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TRIPLE COVERINGS OF THE PROJECTIVE PLANE BRANCHED ALONG QUINTIC CURVES

TADASUKE YASUMURA

ABSTRACT. In this article, we characterize triple coverings over the projective plane $\mathbb{P}^2$ branched along quintic curves with some conditions. The main result is that such triple coverings are induced by projections $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ from certain points.

1. Introduction

In this article, all varieties are assumed to be defined over the field of complex numbers $\mathbb{C}$. Let X be a normal projective variety and Y a smooth projective variety. We say that a morphism $\pi : X \to Y$ is a covering over Y if π is a finite surjective morphism. We define the branch locus Δ_π, as the subset of Y:

$$\Delta_\pi := \{y \in Y \mid \pi \text{ is not locally isomorphic over } y\}.$$

Note that Δ_π is an algebraic subset of pure codimension 1 (see [12]). We denote the function fields of X and Y by $\mathbb{C}(X)$ and $\mathbb{C}(Y)$, respectively. A covering $\pi : X \to Y$ is called a Galois covering if the field extension $\mathbb{C}(X)/\mathbb{C}(Y)$ induced by π is Galois, while it is called a non-Galois covering if $\mathbb{C}(X)/\mathbb{C}(Y)$ is non-Galois. Let G be a finite group. If a covering $\pi : X \to Y$ is a Galois covering with $\mathrm{Gal}(\mathbb{C}(X)/\mathbb{C}(Y)) \simeq G$, then $\pi : X \to Y$ is called a G-covering. A covering $\pi : X \to Y$ is called a triple covering, if $\deg \pi = 3$. Let $\pi : X \to Y$ be a triple covering and y a point of Δ_π. We say that y is a total (resp. simple) branched point, if $\sharp\pi^{-1}(y) = 1$ (resp. $\sharp\pi^{-1}(y) = 2$). Let D be an irreducible component of Δ_π. We say that π is totally branched along D, if all points of D are total branched points, while it is simply branched along D, if there exists a non-empty Zariski open set U_D of D such that all points of U_D are simple branched points. We decompose Δ_π into $\Delta_\pi = D_T + D_S$ such that π is totally (resp. simply) branched along each irreducible component of D_T (resp. D_S).

Let $\pi : X \to \mathbb{P}^2$ be a non-Galois triple covering over the projective plane $\mathbb{P}^2$ with $\deg \Delta_\pi = 5$. Take a general line l on $\mathbb{P}^2$ and consider the covering $\pi^*l \to l$. By using Hurwitz's theorem, we infer that Δ_π satisfies either (i) $\deg D_S = 2$ and $\deg D_T = 3$ or (ii) $\deg D_S = 4$ and $\deg D_T = 1$. We say that π is of Type I (resp. Type II) if Δ_π

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Table 1: Possible list of $D_T + D_S$ (c.f. [10])

Δ_π	D_S	$D_S \cap D_T$	Δ_π	D_S	$D_S \cap D_T$
Δ_1	Q_1	(i)	Δ_{10}	Q_5	(ii)
Δ_2	Q_2		Δ_{11}	Q_6	(iii), a_3
Δ_3	Q_3		Δ_{12}	Q_{12}	
Δ_4	Q_4		Δ_{13}	Q_7	(iii), a_6
Δ_5	Q_5		Δ_{14}	Q_8	(v), a_4
Δ_6	Q_9		Δ_{15}	Q_{10}	(iv), $2a_3$
Δ_7	Q_1	(ii)	Δ_{16}	Q_{13}	
Δ_8	Q_2		Δ_{17}	Q_{11}	(v), a_7
Δ_9	Q_4		Δ_{18}	Q_{14}	(v), an ordinary quadruple point

Δ_π - types of Δ_π;
D_S - types of D_S (Q_i $(1 \leq i \leq 14)$ corresponds to those in Table 2 below);
$D_S \cap D_T$ - the relative position between D_S and D_T and singular points of D_S in $D_S \cap D_T$.

(i) D_T is a bitangent line of D_S at two distinct smooth points.
(ii) D_T is a tangent line of D_S at one smooth point with multiplicity 4.
(iii) D_T is tangent to D_S at one smooth point and passes through one singular point of D_S.
(iv) D_T passes through two distinct singular points of D_S.
(v) D_T meets D_S at just one singular point of D_S.

Table 2: The list of D_S (c.f. [10])

D_S	Irreducible components of D_S	Singularities of D_S
Q_1	an irreducible quartic	$2a_2$
Q_2	an irreducible quartic	$a_1 + 2a_2$
Q_3	an irreducible quartic	$3a_2$
Q_4	an irreducible quartic	a_5
Q_5	an irreducible quartic	e_6
Q_6	an irreducible quartic	$a_2 + a_3$
Q_7	an irreducible quartic	a_6
Q_8	an irreducible quartic	$a_2 + a_4$
Q_9	two irreducible conics	$a_1 + a_5$
Q_{10}	two irreducible conics	$2a_3$
Q_{11}	two irreducible conics	a_7
Q_{12}	a cuspidal cubic and a line	$a_1 + a_2 + a_3$
Q_{13}	an irreducible conic and two lines	$2a_3 + a_1$
Q_{14}	four lines	an ordinary quadruple point

is the case (i) (resp. (ii)). The purpose of this article is to give a characterization for non-Galois triple coverings of Type II. Note that there exists a normal surface $X' \subset X$ such that $\pi : X \to \mathbb{P}^2$ induces a covering $\pi' : X' \to \mathbb{C}^2$ over $\mathbb{C}^2$, where a covering over $\mathbb{C}^2$ means a proper surjective morphism.

Let $\mathcal{D}_{2p}$ be the dihedral group of order $2p$, where p is an odd prime number. In [10], Tokunaga studied $\mathcal{D}_{2p}$-covering branched along quintic curves and he obtain a possible list of branch loci of the non-Galois triple coverings of Type II (see Table 1). In this article, for the types of simple singular points of curves and surfaces, we use those in [1]. Note that we use small letters for curve-singularities to distinguish them from those of surfaces.

Our main result of this article is as follows:

Theorem 1.1. *Let $\pi : X \to \mathbb{P}^2$ be a triple covering of Type* II. *Then* (i) *X is isomorphic to a normal cubic surface in $\mathbb{P}^3$ and* (ii) *$\pi : X \to \mathbb{P}^2$ is a morphism induced by the projection from a point $p \in \mathbb{P}^3 \setminus X$.*

We can present the covering π in more concrete way as follows:

Theorem 1.2. *Let S be the cubic surface in $\mathbb{P}^3$ defined by $F = 0$ and $f_p : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ a projection from $p \in \mathbb{P}^3 \setminus S$. Put $\pi_p := f_p\,|_S : S \to \mathbb{P}^2$. Then F, Δ_{π_p} and p satisfy one of those in Table* 3 *below, up to projective equivalence.*

In Section 2, we summarize the canonical resolution of triple coverings over surfaces based on [8]. In Section 3, we prove Theorem 1.1. In Section 4, we prove Theorem 1.2.

2. Tan's canonical resolution

In this section, we summarize Tan's canonical resolution for triple covering. The reference of this section is [8].

Let $\pi : X \to Y$ be a triple covering over a smooth surface Y. In [8], Tan proved that, after a finite number of blowing-ups, the induced triple covering $\widetilde{\pi} : \widetilde{X} \to \widetilde{Y}$ has the smooth branch locus. Moreover $\widetilde{X}$ is a resolution of X (see [8, Theorem 4.1 and Section 6]). We denote the blowing-ups as follows:

$$\begin{array}{ccccccccccc}
X = \overline{X}_0 & \xleftarrow{\nu_1} & \overline{X}_1 & \xleftarrow{\nu_2} & \overline{X}_2 & \xleftarrow{\nu_3} & \cdots & \xleftarrow{\nu_{k-1}} & \overline{X}_{k-1} & \xleftarrow{\nu_k} & \overline{X}_k = \widetilde{X} \\
\downarrow{\scriptstyle \pi=\overline{\pi}_0} & & \downarrow{\scriptstyle \overline{\pi}_1} & & \downarrow{\scriptstyle \overline{\pi}_2} & & & & \downarrow{\scriptstyle \overline{\pi}_{k-1}} & & \downarrow{\scriptstyle \overline{\pi}_k=\widetilde{\pi}} \\
Y = \overline{Y}_0 & \xleftarrow{\mu_1} & \overline{Y}_1 & \xleftarrow{\mu_2} & \overline{Y}_2 & \xleftarrow{\mu_3} & \cdots & \xleftarrow{\mu_{k-1}} & \overline{Y}_{k-1} & \xleftarrow{\mu_k} & \overline{Y}_k = \widetilde{Y}\,.
\end{array}$$

Here $\mu_i : \overline{Y}_i \to \overline{Y}_{i-1}$ is the blowing-up of $\overline{Y}_{i-1}$ at a singular point p_{i-1} of $\Delta_{\overline{\pi}_{i-1}}$,

Table 3: The list of F, Δ_{π_p} and p

Sing S	F	Δ_{π_p}	p
A_1+2A_2	$WYZ+WX^2+X^3$	Δ_2	$[1:a:b:0]$, $ab\neq -1,0,3$
		Δ_5	$[1:a:b:0]$, $ab=-1$
		Δ_6	$[1:a:b:0]$, $ab=3$
		Δ_{12}	$[1:a:b:0]$, $ab=0$, $a+b\neq 0$
		Δ_{16}	$[1:0:0:0]$
A_1+A_5	$WXY+WZ^2+X^3$	Δ_8	$[1:a:b:0]$, $a+b^2\neq 0$
		Δ_{10}	$[1:a:b:0]$, $a+b^2=0$
$2A_2$	$W^3+kWX^2+WYZ+X^3$ $(4k^3+27\neq 0)$	Δ_1	$[1:a:b:0]$, $ab\neq 0$, $a^4b^4-6a^2b^2k^2-8abk^3-108ab-3k^4\neq 0$
		Δ_4	$[1:a:b:0]$, $ab\neq 0$, $a^4b^4-6a^2b^2k^2-8abk^3-108ab-3k^4=0$
		Δ_{11}	$[1:a:b:0]$,$k\neq 0$, $ab=0$, $a+b\neq 0$
		Δ_{13}	$[1:a:b:0]$, $k=0$, $ab=0$, $a+b\neq 0$
		Δ_{15}	$[1:0:0:0]$, $k\neq 0$
$3A_2$	$WYZ+X^3$	Δ_3	$[1:0:a:b]$, $ab\neq 0$
			$[1:a:0:b]$, $ab\neq 0$
			$[1:a:b:0]$, $ab\neq 0$
A_5	$W^2Z+WXY+WZ^2+X^3$	Δ_7	$[1:a:b:0]$, $27+4a^3+12a^2b^2+12ab^4+4b^6\neq 0$
		Δ_9	$[1:a:b:0]$, $27+4a^3+12a^2b^2+12ab^4+4b^6=0$
E_6	$W^2Y+WZ^2+X^3$	Δ_{14}	$[1:a:b:0]$, $a\neq 0$
		Δ_{17}	$[1:0:b:0]$, $b\neq 0$
$\widetilde{E}_6$	$kW^3+lW^2X+WY^2+X^3$ $(4l^2+27k^3\neq 0)$	Δ_{18}	$p\in H_1\setminus H_2$ $H_1,\ H_2\in\mathcal{H}$, $H_1\neq H_2$

$a,b,k,l\in\mathbb{C}$;
we denote homogeneous coordinates of $\mathbb{P}^2$ by $[X:Y:Z:W]$;
$\mathcal{H}:=\{H_w,H_t,H_{su}\mid t^2+k=0,2u^3+k=0,3ks^2=u^2,s\neq 0,(s,t,u\in\mathbb{C})\}$ if $l=0$;
$\mathcal{H}:=\{H_w,H_{su}\mid 3lu^4-6us^2-1=0,6lus^2+9ks^2-3u^2+l=0,s\neq 0,(s,t,u\in\mathbb{C})\}$ if $l\neq 0$;
$(H_w:=V(W)\setminus V(X)$, $H_t:=V(Y+tW)\setminus V(X)$, $H_{su}:=V(X-sY-uW)\setminus V(3s^3Y+(1+3us^2)W))$;
$\widetilde{E}_6$ means the simple elliptic singularity of type $\widetilde{E}_6$ (for detail of the definition, see [7]);

Sing S - singularities of S;
F - normal forms of defining polynomials of S, up to projective equivalence;
Δ_{π_p} - types of Δ_{π_p};
p - loci of the center for the normal forms.

$\overline{\pi}_i : \overline{X}_i \to \overline{Y}_i$ is the normalization of $\overline{X}_{i-1} \times_{\mu_i} \overline{Y}_i$ and $\nu_i : \overline{X}_i \to \overline{X}_{i-1}$ is a morphism induced by $\overline{\pi}_i : \overline{X}_i \to \overline{Y}_i$. Let $\Delta_{\overline{\pi}_i} = D_{T,i} + D_{S,i}$ be the decomposition of $\Delta_{\overline{\pi}_i}$ such that $\overline{\pi}_i$ is totally branched along each irreducible component of $D_{T,i}$, while it is simply branched along those of $D_{S,i}$. We denote the multiplicity of a point p on a curve C by $m_p(C)$. Let m_i and n_i be integers defined as follows:

$$m_i := \left[\frac{m_{p_i}(D_{S,i})}{2} \right],$$

$$n_i := \begin{cases} m_{p_i}(D_{T,i}) - 1 & \text{if } E_{i+1} \subset \mathrm{Supp}(D_{T,i+1}) \\ m_{p_i}(D_{T,i}) & \text{if otherwise,} \end{cases}$$

where $[\alpha]$ denotes the greatest integer not exceeding α and E_i the exceptional divisor of μ_i. Then we have

$$\begin{aligned} D_{S,k} =& \mu^*(D_S) - 2\sum_{i=0}^{k-1} m_i \mathcal{E}_{i+1}, \\ D_{T,k} =& \mu^*(D_T) - \sum_{i=0}^{k-1} n_i \mathcal{E}_{i+1} \end{aligned}$$

and

$$K_{\widetilde{X}} = \mu^*(K_X) + \sum_{i=0}^{k-1} \mathcal{E}_{i+1},$$

where $\mu := \mu_1 \circ \mu_2 \circ \cdots \circ \mu_k$, $\mathcal{E}_i$ means the total transform of E_i and K_V means the canonical divisor of a smooth projective surface V. Moreover we have the following formulas:

Theorem 2.1 (c.f. Theorem 6.3 in [8]). *Under the above notation, we obtain*

$$\begin{aligned} \chi_{top}(\widetilde{X}) =& 3\chi_{top}(Y) + D_S^2 + D_S \cdot K_Y + 2D_T^2 + 2D_T \cdot K_Y \\ & - \sum_{i=0}^{k-1} 2(m_i - 1)(2m_i + 1) - \sum_{i=0}^{k-1} 2n_i(n_i - 1) + k, \\ K_{\widetilde{X}}^2 =& 3K_Y^2 + \frac{1}{2}D_S^2 + 2D_S \cdot K_Y + \frac{4}{3}D_T^2 + 4D_T \cdot K_Y \\ & - \sum_{i=0}^{k-1} 2(m_i - 1)^2 - \sum_{i=0}^{k-1} \frac{4}{3}n_i(n_i - 3) - k, \end{aligned}$$

where $\chi_{top}(V)$ means the topological Euler number of V.

Put

$$\begin{aligned} N(p,\pi) &:= \{p_i \mid \mu_1 \circ \mu_2 \circ \cdots \circ \mu_i(p_i) = p\}, \\ \delta(p,\pi) &:= \sum_{p_i \in N(p,\pi)} 2(m_i-1)(2m_i+1) + \sum_{p_i \in N(p,\pi)} 2n_i(n_i-1) - {}^{\sharp}N(p,\pi), \\ \kappa(p,\pi) &:= \sum_{p_i \in N(p,\pi)} 2(m_i-1)^2 + \sum_{p_i \in N(p,\pi)} \frac{4}{3} n_i(n_i-3) + {}^{\sharp}N(p,\pi). \end{aligned}$$

Let $\overline{X}$ be the minimal resolution of X. Then we obtain a birational morphism $\varphi : \widetilde{X} \to \overline{X}$. We denote by $\epsilon(p,\pi)$ the number of exceptional curves in $(\mu \circ \widetilde{\pi})^{-1}(p)$ contracted by φ. Since $\chi_{top}(\overline{X}) = \chi_{top}(\widetilde{X}) - \sum_{p \in \mathrm{Sing}(\Delta_\pi)} \epsilon(q,\pi)$ and $K_{\overline{X}}^2 = K_{\widetilde{X}}^2 + \sum_{p \in \mathrm{Sing}(\Delta_\pi)} \epsilon(p,\pi)$, we obtain

$$\begin{aligned} \chi_{top}(\overline{X}) =& 3\chi_{top}(Y) + D_S^2 + D_S \cdot K_Y + 2D_T^2 + 2D_T \cdot K_Y \\ & - \sum_{p \in \mathrm{Sing}(\Delta_\pi)} \delta(p,\pi) + \epsilon(p,\pi) \end{aligned}$$

and

$$\begin{aligned} K_{\overline{X}}^2 =& 3K_Y^2 + \frac{1}{2}D_S^2 + 2D_S \cdot K_Y + \frac{4}{3}D_T^2 + 4D_T \cdot K_Y \\ & - \sum_{p \in \mathrm{Sing}(\Delta_\pi)} \kappa(p,\pi) - \epsilon(p,\pi). \end{aligned}$$

In particular, for the case when $Y = \mathbb{P}^2$, $\deg D_T = 1$ and $\deg D_S = 4$, we obtain

$$\left\{ \begin{array}{ccc} \chi_{top}(\overline{X}) & = & 9 - \sum_{p \in \mathrm{Sing}(\Delta_\pi)} \delta(p,\pi) + \epsilon(p,\pi) \\ K_{\overline{X}}^2 & = & 1/3 - \sum_{p \in \mathrm{Sing}(\Delta_\pi)} \kappa(p,\pi) - \epsilon(p,\pi). \end{array} \right. \tag{1}$$

We end this section by giving the following three facts needed in the next section:

Lemma 2.1 (c.f. Theorem 4.1 in [8]). *Let $\pi : X \to Y$ be a triple covering. If $D_T \cap D_S \neq \emptyset$, then the intersection multiplicity of D_T and D_S at $p \in D_T \cap D_S$ is an even positive integer.*

Lemma 2.2 (c.f. Lemma 6.1 in [8]). *Let $\pi : X \to Y$ be a triple covering such that the branch locus is smooth. If D is an irreducible component of D_T, then the self-intersection number of D is 3-divisible.*

Lemma 2.3 (c.f. Theorem 4.1 in [8]). *Let $\pi : X \to Y$ be a triple covering. Assume that D_T has a singular point p of type a_1 with $p \notin D_S$. Let $\mu : Y' \to Y$ be the blowing-up at p, E the exceptional curve, D_T' the strict transform of D_T and $\pi' : X' \to Y'$ the induced triple covering. Then E satisfies one of the following.*

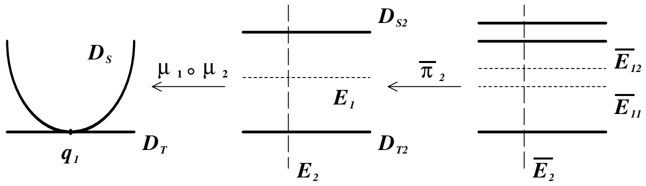

Figure 1: The case where $\pi_1 : S_1 \to \mathbb{P}^2$

(i) *E is not contained in $\Delta_{\pi'}$*

(ii) *π' is totally branched along E. Moreover, let $\mu' : Y'' \to Y'$ be the blowing-up at $q \in E \cap D'_T$ and $\pi'' : X'' \to Y''$ the induced triple covering. Then the exceptional curve of μ' is not contained in $\Delta_{\pi''}$.*

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We use the notation in the previous sections. First, we consider the following three facts needed later. See [5] for the proofs.

Lemma 3.1 (c.f. Lemma 3.1 (iii) in [5]). *Let $p \in D_S \setminus D_T$ be a total branched point. If D_S has only simple singularities as its singularities, then p is a singular point of type a_{3k-1} or e_6.*

Lemma 3.2 (c.f. Proposition 3.1 in [5]). *Let $p \in \mathrm{Sing}(D_S)$ be a simple singular and a simple branched point. Then $\delta(p, \pi) = -{}^\sharp N(p.\pi)$, $\kappa(p, \pi) = {}^\sharp N(p, \pi)$ and $\epsilon(p, \pi) = {}^\sharp N(p, \pi)$.*

Lemma 3.3 (c.f. Proposition 3.3 and Proposition 3.4 in [5]). *Let $p \in D_S$ be a simple singular and a total branched point. Suppose that p is a singular point of type* (i) *a_{6k-4},* (ii) *a_{6k-1} or* (iii) *e_6. Then, corresponding to* (i), (ii) *or* (iii), *we have $\delta(p, \pi) = 2k - 3$, $2k$ or 1, $\kappa(p, \pi) = 6k - 1$, $6k$ or 7, $\epsilon(p, \pi) = 4k$, $4k$ or 5.*

Now, we start to prove Theorem 1.1. Let $\pi_i : S_i \to \mathbb{P}^2$ be a triple covering such that the branch locus is of type Δ_i $(1 \le i \le 18)$. First, we prove the following lemma:

Lemma 3.4. *Let $\overline{S}_i$ be the minimal resolution of S_i. Then we obtain that $\chi_{top}(\overline{S}_i) = 9$ and $K^2_{\overline{S}_i} = 3$ if $1 \le i \le 17$, while $\chi_{top}(\overline{S}_{18}) = K^2_{\overline{S}_{18}} = 0$.*

Proof. We prove Lemma 3.4 only in the cases where $i = 1$ and 11, as the remaining cases can be proved similarly. First, we consider the case where $\pi_1 : S_1 \to \mathbb{P}^2$. Put $\Delta_{\pi_1} = D_T + D_S$ as in Introduction. We denote $D_T \cap D_S = \{q_1, q_2\}$ and $\mathrm{Sing}(D_S) = \{q_3, q_4\}$. Now, we compute $\delta(q_1, \pi_1)$, $\kappa(q_1, \pi_1)$ and $\epsilon(q_1, \pi_1)$. Let $\mu_1 \circ \mu_2 : \overline{Y}_2 \to \mathbb{P}^2$ be two times blowing-ups at q_1 as in Figure 1. Here, D_{T2}, D_{S2} and E_1 mean the strict

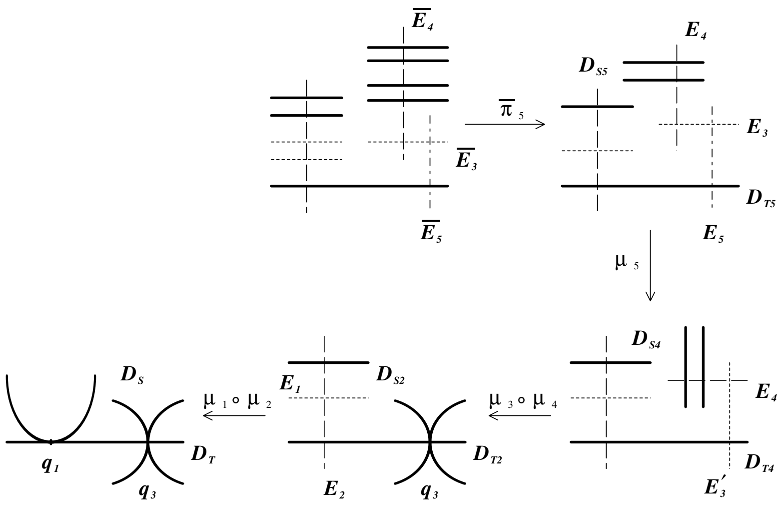

Figure 2: The case where $\pi_{11} : S_{11} \to \mathbb{P}^2$

transforms of D_T, D_S and the exceptional curve of μ_1. E_2 is the exceptional curve of μ_2. Since $m_{q_1}(D_S) = 1$, the induced triple covering $\overline{\pi}_2$ over $\overline{Y}_2$ is simply branched along E_1. By Lemma 2.1, $E_2 \not\subset \Delta_{\overline{\pi}_2}$. Hence, there exist no singular points of $\Delta_{\overline{\pi}_2}$ which are infinitely near points lying over q_1. We denote $\overline{\pi}_2^*(E_1) = 2\,\overline{E}_{11} + \overline{E}_{12}$ and $\overline{\pi}_2^*(E_2) = \overline{E}_2$ (see Figure 1). The self-intersection numbers of $\overline{E}_{11}$, $\overline{E}_{12}$ and $\overline{E}_2$ are -1, -2 and -3, respectively. Therefore, we obtain $\delta(q_1, \pi_1) = -4$, $\kappa(q_1, \pi_1) = -4/3$ and $\epsilon(q_1, \pi_1) = 1$. By the same way, we get $\delta(q_2, \pi_1) = -4$, $\kappa(q_2, \pi_1) = -4/3$ and $\epsilon(q_2, \pi_1) = 1$. By Lemma 3.1, $\sharp\pi_1^{-1}(q_i) = 1$ or 2 $(i = 3, 4)$. From Lemmas 3.2 and 3.3, we have $\delta(q_i, \pi_1) = -1$, $\kappa(q_i, \pi_1) = 5$ and $\epsilon(q_i, \pi_1) = 4$ for the case where $\sharp\pi_1^{-1}(q_i) = 1$, while $\delta(q_i, \pi) = -1$, $\kappa(q_i, \pi_1) = 1$ and $\epsilon(q_i, \pi_1) = 1$ for the case where $\sharp\pi_1^{-1}(q_i) = 2$. For the case where $\sharp\pi_1^{-1}(q_3) = \sharp\pi_1^{-1}(q_4) = 1$, by equations (1) in Section 2, we have $\chi_{top}(\overline{S}_1) = 9$ and $K^2_{\overline{S}_1} = 3$. Since we see $(\chi_{top}(\overline{S}_1) + K^2_{\overline{S}_1})/12 \notin \mathbb{Z}$ for other cases, these cases do not occur. Hence, we obtain Lemma 3.4 for the case where $\pi_1 : S_1 \to \mathbb{P}^2$.

Next, we consider the case where $\pi_{11} : S_{11} \to \mathbb{P}^2$. Again, put $\Delta_{\pi_{11}} = D_T + D_S$. Let $q_1 \in D_S$ such that D_T is tangent to D_S at q_1. We denote singular points of type a_2 and a_3 by q_2 and q_3, respectively. By the same way as the above, we have $\delta(q_1, \pi_{11}) = -4$, $\kappa(q_1, \pi_{11}) = -4/3$ and $\epsilon(q_1, \pi_{11}) = 1$. Let $\mu_1 \circ \mu_2 : \overline{Y}_2 \to \mathbb{P}^2$ be two times blowing-ups at q_1 as in Figure 2. Here, D_{T2}, D_{S2} and E_1 mean the strict transforms of D_T, D_S and the exceptional curve of μ_1. E_2 is the exceptional curve of μ_2. Now, we compute $\delta(q_3, \pi_{11})$, $\kappa(q_3, \pi_{11})$ and $\epsilon(q_3, \pi_{11})$. Let $\mu_3 \circ \mu_4 : \overline{Y}_4 \to \overline{Y}_2$ be two times blowing-ups at q_3 as in Figure 2. We denote D_{T4}, D_{S4} and E_3' by the strict transforms of D_{T2}, D_{S2} and the exceptional curve of μ_3, respectively. E_4 is the exceptional curve of μ_4. Since $(\mu_3 \circ \mu_4)^*(D_{S2}) = D_{S4} + 2E_3' + 4E_4$, the induced triple covering $\overline{\pi}_4$ over $\overline{Y}_4$ is not simply branched along E_3' and E_4. By Lemma 2.1, we see $E_4 \not\subset \Delta_{\overline{\pi}_4}$. Suppose that $E_3' \not\subset \Delta_{\overline{\pi}_4}$. Then after a finite number of blowing-ups $\widehat{\mu} : \widetilde{\widetilde{\mathbb{P}^2}} \to \overline{Y}_4$, we obtain the triple covering $\widetilde{\pi}_{11} : \widetilde{S}_{11} \to \widetilde{\widetilde{\mathbb{P}^2}}$ such that $\Delta_{\widetilde{\pi}_{11}}$ is smooth. Since the self-intersection number of E'_3 is -2, the strict transform of E'_3 on $\widetilde{Y}$ has the self-intersection number -2. This contradicts to Lemma 2.2. Hence, $\overline{\pi}_4$ is totally branched along E'_3. Let $\mu_5 : \overline{Y}_5 \to \overline{Y}_4$ be the blowing-up at $D_{T2} \cap E'_3$ and E_5 the exceptional curve. We denote the strict transforms of D_{T4} and E'_3 by D_{T5} and E_3. Suppose that the induced covering $\overline{\pi}_5$ over $\overline{Y}_5$ is totally branched along E_5. By Lemma 2.3, after two times blowing-ups μ' at $D_{T5} \cap E_5$ and $E_3 \cap E_5$, there exist no singular points of the branch locus of the triple covering induced by μ'. Let D'_T and E''_3 be the strict transforms of D_{T5} and E_3, respectively. Then the self-intersection numbers of D'_T and E''_3 are -4. By the same way as above E'_3, this contradicts to Lemma 2.2. We see that $E_5 \not\subset \Delta_{\overline{\pi}_5}$. We get the self-intersection numbers of E_3, E_4 and E_5 are -3, -2 and -1, respectively. By putting $\overline{\pi}_5^*(E_3) = 3\,\overline{E}_3$, $\overline{\pi}_5^*(E_4) = \overline{E}_4$ and $\overline{\pi}_5^*(E_5) = \overline{E}_5$, the self-intersection numbers of $\overline{E}_3$, $\overline{E}_4$ and $\overline{E}_5$ are -1, -3 and -3, respectively. Since $m_{q_3}(D_{T2}) = 1$ and $m_{q_3}(D_{S2}) = 2$, we obtain $\delta(q_3, \pi_{11}) = -1$, $\kappa(q_3, \pi_{11}) = -1/3$ and $\epsilon(q_3, \pi_{11}) = 1$. Finally, we compute $\chi_{top}(\overline{S}_{11})$ and $K^2_{\overline{S}_{11}}$. As Lemma 3.1, $\sharp\pi_{11}^{-1}(q_2) = 1$ or 2. By the same way as in the case where $\pi_1 : S_1 \to \mathbb{P}^2$, we get $\chi_{top}(\overline{S}_{11}) = 9$ and $K^2_{\overline{S}_{11}} = 3$ for the case where $\sharp\pi_{11}^{-1}(q_2) = 1$, while $(\chi_{top}(\overline{S}_{11}) + K^2_{\overline{S}_{11}})/12 = 4/3$ for the case where $\sharp\pi_{11}^{-1}(q_2) = 2$. Hence,we obtain Lemma 3.4 for the case where $\pi_{11} : S_{11} \to \mathbb{P}^2$. □

Remark 3.1. Let $\pi : X \to \mathbb{P}^2$ is a triple covering such that Δ_π is of type Δ_{18} and $\overline{X} \to X$ the minimal resolution of X. By our proof of Lemma 3.4, $\overline{X}$ is a ruled surface over an elliptic curve such that $\overline{X}$ has the section C_0 with $C_0^2 = -3$. By [7], X is a cone over an elliptic curve. Hence, X is isomorphic to a cubic surface.

Definition 3.1. Let C and S be a smooth projective curve and surface, respectively. If there exists a morphism $f : S \to C$ such that, for any $c \in C$ except finitely many points, $f^{-1}(c)$ is a smooth curve of genus 1, then we call S an elliptic surface. An elliptic surface is relatively minimal, if there exist no exceptional curves of the first kind in any fibers.

Lemma 3.5. *Let $\pi : X \to \mathbb{P}^2$ be a triple covering and $\gamma : \overline{X} \to X$ the minimal resolution of X. We assume that* (i) $K_{\overline{X}}{}^2 = 3$, (ii) $\chi_{top}(\overline{X}) = 9$ *and* (iii), *for any general lines $L \subset \mathbb{P}^2$, the degree of ramification divisor of the induced triple covering of L by π is* 6 (*see* [4, p.301. *Ch.* IV. *Section* 2] *for the definition of the ramification divisor*). *Then*

$$-K_{\overline{X}} \sim (\pi \circ \gamma)^* l,$$

where l is a line on $\mathbb{P}^2$.

Proof. Let x be a point in $\mathbb{P}^2 \setminus \Delta_\pi$. Note that $\sharp\pi^{-1}(x) = 3$. Let $\gamma_x : \overline{X}_x \to \overline{X}$ be the blowing-up at $\pi^{-1}(x)$ and E_i $(i = 1, 2, 3)$ the exceptional curves. Put $\phi := \pi \circ \gamma \circ \gamma_x$. Let Λ_x be a pencil of lines passing through x. For a general $L \in \Lambda_x$, by Hurwitz's theorem, $\phi^* L$ is a curve of genus 1. Hence we obtain an elliptic fibration $f : \overline{X}_x \to \mathbb{P}^1$

induced by Λ_x. Note that $K_{\overline{X}_x}^2 = 0$, $\chi_{top}(\overline{X}_x) = 12$ and $\overline{X}_x$ has a section. $\overline{X}_x$ is a relatively minimal elliptic surface. By the canonical bundle formula, we obtain

$$K_{\overline{X}_x} = -F_f,$$

where F_f is a fiber of $f : \overline{X}_x \to \mathbb{P}^1$. Let l be a line l in Λ_x. Then

$$\begin{array}{rcl} \phi^* l & \sim & F_f + E_1 + E_2 + E_3 \\ & \sim & -K_{\overline{X}_x} + E_1 + E_2 + E_3 \\ & \sim & -\gamma_x^* K_{\overline{X}}. \end{array}$$

Hence, we obtain $(\pi \circ \gamma)^* l \sim -K_{\overline{X}}$. □

Proposition 3.1. *Under the assumption of Lemma* 3.5, $|-K_{\overline{X}}|$ *induces a morphism* $\varphi_{|-K_{\overline{X}}|} : \overline{X} \to \mathbb{P}^3$ *such that* $\overline{X}$ *is birationally equivalent to its image* $\operatorname{Im} \varphi_{|-K_{\overline{X}}|}$ *and that* $\operatorname{Im} \varphi_{|-K_{\overline{X}}|}$ *is a normal cubic surface whose singular points are simple.*

Proof. We use the same notation as in Lemma 3.5. By Lemma 3.5, $-K_{\overline{X}}$ is a nef and big divisor. By Riemann-Roch theorem and Kawamata-Viehweg vanishing theorem ([6], [11]), $h^0(\overline{X}, -K_{\overline{X}}) = 4$. Since $\overline{X}$ is a rational surface, by [3, p.63 Theorem 1 and p.64 Theorem 2], we obtain Proposition 3.1. □

Proposition 3.2. *Under the assumption of Lemma* 3.5, X *is isomorphic to* $\operatorname{Im} \varphi_{|-K_{\overline{X}}|}$ *and* $\pi : X \to \mathbb{P}^2$ *is induced by a projection* $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ *from a point in* $\mathbb{P}^3 \setminus \operatorname{Im} \varphi_{|-K_{\overline{X}}|}$.

Proof. We keep the notation above. Let $[\xi_0 : \xi_1 : \xi_2]$ be homogeneous coordinates of $\mathbb{P}^2$ and l_i $(i = 0, 1, 2)$ lines defined by $\xi_i = 0$ $(i = 0, 1, 2)$, respectively. Since $-K_{\overline{X}} \sim (\gamma \circ \pi)^* l$, there exists a basis $\{\varphi_0, \varphi_1, \varphi_2, \varphi_3\}$ of $H^0(\overline{X}, -K_{\overline{X}})^\vee$ such that φ_i $(i = 0, 1, 2)$ correspond to ξ_i $(i = 0, 1, 2)$, respectively. By taking suitable coordinates of $\mathbb{P}^3$, we have

$$\begin{array}{rccc} \varphi_{|-K_{\overline{X}}|} : & \overline{X} & \to & \mathbb{P}^3 \\ & p & \mapsto & [\varphi_0(p) : \varphi_1(p) : \varphi_2(p) : \varphi_3(p)]. \end{array}$$

By denoting the projection from $[0 : 0 : 0 : 1]$ by $\Pr : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$, the composition $\Pr \circ \varphi_{|-K_{\overline{X}}|}$ is a morphism as $l_0 \cap l_1 \cap l_2 = \emptyset$. Let S_X be the image of $\varphi_{|-K_{\overline{X}}|}$. Now, we obtain the following commutative diagram:

$$\begin{tikzcd} & X \arrow[dr, "\pi"] & \\ \overline{X} \arrow[ur, "\gamma"] \arrow[dr, "\varphi_{|-K_{\overline{X}}|}"'] & & \mathbb{P}^2. \\ & S_X \arrow[ur, "\Pr|_{S_X}"'] & \end{tikzcd}$$

Put $g_1 := \gamma \circ \varphi^{-1}_{|-K_{\overline{X}}|}$ and $g_2 := \varphi_{|-K_{\overline{X}}|} \circ \gamma^{-1}$. Then g_1 and g_2 are birational maps. Suppose that g_1 is not a morphism. Then there exists a fundamental point $s \in S_X$ of g_1. By Zariski's main theorem [4, Ch. V, Theorem 5.2], $g_1(s)$ is a curve on X. We obtain $g_1(s) \cdot \pi^* l \neq 0$ for a line l on $\mathbb{P}^2$. On the other hand, since $\varphi_{|-K_{\overline{X}}|}(\gamma^*(g_1(s))) = s$ and $-K_{\overline{X}} \sim (\mathrm{Pr}\ |_{S_X} \circ \varphi_{|-K_{\overline{X}}|})^* l$, we get $-K_{\overline{X}}.\gamma^*(g_1(s)) = 0$. This is a contradiction. Hence, g_1 is a morphism. By the same way, g_2 is also a morphism. Thus we obtain that X is isomorphic to S_X. □

Thus we complete the proof of Theorem 1.1.

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We keep the notation as before. Let S be a normal cubic surface and $\pi_p := f_p\ |_S$, where $f_p : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ means a projection from $p \in \mathbb{P}^3 \setminus S$. First, we consider the following lemma:

Lemma 4.1. (i) *Δ_{π_p} satisfies one of the following:*

(i-1) *Δ_{π_p} is a sextic curve. π_p is simply branched along each irreducible component of Δ_{π_p}.*

(i-2) *Δ_{π_p} consists of a line L and a quartic curve Q such that π_p is totally branched along L, while it is simply branched along each irreducible component of Q.*

(i-3) *Δ_{π_p} consists of two conics, C_1 and C_2, such that π_p is totally branched along each irreducible component of C_1, while it is simply branched along those of C_2.*

(i-4) *Δ_{π_p} is a cubic curve. π_p is a cyclic triple covering. Thus π_p totally branched along each irreducible component of Δ_{π_p}.*

(ii) *There exists a line $L_T \subset \Delta_{\pi_p}$ such that π_P is totally branched along L_T, if and only if there exists a plane $H_T \subset \mathbb{P}^3$ such that $p \in H_T \setminus S$ and $S \cap H_T$ is a line.*

Proof. (i) We prove only the case of (i-1), as the remaining cases can be proved similarly. Assume that there exists an irreducible component D of Δ_{π_p} such that π_p is totally branched along D. Take a general line l on $\mathbb{P}^2$. Let R be the ramification divisor of the induced triple covering $\pi_p^* l \to l$ over l by π_p. Then we obtain

$$\deg R > 6.$$

On the other hand, let $H \subset \mathbb{P}^3$ be the Zariski closure of $f_p^{-1}(l)$. By Bertini's theorem, $S \cap H = f_p^* l$ is a non-singular cubic curve on H. By Hurwitz's theorem, we obtain

$$\deg R = 6.$$

This is a contradiction.

(ii) Suppose that π_p is totally branched along a line L_T. Then the Zariski closure of $f_p^{-1}(L_T)$ is H_T.

Conversely suppose that there exists such a plane H_T. Then, by putting $L_T := f_p(H_T)$, we obtain (ii) of Lemma 4.1. $\square$

By Lemma 4.1, if $\pi_p : S \to \mathbb{P}^2$ is totally branched along a line, then we obtain $3 \leq \deg \Delta_{\pi_p} \leq 5$. In particular, π_p with $\deg \Delta_{\pi_p} = 3$ is a cyclic triple covering branched along a reduced cubic with a line component, and as such, it must be one of the following possibilities:

Table 4: $\deg \Delta_{\pi_p} = 3$

Δ_{π_p}	Irreducible components	Singularities of Δ_{π_p}
Δ_{C1}	an irreducible conic and a line	$2a_1$
Δ_{C2}	an irreducible conic and a line	a_3
Δ_{C3}	three lines	$3a_1$
Δ_{C4}	three lines	an ordinary triple point

In [9], Tokunaga classified $\mathcal{D}_6$-coverings such that the degrees of branch loci ≤ 4. By [9], we obtain Table 5 for Δ_{π_p} with degree 4 such that π_p is totally branched along a line.

Table 5: $\deg \Delta_{\pi_p} = 4$ (c.f. [9])

Δ_{π_p}	D_S	D_T	Singularities of Δ_{π_p}
Δ_{Q1}	an irreducible conic	two lines	$a_1 + 2a_3$
Δ_{Q2}	two lines	two lines	an ordinary quadruple point

D_S - irreducible components of D_S;
D_T - irreducible components of D_T.

Remark 4.1. By the same way as in Section 3 (resp. Remark 3.1), for each triple covering $\pi : X \to \mathbb{P}^2$ such that Δ_π is of type Δ_{Cj} $(1 \leq j \leq 3)$ or Δ_{Q1} (resp. Δ_{C4} or Δ_{Q2}), X is isomorphic to a cubic surface. So, by characterizing triple coverings such that the branch loci are of type Δ_i $(1 \leq i \leq 18)$, Δ_{Cj} $(1 \leq j \leq 4)$ or Δ_{Qk} $(k = 1, 2)$, we obtain Table 3.

Lemma 4.2. *Let $\pi_p : S \to \mathbb{P}^2$ be a triple covering such that the branch locus is of either type Δ_i $(1 \leq i \leq 18)$, Δ_{Cj} $(1 \leq j \leq 4)$ or Δ_{Qk} $(k = 1, 2)$. The H_T is the plane as in Lemma* 4.1. *Then Δ_{π_p},* $\mathrm{Sing}(S)$ *and* $\mathrm{Sing}(S) \cap H_T$ *fall into one of them in Table* 6 *below. In particular, if Δ_{π_p} is one of such types, then the* *configuration of singularities of S is either A_1+2A_2, A_1+A_5, $2A_2$, $3A_2$, A_5, E_6 or $\widetilde{E}_6$.* $\mathrm{Sing}(S)\cap H_T\neq\emptyset$.

Table 6: Singularities of S

Δ_{π_p}	$\mathrm{Sing}(S)$	$\mathrm{Sing}(S)\cap H_T$	Δ_{π_p}	$\mathrm{Sing}(S)$	$\mathrm{Sing}(S)\cap H_T$
Δ_1	$2A_2$	$2A_2$	Δ_{13}	$2A_2$	$2A_2$
Δ_2	A_1+2A_2	$2A_2$	Δ_{14}	E_6	E_6
Δ_3	$3A_2$	$2A_2$	Δ_{15}	$2A_2$	$2A_2$
Δ_4	$2A_2$	$2A_2$	Δ_{16}	A_1+2A_2	$2A_2$
Δ_5	A_1+2A_2	$2A_2$	Δ_{17}	E_6	E_6
Δ_6	A_1+2A_2	$2A_2$	Δ_{18}	$\widetilde{E}_6$	$\widetilde{E}_6$
Δ_7	A_5	A_5	Δ_{C1}	$2A_2$	$2A_2$
Δ_8	A_1+A_5	A_5	Δ_{C2}	E_6	E_6
Δ_9	A_5	A_5	Δ_{C3}	$3A_2$	$2A_2$
Δ_{10}	A_1+A_5	A_5	Δ_{C4}	$\widetilde{E}_6$	$\widetilde{E}_6$
Δ_{11}	$2A_2$	$2A_2$	Δ_{Q1}	$3A_2$	$2A_2$
Δ_{12}	A_1+2A_2	$2A_2$	Δ_{Q2}	$\widetilde{E}_6$	$\widetilde{E}_6$

Proof. By our proof of Lemma 3.4, we obtain Lemma 4.2. □

Lemma 4.3. *Keep the notation in Lemma* 4.2. *Assume that Δ_{π_p} is not of type Δ_{18}, Δ_{C4} or Δ_{Q2}. Let $q\in\mathrm{Sing}(S)\cap H_T$ and let $l_q\subset\mathbb{P}^2\setminus\Delta_{\pi_p}$ be a line passing through $q':=\pi_p(q)$. Here, H_T means a plane in $\mathbb{P}^3$ as in Lemma* 4.1. *We denote the minimal resolution of S by $\gamma:\overline{S}\to S$. Let $\mathcal{L}_q$ be the strict transform of $\pi_p^*l_q$ by γ, and E_q the exceptional divisor of γ. Put $\Delta_{\pi_p}=D_T+D_S$ as in Introduction.*

(i) *If q' is a smooth point of D_S, then $\sharp\mathcal{L}_q\cap Z_q=2$.*

(ii) *If $q'\in\mathrm{Sing}(D_S)\cup\mathrm{Sing}(D_T)$, then $\sharp\mathcal{L}_q\cap Z_q\neq 2$.*

Proof. Our proof of Lemma 4.3 is case-by-case. We prove only a special case of (i), as the remaining cases of (i) and each case of (ii) are proved similarly.

Suppose that Δ_{π_p} is of type Δ_1. Consider a small neighborhood U of q' with $D_S\cap l_q\cap U=\{q'\}$. By our proof of Lemma 3.4, we obtain Lemma 4.3 for the case of Δ_1 (see Figure 3, where we use the same notation as in Lemma 3.4). □

Lemma 4.4. *We keep the notation as in Lemmas* 4.2 *and* 4.3. *Assume that the tangent cone of q consists of H_T and other plane H_q. Then $\pi_p(q)$ is a smooth point of D_S if $p\notin H_T\cap H_q$, while it is a singular point of D_S or D_T if $p\in H_T\cap H_q$.*

Proof. Let V be a plane with $p,q\in V$ and $V\neq H_T$. The curve $S\cap V$ on V has a singular point at q. First, we suppose $p\notin H_T\cap H_q$. Then $V\neq H_q$. Two distinct lines $V\cap H_T$ and $V\cap H_q$ on V meet $S\cap V$ at q with multiplicity 3, while other lines

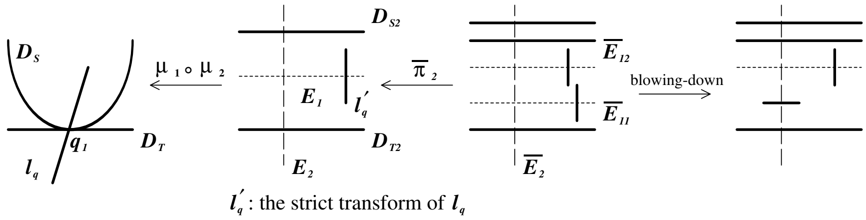

Figure 3: The case where $\pi_1 : X_1 \to \mathbb{P}^2$

through q not contained in S meet $S \cap V$ at q with multiplicity 2. So we obtain an a_1 singularity at q. By Lemma 4.3, $\pi_p(q)$ is a smooth point of D_S.

Next, we suppose $p \in H_T \cap H_q$. Put $V = H_q$. For any lines on V not contained in S, if $q \in L$, then L meets $S \cap V$ at q with multiplicity 3. So $S \cap V = 3l_1$, $2l_1 + l_2$ and $l_1 + l_2 + l_3$, where l_i $(i = 1, 2, 3)$ meen lines through q on H_q. If $S \cap V = 3l_1$, then $\pi_p(q) \in \mathrm{Sing}(D_T)$ as π_p is totally branched along $f_p(V)$.

Consider the case where $S \cap V = 2l_1 + l_2$. We see that π_p is simply blanched along $f_p(V)$, and that $f_p(V)$ meets $L_T = f_p(H_t)$ at $\pi_p(q)$, transversely. By Lemma 2.1 and our assumption, $\pi_p(q) \in \mathrm{Sing}(D_S)$.

Consider the case where $S \cap V = l_1 + l_2 + l_3$. By Lemma 4.3, we obtain $\pi_p(q) \in \mathrm{Sing}(D_S) \cup \mathrm{Sing}(D_T)$.

Hence, we obtain Lemma 4.4. □

In [2], Bruce and Wall classified singular cubic surfaces in $\mathbb{P}^3$ in terms of their singularities. By threr results, we can obtain the normal forms of the cubic surfaces, up to projective equivalence.

We now prove Theorem 1.2. We prove only for the cases where $\mathrm{Sing}(S) = A_1 + 2A_2$, $2A_2$ and $\widetilde{E}_6$, as the remaining cases can be proved similarly. Keep the notaition as before. Let Λ be the set of lines in $\mathbb{P}^3 \setminus H_T$ passing through p and meeting S at just one point. First, we consider the case where $\mathrm{Sing}(S) = A_1 + 2A_2$. In this case, by Table 6, Δ_{π_p} is of either type Δ_2, Δ_5, Δ_6, Δ_{12} or Δ_{16}. We denote two singular points of type A_2 by q_1 and q_2. By [2], the tangent cone of q_i consists of H_T and another plane H_{q_i}. Let q_3 be the singular point of type A_2 and C the tangent cone at q_3. On H_T, the relative position of H_{q_1}, H_{q_2}, C and S is given as in Figure 4 below. By Lemma 4.4, we obtain that Δ_{π_p} is of type Δ_{16} if $p = H_T \cap H_{q_1} \cap H_{q_2}$, while it is of type Δ_{12} if p is contained in either $H_T \cap H_{q_1}$ or $H_T \cap H_{q_2}$. Consider the point $\pi_p(q_3)$. By our proof of Lemma 3.4, $\pi_p(q)$ is a total (resp. simple) branched point for the case where Δ_{π_p} is of type Δ_5 (resp. Δ_2 or Δ_6). If $p \in C$, then Δ_{π_p} is of type Δ_5. Suppose that $p \in H_T \setminus (H_{q_1} \cup H_{q_2} \cup C \cup S)$. By our proof of Lemma 3.4, we obtain that Δ_{π_p} is of type Δ_2 (resp. Δ_6) if $\sharp\Lambda = 2$ (resp. 1) as the number of total branched point not contained in D_T is 2 (resp. 1) for the type Δ_2 (resp. Δ_6). Hence we obtain Theorem 1.2 for the case where $\mathrm{Sing}(S) = A_1 + 2A_2$.

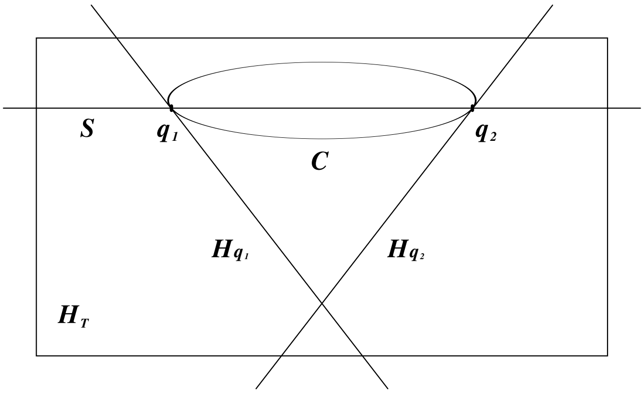

Figure 4: The case where $\mathrm{Sing}(S) = A_1 + 2A_2$

Next, we consider the case where $\mathrm{Sing}(S) = 2A_2$. In this case, Δ_{π_p} is of either type Δ_1, Δ_4, Δ_{11}, Δ_{13}, Δ_{15} or Δ_{C1}. Put $\mathrm{Sing}(S) = \{q_1, q_2\}$. Again, the tangent cone of q_i consists of H_T and another plane H_{q_i}. By [2], put

$$F := W^3 + kWX^2 + WYZ + X^3 \quad (4k^3 + 27 \neq 0).$$

On H_T, the relative position of H_{q_1}, H_{q_2} and S is given as in Figure 5. Suppose

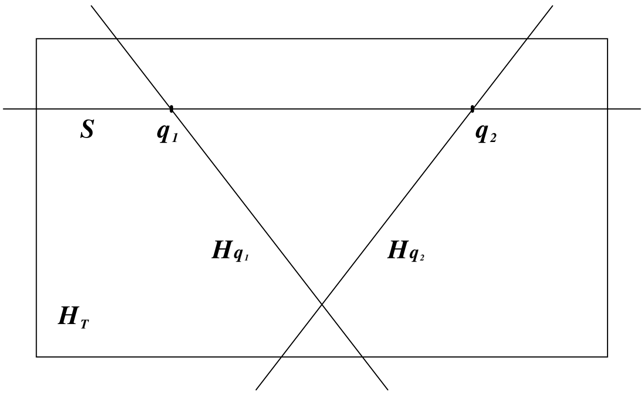

Figure 5: The case where $\mathrm{Sing}(S) = 2A_2$

that $p = H_T \cap H_{q_1} \cap H_{q_2}$. By Lemma 4.4, Δ_{π_p} is of type Δ_{15} or Δ_{C1}. We denote by n_T the number of total branched point not contained in $L_T = f_p(H_T)$. For the type Δ_{15} (resp. Δ_{C1}), we have $n_T = 0$ (resp. ∞). Since $^\sharp\Lambda = 0$ (resp. ∞) if $k \neq 0$ (resp. $k = 0$), we obtain Δ_{15} (resp. Δ_{C1}). Suppose that p is contained in either $H_T \cap H_{q_1}$ or $H_T \cap H_{q_2}$. Then Δ_{π_p} is of either type Δ_{11} or Δ_{13}. By the same way as in the cases of Δ_{15} and Δ_{C1}, we obtain Δ_{11} (resp. Δ_{13}) if $k \neq 0$ (resp. $k = 0$) as $^\sharp\Lambda = 1$

(resp. 0). Assume that $p \notin H_{q_1} \cup H_{q_2}$. Then Δ_{π_p} is of either type Δ_1 or Δ_4. For all k, we have Δ_1 (resp. Δ_4) if $\sharp\Lambda = 2$ (resp. 1) as $n_T = 2$ (resp. 1). Hence we obtain Theorem 1.2 for the case where $\mathrm{Sing}(S) = 2A_2$.

Finally, we consider the case where $\mathrm{Sing}(S) = \widetilde{E}_6$. In this case, Δ_{π_p} is of either type Δ_{18}, Δ_{C4} or Δ_{Q2}. Put $\Delta_{\pi_p} = D_T + D_S$ as in Introduction. D_T consists of one (resp. two, three) line(s) if Δ_{π_p} is of type Δ_{18} (resp. Δ_{Q2}, Δ_{C4}). Let

$$\mathcal{H} := \{H \subset \mathbb{P}^3 \mid H \text{ is a plane such that } H \cap S \text{ is a line}\}.$$

For each $H \in \mathcal{H}$, by (ii) of Lemma 4.1, π_p is totally branched along $f_p(H)$ if $p \in H \setminus S$. Take $H \in \mathcal{H}$ and $p \in H \setminus S$ and put

$$n_{H,p} := \sharp\{H' \in \mathcal{H} \mid p \in H'\}.$$

Considering irreducible components of D_T, we obtain that Δ_{π_p} is of type Δ_{18} (resp. Δ_{Q2}, Δ_{C4}) if $n_{H,p} = 1$ (resp. 2, 3). Hence we obtain Theorem 1.2 for the case where $\mathrm{Sing}(S) = \widetilde{E}_6$.

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(Tadasuke Yasumura) Department of Mathematics and Information Sciences, Tokyo Metropolitan University, 1-1 Minamiohsawa, Hachioji 192–0397, Tokyo, Japan

E-mail address: `yasumura-tadasuke@ed.tmu.ac.jp`

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