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ON A CONJECTURE RELATED TO FURUTA-TYPE INEQUALITIES WITH NEGATIVE POWERS

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ABSTRACT. We discuss an open problem on Furuta-type inequality in the domain called "mysterious delta zone." We show a theorem and a counterexample related to the best possibility of the inequality.

1. INTRODUCTION

A capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$ and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. The following Theorem F is an extension of the celebrated Löwner-Heinz theorem: *$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$.*

Theorem F (Furuta inequality [5]).

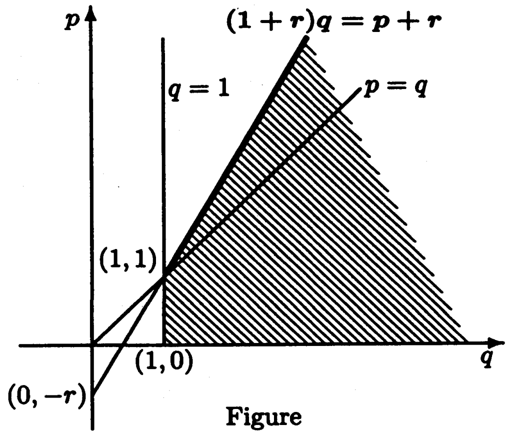
If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $\quad (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $\quad (A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.



Figure

We remark that Theorem F yields Löwner-Heinz theorem when we put $r = 0$ in (i) or (ii) stated above. Alternative proofs of Theorem F are given in [1][12] and also an elementary one-page proof in [6]. It is shown in [14] that the domain drawn for p, q and r in the Figure is best possible one for Theorem F.

It is known that (i) and (ii) in Theorem F remain valid for some negative numbers p, q and r in case A and B are invertible. By a simple observation, the problem to find real numbers p, q and r for which (i) or (ii) holds is reduced to the case $p \geq 0, q > 0$ and $r \in \mathbb{R}$ for (ii). Here we put $r = -t \leq 0$ and q minimum in (ii), then the following results are known.

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Theorem A ([2][13][15][16]). *If $A \geq B \geq 0$ with $A > 0$, then the following inequalities hold:*

(I) $A^{1-t} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{1-t}{p-t}}$ *for* $1 \geq p > t \geq 0$ *with* $p \geq \frac{1}{2}$.

(II) $A^{-t} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{-t}{p-t}}$ *for* $1 \geq t > p \geq 0$ *with* $\frac{1}{2} \geq p$.

(III) $A^{2p-t} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{2p-t}{p-t}}$ *for* $\frac{1}{2} \geq p > t \geq 0$.

(IV) $A^{2p-1-t} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{2p-1-t}{p-t}}$ *for* $1 \geq t > p \geq \frac{1}{2}$.

Extensions of Theorem A are shown in [3][4][9] and [11]. Yoshino [16] initiated an attempt to extend the domain in which the form of Theorem F holds. Afterwards, the domain given by him was enlarged to (I) by Fujii, Kamei and Furuta [2]. Kamei [13] gave simplified proofs of (I) and (III). Tanahashi [15] showed all the inequalities in Theorem A and proved that the outside exponents of (I),(II) and (IV) are best possible. But it has not been proved yet whether the outside exponents of (III) is best possible or not. The domain of (III) is called "mysterious delta zone" and the following problem has been considered.

Problem. *If $A \geq B \geq 0$ with $A > 0$, then does the inequality*

$$A^{1-t} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{1-t}{p-t}} \tag{1.1}$$

hold for all $\frac{1}{2} > p > t > 0$?

Furuta [8] expected that this Problem had a negative answer. It is well known that Tanahashi has obtained the following result related to this Problem by his elaborate and ingenious technique.

Theorem B. *For $\frac{1}{2} > p > t > 0$, there exist $A, B \in B(\mathbb{R}^2)$ such that $A \geq B > 0$ and*

$$B \not\geq A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{1-t}{p-t}} A^{\frac{t}{2}}.$$

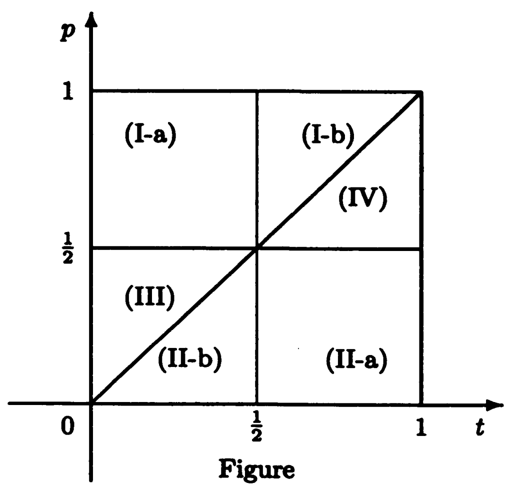
Very recently, Jiang [10] has given a concrete counterexample for $p = \frac{5}{12}$ and $t = \frac{1}{8}$ in Theorem B.

In this paper, we discuss some equivalence relations among the inequalities in Theorem A. Then we show a result and a counterexample related to the Problem.

2. Equivalence relations among inequalities

We divide the domain (I) (resp. (II)) in Theorem A into (I-a) (resp. (II-a)) and (I-b) (resp. (II-b)) as follows. Now there are four *triangular* domains and two *square* domains in the Figure.

We show relations among the inequalities which hold in the four *triangular* domains.



Figure

Theorem 1. *The following assertions hold and they are mutually equivalent:*
(I-b) *If $A \geq B > 0$, then*

$$A^t B^{1-2t} A^t \geq A \geq B \geq A^{\frac{t}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{1-t}{p-t}} A^{\frac{t}{2}} \tag{2.1}$$

for $1 \geq p > t \geq \frac{1}{2}$.
(IV) *If $A \geq B > 0$, then*

$$A^{2p-1} \geq B^{2p-1} \geq B^p A^{-1} B^p \geq A^{\frac{t}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{2p-1-t}{p-t}} A^{\frac{t}{2}} \tag{2.2}$$

for $1 \geq t > p \geq \frac{1}{2}$.

Theorem 2. *The following assertions hold and they are mutually equivalent:*
(III) *If $A \geq B > 0$, then*

$$A^{2p} \geq B^{2p} \geq A^{\frac{t}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{2p-t}{p-t}} A^{\frac{t}{2}} \tag{2.3}$$

for $\frac{1}{2} \geq p > t \geq 0$.
(II-b) *If $A \geq B > 0$, then*

$$B^{-2t} \geq A^{-2t} \geq A^{\frac{-t}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{-t}{p-t}} A^{\frac{-t}{2}} \tag{2.4}$$

for $\frac{1}{2} \geq t > p \geq 0$.

We remark that each of the assertions in Theorem 1 and Theorem 2 is a slightly extension of Theorem A.

We need the following lemma to give proofs of Theorem 1 and Theorem 2.

Lemma 1 ([7]). *Let $A > 0$ and B be an invertible operator. Then*

$$(BAB^*)^\lambda = BA^{\frac{1}{2}}(A^{\frac{1}{2}} B^* B A^{\frac{1}{2}})^{\lambda-1} A^{\frac{1}{2}} B^*$$

holds for any real number λ.

Proof of Theorem 1. (I-b) and (IV) have been already proved in [2][13][15] and [16], so that we have only to prove the equivalence relation between them.
(2.1) in (I-b) is equivalent to the following (2.5) by applying Lemma 1 to the right hand side:

$$A^t B^{1-2t} A^t \geq A \geq B \geq A^t B^{\frac{-p}{2}}(B^{\frac{p}{2}} A^{-t} B^{\frac{p}{2}})^{\frac{1-2t+p}{p-t}} B^{\frac{-p}{2}} A^t. \tag{2.5}$$

(2.5) is equivalent to the following (2.6) by multiplying A^{-t} on the both sides:

$$B^{1-2t} \geq A^{1-2t} \geq A^{-t} B A^{-t} \geq B^{\frac{-p}{2}}(B^{\frac{p}{2}} A^{-t} B^{\frac{p}{2}})^{\frac{2t-1-p}{t-p}} B^{\frac{-p}{2}}. \tag{2.6}$$

Put $p_1 = t, t_1 = p, A_1 = B^{-1}$ and $B_1 = A^{-1}$ in (2.6). Then (2.6) is equivalent to the following (2.7):

$$A_1^{2p_1-1} \geq B_1^{2p_1-1} \geq B_1^{p_1} A_1^{-1} B_1^{p_1} \geq A_1^{\frac{t_1}{2}}(A_1^{\frac{-t_1}{2}} B_1^{p_1} A_1^{\frac{-t_1}{2}})^{\frac{2p_1-1-t_1}{p_1-t_1}} A_1^{\frac{t_1}{2}}. \tag{2.7}$$

It is obvious that $A \geq B > 0$ is equivalent to $A_1 \geq B_1 > 0$ and $1 \geq p > t \geq \frac{1}{2}$ is equivalent to $1 \geq t_1 > p_1 \geq \frac{1}{2}$. Hence (I-b) is equivalent to (IV) since (I-b) is equivalent to (2.7) and (2.7) corresponds to (2.2) in (IV). □

Proof of Theorem 2. (III) and (II-b) have been already proved in [13] and [15], so that we have only to prove the equivalence relation between them.
(2.3) in (III) is equivalent to the following (2.8) by applying Lemma 1 to the right hand side:

$$A^{2p} \geq B^{2p} \geq B^{\frac{p}{2}}(B^{\frac{p}{2}}A^{-t}B^{\frac{p}{2}})^{\frac{p}{p-t}}B^{\frac{p}{2}}. \tag{2.8}$$

Put $p_1 = t, t_1 = p, A_1 = B^{-1}$ and $B_1 = A^{-1}$ in (2.8). Then (2.8) is equivalent to the following (2.9):

$$B_1^{-2t_1} \geq A_1^{-2t_1} \geq A_1^{\frac{-t_1}{2}}(A_1^{\frac{-t_1}{2}}B_1^{p_1}A_1^{\frac{-t_1}{2}})^{\frac{-t_1}{p_1-t_1}}A_1^{\frac{-t_1}{2}}. \tag{2.9}$$

It is obvious that $A \geq B > 0$ is equivalent to $A_1 \geq B_1 > 0$ and $\frac{1}{2} \geq p > t \geq 0$ is equivalent to $\frac{1}{2} \geq t_1 > p_1 \geq 0$. Hence (III) is equivalent to (II-b) since (III) is equivalent to (2.9) and (2.9) corresponds to (2.4) in (II-b). □

3. Result and counterexamples related to the open problem

By using the same technique as in the preveous section, we obtain the following more precise result than Theorem B.

Theorem 3. *Let $\frac{1}{2} > p > t > 0$. For any $\alpha > 0$, there exist $A, B \in B(\mathbb{R}^2)$ such that $A \geq B > 0$ and*

$$B^{2p+\alpha} \not\geq A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{2p-t+\alpha}{p-t}}A^{\frac{t}{2}}.$$

Proof of Theorem 3. First of all, we recall the following (3.1):

$$A^{-t} \geq (A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{-t}{p-t}} \quad \text{for } \frac{1}{2} > t > p > 0 \text{ and } A \geq B > 0, \tag{3.1}$$

and that the outside exponents of (3.1) is best possible proved in [15].

Let $\alpha > 0$. Assume

$$B^{2p+\alpha} \geq A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{2p-t+\alpha}{p-t}}A^{\frac{t}{2}} \quad \text{for } \frac{1}{2} > p > t > 0 \text{ and } A \geq B > 0. \tag{3.2}$$

Applying Lemma 1 to the right hand side of (3.2), we have

$$B^{2p+\alpha} \geq B^{\frac{p}{2}}(B^{\frac{p}{2}}A^{-t}B^{\frac{p}{2}})^{\frac{p+\alpha}{p-t}}B^{\frac{p}{2}}. \tag{3.3}$$

Multiplying $B^{\frac{-p}{2}}$ on the both sides and then taking inverses, we have

$$(B^{\frac{-p}{2}}A^tB^{\frac{-p}{2}})^{\frac{-p-\alpha}{t-p}} \geq B^{-p-\alpha}. \tag{3.4}$$

Put $p_1 = t, t_1 = p, A_1 = B^{-1}$ and $B_1 = A^{-1}$ in (3.4). Then (3.4) is equivalent to the following (3.5):

$$A_1^{-(t_1+\alpha)} \geq (A_1^{\frac{-t_1}{2}}B_1^{p_1}A_1^{\frac{-t_1}{2}})^{\frac{-(t_1+\alpha)}{p_1-t_1}} \quad \text{for } \frac{1}{2} > t_1 > p_1 > 0 \text{ and } A_1 \geq B_1 > 0. \tag{3.5}$$

This is a contradiction to the best possibility of (3.1). This contradiction proves Theorem 3. □

Remark. We remark that Theorem B is obtained by putting $\alpha = 1-2p > 0$ in Theorem 3. And we show the following concrete counterexample to the Problem. For the sake of convenience, we define $X(\alpha)$ for $\alpha \geq 0$ as follows:

$$X(\alpha) = A^{2p+\alpha} - A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{2p-t+\alpha}{p-t}}A^{\frac{t}{2}}.$$

Then we remark that (1.1) in the Problem holds if and only if $X(1-2p) \geq 0$ holds.

Counterexample. Let $p = 0.3$ and $t = 0.15$, then $\frac{1}{2} > p > t > 0$. Put

$$A = \begin{pmatrix} 18926 & 2549 & 26988 \\ 2549 & 38479 & 3638 \\ 26988 & 3638 & 38524 \end{pmatrix}, \qquad B = \begin{pmatrix} 19 & 0 & 0 \\ 0 & 38133 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then it is easily verified that $A \geq B > 0$ since the eigenvalues of $A - B$ are $57773.5671\ldots$, $0.0083\ldots$ and $2.4245\ldots$.

For an example, put $\alpha_1 = 1 - 2p = 0.4$, then

$$X(\alpha_1) = A - A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{1-t}{p-t}}A^{\frac{t}{2}}$$

$$= \begin{pmatrix} 18916.25\ldots & 2587.35\ldots & 26990.14\ldots \\ 2587.35\ldots & 432.25\ldots & 3655.53\ldots \\ 26990.14\ldots & 3655.53\ldots & 38523.50\ldots \end{pmatrix}.$$

The eigenvalues of $X(\alpha_1)$ are $57785.0756\ldots, 87.9132\ldots$ and $-0.9723\ldots$, therefore $X(\alpha_1) \not\geq 0$, that is, $A^{1-t} \not\geq (A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{1-t}{p-t}}$. Thus, this is a counterexample to the Problem.

Furthermore, for another example, put $\alpha_2 = 0.37 < 0.4 = 1 - 2p = \alpha_1$, then

$$X(\alpha_2) = A^{0.97} - A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{0.97-t}{p-t}}A^{\frac{t}{2}}$$

$$= \begin{pmatrix} 13614.65\ldots & 1817.80\ldots & 19425.33\ldots \\ 1817.8\ldots & 300.01\ldots & 2567.35\ldots \\ 19425.33\ldots & 2567.35\ldots & 27728.05\ldots \end{pmatrix}.$$

The eigenvalues of $X(\alpha_2)$ are $41578.4615\ldots, 64.2655\ldots$ and $-0.0014\ldots$, therefore $X(\alpha_2) \not\geq 0$, that is, $A^{0.97-t} \not\geq (A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{0.97-t}{p-t}}$.

This counterexample gives negative and partial answer to the Problem. We expect the following conjecture but it still remains an open problem.

Conjecture. *Let $\frac{1}{2} > p > t > 0$. For any $\alpha > 0$, there exist $A, B \in B(H)$ such that $A \geq B > 0$ and*

$$A^{2p-t+\alpha} \not\geq (A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{2p-t+\alpha}{p-t}}.$$

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