary 49J53, 54C60; Secondary 90C29.
Key words and phrases. Set-valued analysis, semicontinuity, cone-continuity, nonlinear scalarization, set-relation.
This work is based on research 21540121 supported by Grant-in-Aid for Scientific Research (C) from Japan Society for the Promotion of Science.

$\mathbb{R}_+ := \{r \in \mathbb{R} : r \geq 0\}$. Regarding $\mathbb{R}_+$ as an ordering cone, the lower semicontinuity of f is characterized by an order structure of real numbers. In the case of vector-valued functions, Luc [7] introduces the notion of cone-continuity as follows: given a convex cone C in a vector space Y, a vector-valued function $f : X \to Y$ is C-continuous at $\bar{x} \in X$ if for any neighborhood V of $f(\bar{x})$, there exists a neighborhood U of $\bar{x}$ such that $f(x) \in V + C$ for all $x \in U$. By the same discussion in the case of real-valued functions, the C-continuity of f is characterized via the order structure induced by the ordering cone C. In the case of set-valued maps, various notions of cone-(semi)continuity are introduced in [2]. By using an order structure with an ordering cone, we can regard the notion of cone-continuity for set-valued maps as an analogous concept with semicontinuity of real-valued functions.

In the paper, we focus on the case of set-valued maps, and consider two types of composite functions of a set-valued map and each of certain scalarizing functions, which are proposed in [6]. Then, we investigate the inheritance properties on cone-continuity of parent set-valued maps via this kind of scalarization.

The organization of the paper is as follows. In Section 2, we introduce a mathematical methodology [4] on comparison between two sets in an ordered vector space and several definitions of continuity and cone-continuity for set-valued maps (see [2]). Moreover, we consider relationships between continuity notions for set-valued maps and semicontinuity for real-valued functions. In Section 3, we introduce two types of nonlinear scalarizing functions for sets proposed in [6]. Also we investigate how certain kinds of cone-continuity for set-valued maps are inherited to composite functions with the scalarizing functions.

2. Mathematical Preliminaries

Let Y be a real topological vector space with the vector ordering $\leq_C$ induced by a proper convex cone C ($C \neq \emptyset$, $C \neq Y$ and $C + C = C$) with nonempty topological interior as follows:

$$x \leq_C y \text{ if } y - x \in C \text{ for } x, y \in Y.$$

It is well known that $\leq_C$ is reflexive and transitive where C is a convex cone, and that $\leq_C$ has invariable properties to vector space structure as translation and scalar multiplication. Then, the space Y is called an *ordered topological vector space*. In particular, if C is pointed, then $\leq_C$ is antisymmetric, and hence Y is a partially ordered topological vector space.

Throughout the paper, we assume that X is a real topological vector space, Y a real ordered topological vector space and F a set-valued map from X into $2^Y \setminus \{\emptyset\}$, respectively. For any $A \subset Y$, we denote the interior, closure and complement of A by $\operatorname{int} A$, $\operatorname{cl} A$ and A^c, respectively.

At first, we review some basic concepts of set-relation and several definitions of continuity and cone-continuity for set-valued maps.

Definition 2.1 (set-relation, [4]). For nonempty sets A, $B \subset Y$ and convex cone C in Y, we write

$A \leq_C^{(1)} B$ by $A \subset \bigcap_{b \in B}(b - C)$, equivalently $B \subset \bigcap_{a \in A}(a + C)$;
$A \leq_C^{(2)} B$ by $A \cap \left(\bigcap_{b \in B}(b - C)\right) \neq \emptyset$;
$A \leq_C^{(3)} B$ by $B \subset (A + C)$;
$A \leq_C^{(4)} B$ by $\left(\bigcap_{a \in A}(a + C)\right) \cap B \neq \emptyset$;
$A \leq_C^{(5)} B$ by $A \subset (B - C)$;
$A \leq_C^{(6)} B$ by $A \cap (B - C) \neq \emptyset$, equivalently $(A + C) \cap B \neq \emptyset$.

Proposition 2.1 ([4]). *For nonempty sets $A, B \subset Y$, the following statements hold.*

$A \leq_C^{(1)} B$ *implies* $A \leq_C^{(2)} B$; $A \leq_C^{(1)} B$ *implies* $A \leq_C^{(4)} B$;
$A \leq_C^{(2)} B$ *implies* $A \leq_C^{(3)} B$; $A \leq_C^{(4)} B$ *implies* $A \leq_C^{(5)} B$;
$A \leq_C^{(3)} B$ *implies* $A \leq_C^{(6)} B$; $A \leq_C^{(5)} B$ *implies* $A \leq_C^{(6)} B$.

Proposition 2.2 ([5]). *For nonempty sets $A, B \subset Y$, the following statements hold.*

(i) *For each $j = 1, \ldots, 6$,*
$A \leq_C^{(j)} B$ *implies* $A + y \leq_C^{(j)} B + y$ *for* $y \in Y$, *and*
$A \leq_C^{(j)} B$ *implies* $\alpha A \leq_C^{(j)} \alpha B$ *for* $\alpha > 0$;
(ii) *For each $j = 1, \ldots, 5$, $\leq_C^{(j)}$ is transitive;*
(iii) *For each $j = 3, 5, 6$, $\leq_C^{(j)}$ is reflexive.*

Proposition 2.3 ([5]). *For nonempty subsets $V, V' \subset Y$ and direction $k \in C \setminus (-\operatorname{cl} C)$, the following statements hold.*

(i) *For each $j = 1, \ldots, 6$, $V \leq_C^{(j)} tk + V'$ implies $V \leq_C^{(j)} sk + V'$ for any $s \geq t$;*
(ii) *For each $j = 1, \ldots, 6$, $tk + V' \leq_C^{(j)} V$ implies $sk + V' \leq_C^{(j)} V$ for any $s \leq t$.*

Next, we recall usual definitions of continuity for set-valued maps.

Definition 2.2 (lower continuous, [2]). A set-valued map F is said to be *lower continuous* (*l.c.*, for short) *at* $\bar{x}$ if for every open set $V \subset Y$ with $F(\bar{x}) \cap V \neq \emptyset$, there exists an open neighborhood U of $\bar{x}$ such that $F(x) \cap V \neq \emptyset$ for all $x \in U$. We shall say that F is *lower continuous on* X if F is lower continuous at every point $x \in X$.

Definition 2.3 (upper continuous, [2]). A set-valued map F is said to be *upper continuous* (*u.c.*, for short) *at* $\bar{x}$ if for every open set $V \subset Y$ with $F(\bar{x}) \subset V$, there exists an open neighborhood U of $\bar{x}$ such that $F(x) \subset V$ for all $x \in U$. We shall say that F is *upper continuous on* X if F is upper continuous at every point $x \in X$.

Classically, we find the terms "lower semicontinuous" and "upper semicontinuous" for these notions. Instead, in this paper, we use the terms "lower continuous" and "upper continuous" along the lines of [2], because both notions above coincide with the usual continuity of single-valued functions when the set-valued map is singleton, that is, $F(x) = \{f(x)\}$ for some function $f : X \to Y$.

For cone-continuity of set-valued maps, there are many concepts; see [3] in 1999, [1] in 2000 and [2] in 2003. In this paper, we use the following typical definitions of cone-continuity for set-valued maps based on [2].

Definition 2.4 (C-lower continuous, [2])**.** A set-valued map F from X into $2^Y \setminus \{\emptyset\}$ is said to be *C-lower continuous* (C*-l.c.*, for short) *at* $\bar{x}$ if for every open set $V \subset Y$ with $F(\bar{x}) \cap V \neq \emptyset$, there exists an open neighborhood U of $\bar{x}$ such that $F(x) \cap (V + C) \neq \emptyset$ for all $x \in U$. We shall say that F is *C-lower continuous on* X if F is C-lower continuous at every point $x \in X$.

Definition 2.5 (C-upper continuous, [2])**.** A set-valued map F from X into $2^Y \setminus \{\emptyset\}$ is said to be *C-upper continuous* (C*-u.c.*, for short) *at* $\bar{x}$ if for every open set $V \subset Y$ with $F(\bar{x}) \subset V$, there exists an open neighborhood U of $\bar{x}$ such that $F(x) \subset V + C$ for all $x \in U$. We shall say that F is *C-upper continuous on* X if F is C-upper continuous at every point $x \in X$.

Remark 2.1. When $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, C-lower continuity and C-upper continuity for singleton set-valued maps coincide with the usual lower semicontinuity for real-valued functions. Also, $(-C)$-lower continuity and $(-C)$-upper continuity for singleton set-valued maps coincide with the usual upper semicontinuity for real-valued functions. By symbolic interpretation, C and $-C$ correspond to "lower" and "upper," respectively.

3. Relationships between Cone-Semicontinuity of Set-Valued Maps and Semicontinuity of Real-Valued Functions

At first, we introduce the definition of two types of nonlinear scalarizing functions for sets proposed by a unified approach in [5]. Let V and V' be nonempty subsets of Y, and direction $k \in \operatorname{int} C$. For each $j = 1, \ldots, 6$, $I^{(j)}_{k,V'} : 2^Y \setminus \{\emptyset\} \to \mathbb{R} \cup \{\pm\infty\}$ and $S^{(j)}_{k,V'} : 2^Y \setminus \{\emptyset\} \to \mathbb{R} \cup \{\pm\infty\}$ are defined by

$$I^{(j)}_{k,V'}(V) := \inf \left\{ t \in \mathbb{R} \;\middle|\; V \leq^{(j)}_C tk + V' \right\},$$

$$S^{(j)}_{k,V'}(V) := \sup \left\{ t \in \mathbb{R} \;\middle|\; tk + V' \leq^{(j)}_C V \right\},$$

respectively. These functions are called *unified types of scalarizing functions* for sets.

In this section, we introduce relationships between cone-continuity of set-valued maps and semicontinuity of certain composite functions with the unified types of scalarizing functions. This is along the lines of [5, 6] but different from the approach of [8]. For any $x \in X$ and for each $j = 1, \ldots, 6$, we consider the following composite functions:

$$(I^{(j)}_{k,V'} \circ F)(x) := I^{(j)}_{k,V'}(F(x)),$$
$$(S^{(j)}_{k,V'} \circ F)(x) := S^{(j)}_{k,V'}(F(x)).$$

Then, we can directly discuss inheritance properties on cone-continuity of parent set-valued map F to semicontinuity of $I^{(j)}_{k,V'} \circ F$ and $S^{(j)}_{k,V'} \circ F$ in an analogous fashion to linear scalarizing function like inner product. For this end, we consider the following level sets;

$$\mathrm{lev}^l_r(f) := \{x \in X | f(x) \leq r\},$$
$$\mathrm{lev}^u_r(f) := \{x \in X | r \leq f(x)\},$$

where $f : X \to \mathbb{R} \cup \{\pm\infty\}$. Then, we show how certain kinds of cone-continuity for parent set-valued maps are inherited to these composite functions with the unified types of scalarizing functions.

Theorem 3.1 ([6]). *Let F be a set-valued map and $k \in \mathrm{int}\, C$. Then, the following statements hold.*

(i) *For each $j = 1, 4, 5$,*
  (a) *if F is lower continuous on X, then $I^{(j)}_{k,V'} \circ F$ is lower semicontinuous on X,*
  (b) *if F is upper continuous on X, then $I^{(j)}_{k,V'} \circ F$ is upper semicontinuous on X.*

(ii) *For each $j = 2, 3, 6$,*
  (c) *if F is lower continuous on X, then $I^{(j)}_{k,V'} \circ F$ is upper semicontinuous on X,*
  (d) *if F is upper continuous on X, then $I^{(j)}_{k,V'} \circ F$ is lower semicontinuous on X.*

Theorem 3.2 ([6]). *Let F be a set-valued map and $k \in \mathrm{int}\, C$. Then, the following statements hold.*

(i) *For each $j = 1, 2, 3$,*
  (a) *if F is lower continuous on X, then $S^{(j)}_{k,V'} \circ F$ is upper semicontinuous on X,*
  (b) *if F is upper continuous on X, then $S^{(j)}_{k,V'} \circ F$ is lower semicontinuous on X.*

(ii) *For each $j = 4, 5, 6$,*

(c) *if F is lower continuous on X, then $S^{(j)}_{k,V'} \circ F$ is lower semicontinuous on X,*
(d) *if F is upper continuous on X, then $S^{(j)}_{k,V'} \circ F$ is upper semicontinuous on X.*

To show main results, we give the following lemma.

Lemma 3.1. *Let A be a subset in Y and C a convex cone in Y. Then, the following statements hold.*

(i) $\{\mathrm{cl}\,(A+C)\}^c = \{\mathrm{cl}\,(A+C)\}^c - C$ *and* $\{\mathrm{cl}\,(A-C)\}^c = \{\mathrm{cl}\,(A-C)\}^c + C$*;*
(ii) $\mathrm{int}\,(A+C) = \mathrm{int}\,(A+C) + C$ *and* $\mathrm{int}\,(A-C) = \mathrm{int}\,(A-C) - C$*.*

Proof. Since C is a convex cone, we can prove easily by the definitions of the closure and interior. □

Theorem 3.3. *Let F be a set-valued map, C a convex cone in Y and $k \in \mathrm{int}\,C$. Then, the following statements hold.*

(i) *For each $j = 1, 4, 5$,*
  (a) *if F is C-lower continuous on X, then $I^{(j)}_{k,V'} \circ F$ is lower semicontinuous on X,*
  (b) *if F is $(-C)$-upper continuous on X, then $I^{(j)}_{k,V'} \circ F$ is upper semicontinuous on X.*
(ii) *For each $j = 2, 3, 6$,*
  (c) *if F is $(-C)$-lower continuous on X, then $I^{(j)}_{k,V'} \circ F$ is upper semicontinuous on X,*
  (d) *if F is C-upper continuous on X, then $I^{(j)}_{k,V'} \circ F$ is lower semicontinuous on X.*

Proof. The proof throughout the whole of the theorem is given by the same method, and so we shall prove in cases of $j = 3, 5$.

First, we prove (a) and (d). For $j = 3, 5$, we show that

$$\mathrm{lev}^l_r(I) := \{x \in X \mid (I^{(j)}_{k,V'} \circ F)(x) \le r\}$$

is closed for any $r \in \mathbb{R}$, that is, for any net $\{x_\alpha\}_{\alpha \in J} \subset \mathrm{lev}^l_r(I)$,

$$x_\alpha \to \bar{x} \Rightarrow \bar{x} \in \mathrm{lev}^l_r(I),$$

where J is a directed set. Assume that there exist $\bar{r} \in \mathbb{R}$, $\{x_\beta\}_{\beta \in J} \subset \mathrm{lev}^l_{\bar{r}}(I)$, and $\bar{x} \in X$ such that

$$x_\beta \to \bar{x} \quad \text{and} \quad \bar{x} \notin \mathrm{lev}^l_{\bar{r}}(I).$$

Let $t_{\bar{x}} := I^{(j)}_{k,V'} \circ F(\bar{x})$. Then there exist $\epsilon > 0$ and $\delta > 0$ such that $\bar{r} < \bar{r} + \epsilon < \bar{r} + \epsilon + \delta < t_{\bar{x}}$ because $\bar{x} \notin \mathrm{lev}^l_r(I)$. Let $t_\beta := I^{(j)}_{k,V'} \circ F(x_\beta)$ for any $\beta \in J$. Then

$t_\beta \leq \bar{r}$. Therefore, $t_\beta \leq \bar{r} < \bar{r}+\epsilon < \bar{r}+\epsilon+\delta < t_{\bar{x}}$ and so we obtain

$$F(\bar{x}) \not\leq_C^{(j)} (\bar{r}+\epsilon+\delta)k + V' \quad \text{and} \quad F(x_\beta) \leq_C^{(j)} (\bar{r}+\epsilon)k + V'. \tag{3.1}$$

(a) we consider the case of $j = 5$. By (3.1) and the definition of type (5) set-relation, we have

$$F(\bar{x}) \not\subset (\bar{r}+\epsilon+\delta)k + V' - C \quad \text{and} \quad F(x_\beta) \subset (\bar{r}+\epsilon)k + V' - C. \tag{3.2}$$

Since C is a convex cone, $k \in \operatorname{int} C$ and $\delta > 0$,

$$\operatorname{cl}((\bar{r}+\epsilon)k + V' - C) \subset (\bar{r}+\epsilon+\delta)k + V' - C,$$

and then

$$\{(\bar{r}+\epsilon+\delta)k + V' - C\}^c \subset \{\operatorname{cl}((\bar{r}+\epsilon)k + V' - C)\}^c. \tag{3.3}$$

Hence, by (3.2) and (3.3), we have

$$F(\bar{x}) \cap (\{\operatorname{cl}((\bar{r}+\epsilon)k + V' - C)\}^c) \neq \emptyset,$$

and

$$F(x_\beta) \cap (\{\operatorname{cl}((\bar{r}+\epsilon)k + V' - C)\}^c) = \emptyset.$$

By Lemma 3.1, we obtain

$$\{\operatorname{cl}((\bar{r}+\epsilon)k + V' - C)\}^c = \{\operatorname{cl}((\bar{r}+\epsilon)k + V' - C)\}^c + C.$$

Consequently, we have

$$F(\bar{x}) \cap (\{\operatorname{cl}((\bar{r}+\epsilon)k + V' - C)\}^c) \neq \emptyset,$$

and

$$F(x_\beta) \cap (\{\operatorname{cl}((\bar{r}+\epsilon)k + V' - C)\}^c + C) = \emptyset.$$

This is a contradiction to the C-lower continuity of F on X. Consequently, $I^{(5)}_{k,V'} \circ F$ is lower semicontinuous on X.

(d) we consider the case of $j = 3$. By (3.1) and the definition of type (3) set-relation, we obtain

$$(\bar{r}+\epsilon+\delta)k + V' \not\subset F(\bar{x}) + C \quad \text{and} \quad (\bar{r}+\epsilon)k + V' \subset F(x_\beta) + C. \tag{3.4}$$

Assume that $F(x_\beta) \subset F(\bar{x}) - \delta k + C$, then we obtain $(\bar{r}+\epsilon)k + V' \subset F(x_\beta) + C \subset F(\bar{x}) - \delta k + C$, hence, $(\bar{r}+\epsilon+\delta)k + V' \subset F(\bar{x}) + C$. This is a contradiction to (3.4), and so we have

$$F(x_\beta) \not\subset F(\bar{x}) - \delta k + C. \tag{3.5}$$

Moreover, since C is a convex cone, $k \in \operatorname{int} C$ and $\delta > 0$,

$$F(\bar{x}) \subset F(\bar{x}) + C \subset \operatorname{int}(F(\bar{x}) - \delta k + C). \tag{3.6}$$

Hence, by (3.5) and (3.6), we have $F(\bar{x}) \subset \text{int}\,(F(\bar{x}) - \delta k + C)$ and $F(x_\beta) \not\subset \text{int}\,(F(\bar{x}) - \delta k + C)$. By Lemma 3.1, we obtain

$$\text{int}\,(F(\bar{x}) - \delta k + C) = \text{int}\,(F(\bar{x}) - \delta k + C) + C.$$

Consequently, we have

$$F(\bar{x}) \subset \text{int}\,(F(\bar{x}) - \delta k + C) \quad \text{and} \quad F(x_\beta) \not\subset \text{int}\,(F(\bar{x}) - \delta k + C) + C.$$

This is a contradiction to the C-upper continuity of F on X. Consequently, $I^{(3)}_{k,V'} \circ F$ is lower semicontinuous on X.

Second, we prove (b) and (c). For each $j = 3, 5$, we show that

$$\text{lev}^u_r(I) := \{x \in X | r \leq (I^{(j)}_{k,V'} \circ F)(x)\}$$

is closed for any $r \in \mathbb{R}$, that is, for any $\{x_\alpha\}_{\alpha \in J} \subset \text{lev}^u_r(I)$,

$$x_\alpha \to \bar{x} \Rightarrow \bar{x} \in \text{lev}^u_r(I),$$

where J is a directed set. Assume that there exist $\bar{r} \in \mathbb{R}$, $\{x_\beta\}_{\beta \in J} \subset \text{lev}^u_{\bar{r}}(I)$, and $\bar{x} \in X$ such that

$$x_\beta \to \bar{x} \quad \text{and} \quad \bar{x} \notin \text{lev}^u_{\bar{r}}(I).$$

Let $t_{\bar{x}} := I^{(j)}_{k,V'} \circ F(\bar{x})$. Then there exist $\epsilon > 0$ and $\delta > 0$ such that $t_{\bar{x}} < \bar{r} - \epsilon < \bar{r} - \epsilon + \delta < \bar{r}$ because $\bar{x} \notin \text{lev}^u_{\bar{r}}(I)$. Let $t_\beta := I^{(j)}_{k,V'} \circ F(x_\beta)$ for any $\beta \in J$. Then $\bar{r} \leq t_\beta$. Therefore, $t_{\bar{x}} < \bar{r} - \epsilon < \bar{r} - \epsilon + \delta < \bar{r} \leq t_\beta$ and so we obtain

$$F(\bar{x}) \leq^{(j)}_C (\bar{r} - \epsilon)k + V' \quad \text{and} \quad F(x_\beta) \not\leq^{(j)}_C (\bar{r} - \epsilon + \delta)k + V'. \tag{3.7}$$

(b) we consider the case of $j = 5$. By (3.7) and the definition of type (5) set-relation, we have

$$F(\bar{x}) \subset (\bar{r} - \epsilon)k + V' - C \quad \text{and} \quad F(x_\beta) \not\subset (\bar{r} - \epsilon + \delta)k + V' - C. \tag{3.8}$$

Since C is a convex cone, $k \in \text{int}\, C$ and $\delta > 0$,

$$(\bar{r} - \epsilon)k + V' - C \subset \text{int}\,((\bar{r} - \epsilon + \delta)k + V' - C). \tag{3.9}$$

Hence, by (3.8) and (3.9), we have

$$F(\bar{x}) \subset \text{int}\,((\bar{r} - \epsilon + \delta)k + V' - C) \quad \text{and} \quad F(x_\beta) \not\subset \text{int}\,((\bar{r} - \epsilon + \delta)k + V' - C).$$

By Lemma 3.1, we obtain

$$\text{int}\,((\bar{r} - \epsilon + \delta)k + V' - C) = \text{int}\,((\bar{r} - \epsilon + \delta)k + V' - C) - C.$$

Consequently, we have

$$F(\bar{x}) \subset \text{int}\,((\bar{r} - \epsilon + \delta)k + V' - C) \quad \text{and} \quad F(x_\beta) \not\subset \text{int}\,((\bar{r} - \epsilon + \delta)k + V' - C) - C.$$

This is a contradiction to the $(-C)$-upper continuity of F on X. Consequently, $I^{(5)}_{k,V'} \circ F$ is upper semicontinuous on X.

(c) we consider the case of $j = 3$. By (3.7) and the definition of type (3) set-relation, we obtain

$$(\bar{r} - \epsilon)k + V' \subset F(\bar{x}) + C \quad \text{and} \quad (\bar{r} - \epsilon + \delta)k + V' \not\subset F(x_\beta) + C. \tag{3.10}$$

Assume that $F(\bar{x}) \subset F(x_\beta) - \delta k + C$, then we obtain $(\bar{r} - \epsilon)k + V' \subset F(\bar{x}) + C \subset F(x_\beta) - \delta k + C$, hence, we have $(\bar{r} - \epsilon + \delta)k + V' \subset F(x_\beta) + C$. This is a contradiction to (3.10), and so we have

$$F(\bar{x}) \not\subset F(x_\beta) - \delta k + C. \tag{3.11}$$

Moreover, since C is a convex cone, $k \in \operatorname{int} C$ and $\delta > 0$,

$$F(x_\beta) \subset \operatorname{cl}(F(x_\beta) + C) \subset F(x_\beta) - \delta k + C. \tag{3.12}$$

Hence, by (3.11) and (3.12), we have $F(\bar{x}) \cap (\{\operatorname{cl}(F(x_\beta) + C)\}^c) \neq \emptyset$ and $F(x_\beta) \cap (\{\operatorname{cl}(F(x_\beta) + C)\}^c) = \emptyset$. By Lemma 3.1, we obtain

$$\{\operatorname{cl}(F(x_\beta) + C)\}^c = \{\operatorname{cl}(F(x_\beta) + C)\}^c - C.$$

Consequently, we have

$$F(\bar{x}) \cap (\{\operatorname{cl}(F(x_\beta) + C)\}^c) \neq \emptyset \quad \text{and} \quad F(x_\beta) \cap (\{\operatorname{cl}(F(x_\beta) + C)\}^c - C) = \emptyset.$$

This is a contradiction to the $(-C)$-lower continuity of F on X. Consequently, $I^{(3)}_{k,V'} \circ F$ is upper semicontinuous on X.

□

Theorem 3.4. *Let F be a set-valued map, C a convex cone in Y and $k \in \operatorname{int} C$. Then, the following statements hold.*

(i) *For each $j = 1, 2, 3$,*
  (a) *if F is $(-C)$-lower continuous on X, then $S^{(j)}_{k,V'} \circ F$ is upper semicontinuous on X,*
  (b) *if F is C-upper continuous on X, then $S^{(j)}_{k,V'} \circ F$ is lower semicontinuous on X.*

(ii) *For each $j = 4, 5, 6$,*
  (c) *if F is C-lower continuous on X, then $S^{(j)}_{k,V'} \circ F$ is lower semicontinuous on X,*
  (d) *if F is $(-C)$-upper continuous on X, then $S^{(j)}_{k,V'} \circ F$ is upper semicontinuous on X.*

Proof. By the same way as the proof of Theorem 3.3, the statements are proved. □

By Theorems 3.1–3.4, we summarize the inheritance properties on continuity and cone-continuity of parent set-valued maps via the unified types of scalarizing functions in Table 3.1. By symbolic interpretation, (semi-)continuity notions with prefixes C and $-C$ are inherited to the semicontinuity with "lower" and "upper," respectively.

TABLE 3.1. Inherited properties on semicontinuity of set-valued maps via scalarization.

F	$I^{(j)}_{k,V'} \circ F$		$S^{(j)}_{k,V'} \circ F$	
	$j = 1, 4, 5$	$j = 2, 3, 6$	$j = 4, 5, 6$	$j = 1, 2, 3$
l.c. on X	l.s.c. on X	u.s.c. on X	l.s.c. on X	u.s.c. on X
u.c. on X	u.s.c. on X	l.s.c. on X	u.s.c. on X	l.s.c. on X
C-l.c. on X	l.s.c. on X	$(*)$	l.s.c. on X	$(*)$
C-u.c. on X	$(*)$	l.s.c. on X	$(*)$	l.s.c. on X
$(-C)$-l.c. on X	$(*)$	u.s.c. on X	$(*)$	u.s.c. on X
$(-C)$-u.c. on X	u.s.c. on X	$(*)$	u.s.c. on X	$(*)$

Example 3.1. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$. We consider a set-valued map $F : X \to 2^Y$ defined by

$$F(x) := \begin{cases} \left[\begin{pmatrix} x \\ x \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix}\right] & (x \leq -1), \\ \left[\begin{pmatrix} x \\ x+2 \end{pmatrix}, \begin{pmatrix} x \\ 3 \end{pmatrix}\right] & (-1 < x < 1), \\ \left[\begin{pmatrix} x-1 \\ 0 \end{pmatrix}, \begin{pmatrix} x-1 \\ x \end{pmatrix}\right] & (1 \leq x), \end{cases}$$

where $[a, b] := \{c \in Y \mid a \leq_C c, c \leq_C b\}$. It is easy to check that F is C-upper continuous on X. Let $k = \binom{1}{1}$ and $V' = \left[\binom{0}{0}, \binom{1}{1}\right]$, and hence we have

$$(I^{(3)}_{k,V'} \circ F)(x) = \begin{cases} x & (x \leq -1), \\ x+2 & (-1 < x < 1), \\ x-1 & (1 \leq x). \end{cases}$$

Hence $I^{(3)}_{k,V'} \circ F$ is lower semicontinuous on X.

Remark 3.1. If F is neither lower continuous on X nor upper continuous on X, we can not apply the results in [6] to the composite functions of F and each of the unified types of scalarizing functions. However, by Theorems 3.3 and 3.4, we get a clue to confirm cone-continuity of a parent set-valued map F by checking semicontinuity of the scalarizing functions.

Example 3.2. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$. We consider a set-valued map $G : X \to 2^Y$ defined by

$$G(x) := \begin{cases} \left[\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ x^2 \end{pmatrix}\right] & (x \leq -1), \\ \left[\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ x+3 \end{pmatrix}\right] & (-1 < x < 1), \\ \left[\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ 5x \end{pmatrix}\right] & (1 \leq x), \end{cases}$$

where $[a, b] := \{c \in Y \mid a \leq_C c, c \leq_C b\}$. It is easy to check that G is C-upper continuous on X. Let $k = \binom{1}{1}$ and $V' = \left[\binom{0}{0}, \binom{1}{1}\right]$, and then we have

$$(I^{(5)}_{k,V'} \circ G)(x) = \begin{cases} x^2 - 1 & (x \leq -1), \\ x + 2 & (-1 < x < 1), \\ 5x - 1 & (1 \leq x). \end{cases}$$

Hence $I^{(5)}_{k,V'} \circ G$ is neither lower semicontinuous nor upper semicontinuous on X.

Example 3.3. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$. We consider a set-valued map $H : X \to 2^Y$ defined by

$$H(x) := \begin{cases} \left[\begin{pmatrix} x-1 \\ 0 \end{pmatrix}, \begin{pmatrix} x-1 \\ -x \end{pmatrix}\right] & (x < -1), \\ \left[\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] & (-1 \leq x \leq 0), \\ \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x+2 \\ x+2 \end{pmatrix}\right] & (0 < x), \end{cases}$$

where $[a,b] := \{c \in Y \mid a \leq_C c, c \leq_C b\}$. It is easy to check that H is $(-C)$-lower continuous on X. Let $k = \binom{1}{1}$ and $V' = \left[\binom{0}{0}, \binom{1}{1}\right]$, and then we have

$$(S^{(5)}_{k,V'} \circ H)(x) = \begin{cases} x-2 & (x < -1), \\ -1 & (-1 \leq x \leq 0), \\ x+1 & (0 < x). \end{cases}$$

Hence $S^{(5)}_{k,V'} \circ H$ is neither lower semicontinuous nor upper semicontinuous on X.

Remark 3.2. Each cell with $(*)$ in Table 3.1 is undetermined on semicontinuity for the scalarizing functions. By Examples 3.2 and 3.3, $I^{(5)}_{k,V'} \circ G$ and $S^{(5)}_{k,V'} \circ H$ are neither lower semicontinuous on X nor upper semicontinuous on X.

Acknowledgements. The authors are grateful to the referees for their valuable comments and suggestions which have contributed to the final preparation of the paper.

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Received July 21, 2010
Revised August 12, 2010