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On Heinz-Kato type characterizations of the Furuta inequality

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Dedicated to Professor Hideko Lin on her retirement

ABSTRACT. We present several Heinz-Kato type characterizations of the Furuta inequality in this paper. We also show that every hyponormal operator satisfies the Heinz-Kato type inequalities.

Throughout the paper the capital letters always mean bounded linear operators acting on a Hilbert space H. T is positive in case $(Tx, x) \geq 0$ for all $x \in H$, and $T \geq 0$ in notation. If S and T are Hermitian, we write $T \geq S$ in case $T - S \geq 0$. $T = U|T|$ is the polar decomposition of T with U the partial isometry, and $|T|$ the positive square root of the positive operator T^*T.

A celebrated inequality of Furuta (1987) [2] states that

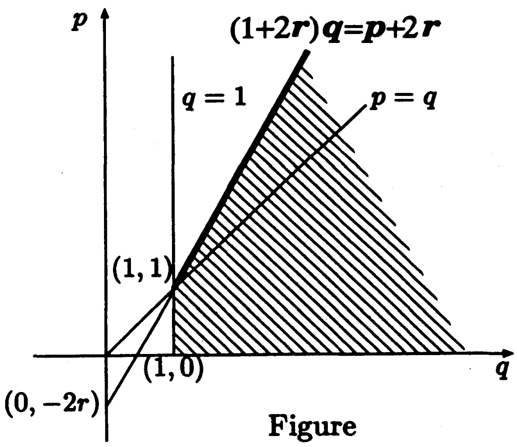
if $A \geq B \geq 0$, then for each $r \geq 0$

(i) $(B^r A^p B^r)^{1/q} \geq (B^r B^p B^r)^{1/q}$

and

(ii) $(A^r A^p A^r)^{1/q} \geq (A^r B^p A^r)^{1/q}$

hold for $p \geq 0$ and $q \geq 1$ with $(1+2r)q \geq p+2r$.



Figure

The essential signification of this inequality is surely the "best possible" figure on p, q, and r [8]. If we write $(p+2r)/q = (1+2r)\alpha$, then the inequality (i) becomes $(B^r A^p B^r)^{(1+2r)\alpha)/(p+2r)} \geq B^{(1+2r)\alpha}$ for $p \geq 1$, $r \geq 0$, and $\alpha \in [0,1]$. This inequality was characterized by Furuta himself as follows.

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Proposition [3, Theorem 1 and Theorem A]. Let $A, B \geq O$, and let $T = U \mid T \mid$ be the polar decomposition. Then the following are equivalent for $p, q \geq 1$, $r, s \geq 0$, and $\alpha, \beta \in [0,1]$.

(1) $(B^r A^p B^r)^{(1+2r)\alpha/(p+2r)} \geq B^{(1+2r)\alpha}$ if $A \geq B$ (Furuta's inequality);

(2) $\mid (T \mid T \mid^{(1+2r)\alpha+(1+2s)\beta-1} x, y) \mid^2$

$\leq ((\mid T \mid^{2r} A^{2p} \mid T \mid^{2r})^{(1+2r)\alpha/(p+2r)}x, x)((\mid T^* \mid^{2s} B^{2q} \mid T^* \mid^{2s})^{(1+2s)\beta/(q+2s)}y, y)$

for all $x, y \in$H, $\mid T \mid^2 \leq A^2$, $\mid T^* \mid^2 \leq B^2$, and $(1+2r)\alpha + (1+2s)\beta \geq 1$.

In this paper we shall give Heinz-Kato type characterizations of the inequality: $A^{(1+2r)\alpha} \geq (A^r B^p A^r)^{(1+2r)\alpha/(p+2r)}$ if $A \geq B \geq O$, $p \geq 1$, $r \geq 0$, and $\alpha \in [0,1]$, which is equivalent to the inequality (1) in Proposition. In the same spirit we give more characterizations of the inequality (1) in the Proposition above. Finally, We also show that every hyponormal operator satisfies the Heinz-Kato type inequalities.

Theorem 1. Let $A, B \geq O$, and let $T = U \mid T \mid$ be the polar decomposition. Then the following are equivalent for all $x, y \in$H, $r, s \geq 0$, $p, q \geq 1$, $\alpha, \beta \in [0,1]$, and if $\mid T \mid^2 \leq A^2$ and $\mid T^* \mid^2 \leq B^2$ for each one of the inequalities (2) through (7).

(1) $A^{(1+2r)\alpha} \geq (A^r B^p A^r)^{(1+2r)\alpha/(p+2r)}$ if $A \geq B$ (Furuta inequality);

(2) $\mid ((B^{2s} \mid T^* \mid^{2q} B^{2s})^{(1+2s)\beta/2(q+2s)} U (A^{2r} \mid T \mid^{2p} A^{2r})^{(1+2r)\alpha/2(p+2r)} x, y) \mid$
$\leq \| A^{(1+2r)\alpha} x \| \| B^{(1+2s)\beta} y \|$;

(3) $\mid ((B^{2s} \mid T^* \mid^{2q} B^{2s})^{(1+2s)\beta/2(q+2s)} U (A^{2r} \mid T \mid^{2p} A^{2r})^{(1+2r)\alpha/2(p+2r)} x, y) \mid$
$\leq \| A^{(1+2r)\alpha} x \| \| (B^{2s} \mid T^* \mid^{2q} B^{2s})^{(1+2s)\beta/2(q+2s)} y \|$;

(4) $\mid ((B^{2s} \mid T^* \mid^{2q} B^{2s})^{(1+2s)\beta/2(q+2s)} U (A^{2r} \mid T \mid^{2p} A^{2r})^{(1+2r)\alpha/2(p+2r)} x, y) \mid$
$\leq \| (A^{2r} \mid T \mid^{2p} A^{2r})^{(1+2r)\alpha/2(p+2r)} x \| \| B^{(1+2s)\beta} y \|$;

(5) $\mid ((B^{2s} \mid T^* \mid^{2q} B^{2s})^{(1+2s)\beta/2(q+2s)} (A^{2r} \mid T \mid^{2p} A^{2r})^{(1+2r)\alpha/2(p+2r)} x, y) \mid$
$\leq \| A^{(1+2r)\alpha} x \| \| B^{(1+2s)\beta} y \|$;

(6) $\mid ((B^{2s} \mid T^* \mid^{2q} B^{2s})^{(1+2s)\beta/2(q+2s)} (A^{2r} \mid T \mid^{2p} A^{2r})^{(1+2r)\alpha/2(p+2r)} x, y) \mid$
$\leq \| A^{(1+2r)\alpha} x \| \| (B^{2s} \mid T^* \mid^{2q} B^{2s})^{(1+2s)\beta/2(q+2s)} y \|$;

(7) $\mid ((B^{2s} \mid T^* \mid^{2q} B^{2s})^{(1+2s)\beta/2(q+2s)} (A^{2r} \mid T \mid^{2p} A^{2r})^{(1+2r)\alpha/2(p+2r)} x, y) \mid$
$\leq \| (A^{2r} \mid T \mid^{2p} A^{2r})^{(1+2r)\alpha/2(p+2r)} x \| \| B^{(1+2s)\beta} y \|$.

Proof. (1) implies (2). Remark that if $\mid T \mid^2 \leq A^2$, then $(A^{2r} \mid T \mid^{2p} A^{2r})^{(1+2r)\alpha/(p+2r)} \leq A^{2(1+2r)\alpha}$ by (1). And since $\mid T^* \mid^2 \leq B^2$, so $(B^{2s} \mid T^* \mid^{2q} B^{2s})^{(1+2s)\beta/(q+2s)} \leq B^{2(1+2s)\beta}$ by (1), again. To prove the inequality (2) we proceed as follows:

$\mid ((B^{2s} \mid T^* \mid^{2q} B^{2s})^{(1+2s)\beta/2(q+2s)} U (A^{2r} \mid T \mid^{2p} A^{2r})^{(1+2r)\alpha/2(p+2r)} x, y) \mid$

$\leq \| U(A^{2r} \mid T \mid^{2p} A^{2r})^{(1+2r)\alpha/2(p+2r)} x \| \| (B^{2s} \mid T^* \mid^{2q} B^{2s})^{(1+2s)\beta/2(q+2s)} y \|$
$= ((A^{2r} \mid T \mid^{2p} A^{2r})^{(1+2r)\alpha/(p+2r)} x, x)^{1/2} ((B^{2s} \mid T^* \mid^{2q} B^{2s})^{(1+2s)\beta/(q+2s)} y, y)^{1/2}$
$\leq (A^{2(1+2r)\alpha} x, x)^{1/2} (B^{2(1+2s)\beta} y, y)^{1/2}$ (by the remark above)
$= \| A^{(1+2r)\alpha} x \| \| B^{(1+2s)\beta} y \|$.

(1) implies (3) and (4). Each implication is easily seen from the proof above.

(2) implies (1). In the inequality (2) let $B = A$, and let $T = B$ so that $U = I$, the identity operator. Also let $s = r$, $\beta = \alpha$, $q = p$, and $y = x$. Then we have

$\mid ((A^{2r} B^{2p} A^{2r})^{(1+2r)\alpha/(p+2r)} x, x) \mid \leq (A^{2(1+2r)\alpha} x, x)$

under the condition that $A^2 \geq B^2$. Hence, (1) holds true.

(3) implies (1). In the inequality (3) let us assume the same substitutions as in the proof "(2) implies (1)". Then the inequality becomes

$\| (A^{2r} B^{2p} A^{2r})^{(1+2r)\alpha/2(p+2r)} x \|^2$
$\leq (A^{2(1+2r)\alpha} x, x)^{1/2} \| (A^{2r} B^{2p} A^{2r})^{(1+2r)\alpha/2(p+2r)} x \|$.

It follows that

$((A^{2r} B^{2p} A^{2r})^{(1+2r)\alpha/(p+2r)} x, x) \leq (A^{2(1+2r)\alpha} x, x)$

under the condition that $A^2 \geq B^2$. Hence, (1) holds true.

(4) implies (1). The proof is similar to "(3) implies (1)", and we shall omit.

Note that $U^* U = I$, and so in the proof above it is readily seen that the operator U may be omitted without affecting equivalence of inequalities. More precisely, we have inequalities (5), (6) and (7) which are all equivalent to (1), and this completes the proof of the theorem 1.

If we adopt the same technique as in the Theorem 1, we can say more about the Proposition in above. i.e., more characterizations of (1) in Proposition will be given next.

Theorem 2. Let $A, B \geq O$, and let $T = U \mid T \mid$ be the polar decomposition. Then the following are equivalent for all $x, y \in$H, $r, s \geq 0$, $p, q \geq 1$, $\alpha, \beta \in [0, 1]$, and if $\mid T \mid^2 \leq A^2$ and $\mid T^* \mid^2 \leq B^2$ for each one of the inequalities (2) through (7).

(1) $(B^r A^p B^r)^{(1+2r)\alpha/(p+2r)} \geq B^{(1+2r)\alpha}$ if $A \geq B$ (Furuta's inequality);

(2) $\mid (\mid T^* \mid^{(1+2s)\beta} U \mid T \mid^{(1+2r)\alpha} x, y) \mid$
$\leq \| (\mid T \mid^{2r} A^{2p} \mid T \mid^{2r})^{(1+2r)\alpha/2(p+2r)} x \| \| (\mid T^* \mid^{2s} B^{2q} \mid T^* \mid^{2s})^{(1+2s)\beta/2(q+2s)} y \|$;

(3) $\mid (\mid T^* \mid^{(1+2s)\beta} U \mid T \mid^{(1+2r)\alpha} x, y) \mid$
$\leq \| (\mid T \mid^{2r} A^{2p} \mid T \mid^{2r})^{(1+2r)\alpha/2(p+2r)} x \| \| \mid T^* \mid^{(1+2s)\beta} y \|$;

(4) $\mid (\mid T^* \mid^{(1+2s)\beta} U \mid T \mid^{(1+2r)\alpha} x, y) \mid$
$\leq \| \mid T \mid^{(1+2r)\alpha} x \| \| (\mid T^* \mid^{2s} B^{2q} \mid T^* \mid^{2s})^{(1+2s)\beta/2(q+2s)} y \|$;

(5) $|(|T^*|^{(1+2s)\beta}|T|^{(1+2r)\alpha}x,y)|$
$\leq \| (|T|^{2r}A^{2p}|T|^{2r})^{(1+2r)\alpha/2(p+2r)}x\| \| (|T^*|^{2s}B^{2q}|T^*|^{2s})^{(1+2s)\beta/2(q+2s)}y\|$;
(6) $|(|T^*|^{(1+2s)\beta}|T|^{(1+2r)\alpha}x,y)|$
$\leq \| (|T|^{2r}A^{2p}|T|^{2r})^{(1+2r)\alpha/2(p+2r)}x\| \| |T^*|^{(1+2s)\beta}y\|$;
(7) $|(|T^*|^{(1+2s)\beta}|T|^{(1+2r)\alpha}x,y)|$
$\leq \| |T|^{(1+2r)\alpha}x\| \| (|T^*|^{2s}B^{2q}|T^*|^{2s})^{(1+2s)\beta/2(q+2s)}y\|$.

Proof. Since $|T|^2 \leq A^2$ and $|T^*|^2 \leq B^2$, the inequality (1) implies that $|T|^{2(1+2r)\alpha} \leq (|T|^{2r}A^{2p}|T|^{2r})^{(1+2r)\alpha/(p+2r)}$, and $|T^*|^{2(1+2s)\beta} \leq (|T^*|^{2s}B^{2q}|T^*|^{2s})^{(1+2s)\beta/(q+2s)}$, respectively. The rest of the proof may be carried out as in the Theorem 1.

Notice that the inequality (2) in the Theorem 2 is exactly the same as that of (2) in the Proposition, since $|T^*|^{(1+2s)\beta}U|T|^{(1+2r)\alpha} = T|T|^{(1+2r)\alpha+(1+2s)\beta-1}$.

Recall that T is said to be p-hyponormal if $(TT^*)^p \leq (T^*T)^p$ holds for $0 < p \leq 1$, and 1-hyponormal is nothing but hyponormal, of course. Clearly, if T is hyponormal, then it is p-hyponormal by the Löwner-Heinz inequality i.e., $A^\alpha \geq B^\alpha$ if $A \geq B$. We shall give an application of the Theorem 1 and 2 next.

Corollary . If T is hyponormal, then the following Heinz-Kato type inequalities hold for all $x, y \in$H, $r, s \geq 0$, $p, q \geq 1$, $\alpha, \beta \in [0,1]$, and $T = U|T|$ the polar decomposition.

(1) $|(((|T|^{2s}|T^*|^{2q}|T|^{2s})^{(1+2s)\beta/2(q+2s)}U|T|^{(1+2r)\alpha}x,y)|$
$\leq \| |T|^{(1+2r)\alpha}x\| \| |T|^{(1+2s)\beta}y\|$;
(2) $|(((|T|^{2s}|T^*|^{2q}|T|^{2s})^{(1+2s)\beta/2(q+2s)}|T|^{(1+2r)\alpha}x,y)|$
$\leq \| |T|^{(1+2r)\alpha}x\| \| |T|^{(1+2s)\beta}y\|$;
(3) $|(|T^*|^{(1+2s)\beta}U|T|^{(1+2r)\alpha}x,y)|$
$\leq \| |T|^{(1+2r)\alpha}x\| \| (|T^*|^{2s}|T|^{2q}|T^*|^{2s})^{(1+2s)\beta/2(q+2s)}y\|$;
(4) $|(|T^*|^{(1+2s)\beta}|T|^{(1+2r)\alpha}x,y)|$
$\leq \| |T|^{(1+2r)\alpha}x\| \| (|T^*|^{2s}|T|^{2q}|T^*|^{2s})^{(1+2s)\beta/2(q+2s)}y\|$.

Proof. Let $A = B = |T|$ in the Theorem 1 and 2. Then the conditions in the Theorem 1 and 2 becomes $|T^*|^2 \leq |T|^2$, which means that T is hyponormal. Therefore, all inequalities in above hold true.

Incidently, it was proved in [5, Theorem 2] that T is p-hyponormal if and only if the inequality $|(T|T|^{2p-1}x,y)| \leq \| |T|^p x\| \| |T|^p y\|$ holds for all $x, y \in$H. This inequality is obtained from (1), or (3) in Corollary when $r = s = 0$, and $\alpha = \beta = p \in [0,1]$.

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