Nihonkai Math. J.
Vol.9 (1998), 171–186

Quasilinear Reaction-Diffusion Systems in L^∞ spaces

Joseph Jude PEIRIS and Toshitaka MATSUMOTO

0. Introduction

The present paper is concerned with a system of quasilinear degenerate reaction-diffusion equations of the form

$$
\text{(RDS)}\quad
\begin{aligned}
\partial_t u_1 &= \beta_1(\cdot,\nabla u_1)\Delta u_1 - d_1(t,\cdot)u_1u_4 - d_2(t,\cdot)u_1u_3\\
\partial_t u_2 &= \beta_2(\cdot,\nabla u_2)\Delta u_2 - d_3(t,\cdot)u_2u_4 + d_2(t,\cdot)u_1u_3\\
\partial_t u_3 &= \phantom{\beta_2(\cdot,\nabla u_2)\Delta u_2 - {}} d_3(t,\cdot)u_2u_4 - d_2(t,\cdot)u_1u_3\\
\partial_t u_4 &= - d_1(t,\cdot)u_1u_4 - d_3(t,\cdot)u_2u_4.
\end{aligned}
$$

in the product space $L^\infty(\Omega)^4$. It is often natural to treat reaction-diffusion systems in such L^∞ spaces over the cylindrical domain $(0,\tau)\times\Omega$ under the boundary condition of Robin type

$$
\text{(BC)}\qquad \frac{\partial u_i}{\partial\nu}(t,x) + a_i(x)u_i(x) = 0 \quad \text{for } (t,x)\in(0,\tau)\times\partial\Omega \text{ and } i=1,2
$$

and the initial condition

$$
\text{(IC)}\qquad u_i(0,x) = w_i(x) \geq 0 \quad \text{for } x\in\Omega \text{ and } i=1,2,3,4.
$$

Here Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $[0,\tau]$ is a fixed time-interval, $\beta_i(x,\xi)$ are a nonnegative continuous functions defined in $\Omega\times\mathbb{R}^n$, ν denotes the outward unit normal to $\partial\Omega$ and $a_i(x)$ is a continuously differentiable positive function on $\partial\Omega$. We here direct our attention to a boundary condition of Robin type, although various types of boundary conditions may be imposed.

This problem was formulated as a mathematical model describing the time evolution of chemical reaction of four kinds of molecules existing in a non-Newtonian fluid. In the semilinear case, that is $\beta_1\equiv\beta_2\equiv 1$, Mimura and Nakaoka [13] treated (RDS)-(BC)-(IC) in the space of uniformly continuous bounded functions in $\mathbb{R}^n$ under the condition that

d_i, $i = 1, 2, 3$, are positive constants, and they showed the asymptotic behavior of the solution. For the detailed arguments and other related problems, we refer to the paper by Fife [3], Kobayasi and Oharu [11] and Matsumoto [14].

As for the quasilinear case, Serizawa [16] treated a quasilinear problem

$$u_t(t,x) = \beta(u_x(t,x))u_{xx}(t,x)$$

in $C[0,1]$ under some nonlinear boundary condition. More generally, a class of quasilinear equations of the form

$$\text{(QL)} \qquad \begin{aligned} u_t(t,x) &= \beta(x, u_x(t,x))u_{xx}(t,x) \\ u(0,x) &= u_0(x) \end{aligned}$$

are considered in $C[0,1]$ by Burch and Goldstein [1] and Goldstein and Lin [5]. Especially, in [5], they treated (QL) under various boundary conditions in the case where $\beta(x,\xi)$ may vanish on the boundary $\partial\Omega$ and showed the existence, uniqueness of the semigroup solutions to (QL). Dorroh and Rieder [2] then extended the results given in [5] to the equation

$$\begin{aligned} u_t(t,x) &= \beta(t,x,u(t,x),u_x(t,x))u_{xx}(t,x) + \psi(t,x,u(t,x),u_x(t,x)) \\ u(0,x) &= u_0(x) \end{aligned}$$

under time-dependent boundary conditions. For the detailed arguments and also the treatment of multidimensional case, we refer to Goldstein and Lin [6, 7], Goldstein *et al.* [8] and Lin [12].

In this paper we treat the quasilinear problem for (RDS) in the case where $\beta_i(x,\xi)$ may vanish on the boundary $\partial\Omega$ under the boundary and initial conditions (BC)-(IC) in the Banach space $(L^\infty(\Omega))^4$. It should be noted here that $u_3(\cdot)$ and $u_4(\cdot)$ are expected to be only $L^\infty(\Omega)$-valued functions because of the degeneracy in (RDS). We demonstrate in our main result that there exists a weakly-star differentiable solutions to (RDS)-(BC)-(IC) if the initial data is smooth.

This paper is organized as follows: In Section 1 the basic results are outlined so that they may be applied to our problem. In Section 2, we introduce quasilinear diffusion operator in $L^\infty(\Omega)$ and investigate the m-dissipativity of such operators. Section 3 discusses the abstract quasilinear problem associated with (RDS) in the product space $(L^\infty(\Omega))^4$ and contains our main result.

1. Preliminaries

In this section we introduce notation and some basic results which are applied to our results. In what follows, Ω is assumed to be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. For $1 \le p \le \infty$ and $m \ge 1$, $L^p(\Omega)$ and $W^{m,p}(\Omega)$ denote the usual Lebesgue and Sobolev spaces, respectively. The norm of $L^p(\Omega)$ is denoted by $|\cdot|_p$ for $1 \le p \le \infty$.

In order to prove our results we apply the characterization of the duality mapping of $L^\infty(\Omega)$ discussed by Peiris [15]. It is well-known that the dual space of $L^\infty(\Omega)$ is identified with the space $ba(\Omega)$ of finitely additive bounded measures on Ω which vanish on sets of Lebesgue measure zero. For $v \in L^\infty(\Omega)$ and $\mu \in ba(\Omega)$ the value of μ at v is written as $\langle v, \mu \rangle$. The normalized duality mapping of $L^\infty(\Omega)$ is a multi-valued mapping F_0 from $L^\infty(\Omega)$ into $ba(\Omega)$ which assigns to each $v \in L^\infty(\Omega)$ a subset of $ba(\Omega)$ defined by

$$F_0(v) = \{\mu \in ba(\Omega) : \langle v, \mu \rangle = |v|, \ \|\mu\|_{ba(\Omega)} = 1\}.$$

It is seen that each value of $F_0(v)$ is a weakly-star compact and convex subset of $ba(\Omega)$ for each $v \in L^\infty(\Omega)$. A finitely additive measure $\mu \in ba(\Omega)$ is said to be a 0-1 (or 0-(-1)) measure if μ assumes only the values 0 and 1 (or 0 and (-1)). It is shown in [15] that $F_0(v)$ contains at least one 0-1 (or 0-(-1)) measure for $v \in L^\infty(\Omega)$.

The following two propositions play an essential role in the proof of Theorem 5.

PROPOSITION 1. ([15]) *Let μ be a 0-1 measure in $ba(\Omega)$.*

(i) *There exists $a \in \overline{\Omega}$ such that $\mu_{|C(\overline{\Omega})} = \delta_a$ in the sense that μ is a Hahn-Banach extention of the point mass δ_a concentrated at a.*

(ii) *μ is multiplicative in the sense that $\langle vw, \mu \rangle = \langle v, \mu \rangle \langle w, \mu \rangle$ for any $v, w \in L^\infty(\Omega)$.*

PROPOSITION 2. ([15]) *Let $u \in W^{1,p}(\Omega)$ for some $p > n$ and $\Delta u \in L^\infty(\Omega)$. If u has a non-negative maximum at a point $a \in \Omega$, then there exists a 0-1 measure $\mu_a \in F_0(u)$ such that the essential support is concentrated at a and $\langle \Delta u, \mu_a \rangle \le 0$.*

We next introduce a class of nonlinear operators in a Banach space $(X, |\cdot|)$ and the associated nonlinear problems of the form

$$\text{(NP)} \qquad \frac{du(t)}{dt} = \mathfrak{A}(t)u(t), \quad s < t < \tau; \ u(s) = v, \quad 0 \le s < \tau.$$

In what follows, $(X^*, |\cdot|)$ denotes the dual space of X. For $x \in X$ and $f \in X^*$ the value of f at v is written as $\langle v, f \rangle$. The duality mapping of X is denoted by F. For $v, w \in X$ the symbol $\langle v, w \rangle_i$ stands for the infimum of the set $\{\langle v, f \rangle : f \in F(w)\}$. The symbol $Lip([0,\tau]; X)$ denotes the space of X-valued functions which are Lipschitz continuous on $[0, \tau]$. We also denote by $BV([0,\tau]; X)$ the set of X-valued functions of bounded variation.

We then consider a general class of nonlinear evolution operators. Let $\mathfrak{A}(t)$, $0 \le t \le \tau$, be possibly nonlinear operators in X which are defined on subsets $D(\mathfrak{A}(t))$, $0 \le t \le \tau$, respectively. We here assume that $D \equiv \overline{D(\mathfrak{A}(t))}$ for $t \in [0, \tau]$. This assumption is too restricted but sufficient for the application to the quasilinear problem under consideration. To restrict the time dependence of the family of nonlinear operators $\mathfrak{A}(t)$, we introduce the following family of nonnegative functions defined on all of $[0,\tau]^2$. By $\mathcal{F}$ is meant the class of all $\theta \in C([0,\tau]^2)$ such that for $0 \le s \le t \le \tau$

$$\theta(s,t) = \theta(t,s) \quad \text{and} \quad \theta(s,s) = 0.$$

A function $\theta \in \mathcal{F}$ is said to belong to the class $\mathcal{F}_{BV}$, if there exists a constant $M > 0$ such that

$$\sum_{k=1}^{N} \theta(t_k, t_{k-1}) \leq M$$

for all partitions $\Delta = \{0 = t_0 < t_1 < \ldots < t_N = \tau\}$ of $[0, \tau]$. Moreover, in order to restrict the quasi-dissipativity in a local sense of $\mathfrak{A}(t)$, we employ a continuous functional $\varphi : X \to [0, \infty]$ such that $D \subset D(\varphi) \equiv \{v \in X : \varphi(v) < \infty\}$.

For a family $\{\theta_\alpha : \alpha > 0\}$ of functions in $\mathcal{F}$, a subset D of X and a functional φ as mentioned above, we introduce a class of nonlinear operators $\mathfrak{A}(t)$ with which we are concerned in this paper.

DEFINITION 3. A one-parameter family $\{\mathfrak{A}(t)\}$ of nonlinear operators is said to belong to the class $\mathfrak{U}(D, \varphi)$, if $\mathfrak{A}(t)$ are nonlinear operators satisfying the conditions (H1) and (H2) below:

(H1) For each $\alpha > 0$, the set $D_\alpha \equiv \{v \in D : \varphi(v) \leq \alpha\}$ is closed in X.

(H2) For each $\alpha > 0$ and each $t \in [0, \tau]$, the nonlinear operator $\mathfrak{A}(t)$ is locally quasi-dissipative in the sense that

$$\langle \mathfrak{A}(s)v - \mathfrak{A}(t)w,\ v - w\rangle_i \leq \omega_\alpha |v - w|^2 + \theta_\alpha(s, t)|v - w|$$

for $v \in D(\mathfrak{A}(s)) \cap D_\alpha$, $w \in D(\mathfrak{A}(t)) \cap D_\alpha$, some constant $\omega_\alpha \in \mathbb{R}$ and some $\theta_\alpha \in \mathcal{F}$.

A two-parameter family $\mathcal{U} \equiv \{U(t, s) : 0 \leq s \leq t \leq \tau\}$ of possibly nonlinear operators from D into D is called an *evolution operator on* D, if it has the two properties below:

(E1) $$U(r, r)v = v \quad \text{and} \quad U(t, s)U(s, r)v = U(t, r)v$$

for $0 \leq r \leq s \leq t \leq \tau$ and $v \in D$.

(E2) For each $s \in [0, \tau)$ and each $v \in D$, $U(\cdot, s)v \in C([s, \tau]; X)$.

Because of the localized quasi-dissipativity condition (H2), the nonlinear problem (NP) may admit only local solutions. Hence it is necessary to restrict the growth of the solutions in order to discuss the solutions of (NP) on $[s, \tau]$. In this paper we employ a typical growth condition in terms of the real-valued function $\varphi(u(\cdot))$, namely,

(EG) $$\varphi(u(t)) \leq e^{a(t-s)}(\varphi(v) + b(t - s)), \qquad t \in [s, \tau],$$

where a and b are nonnegative constants. This type of growth condition may be called the *exponential growth condition.*

In more general situation Kobayasi *et al.* [10] gives a sufficient condition for a one-parameter family of nonlinear operators $\{\mathfrak{A}(t)\}$ to generate an evolution operator $\mathcal{U}$ on D associated with (NP) and satisfying the growth condition (EG). The next theorem is a special case of Theorem 4.1 in [10].

THEOREM 4. (GENERATION THEOREM) *Let $\{\mathfrak{A}(t)\}$ be a one-parameter family of non-linear operators belonging to the class $\mathfrak{U}(D,\varphi)$. Let $a,\ b \geq 0$. Then condition* (I) *below implies* (II)*:*

(I) *For $s \in [0,\tau)$, $\alpha > 0$ there is $\lambda(s,\alpha) > 0$ such that for $v \in D_\alpha$ and $\lambda \in (0, \lambda(s,\alpha))$ there exists v_λ in $D(\mathfrak{A}(s+\lambda))$ satisfying*

(I.a) $$v_\lambda - \lambda\mathfrak{A}(s+\lambda)v_\lambda = v,$$

(I.b) $$\varphi(v_\lambda) \leq (1-a\lambda)^{-1}(\varphi(v)+b\lambda).$$

(II) *There exists an evolution operator $\mathcal{U} \equiv \{U(t,s) : t \geq 0\}$ on D such that*

(II.a) *for $s \in [0,\tau)$, $\alpha > 0$ and $\gamma \geq e^{a(\tau-s)}(\alpha + b(\tau - s))$, there exists a constant $\omega_\gamma \in \mathbb{R}$ satisfying*

$$|U(t,s)v - U(t,s)w| \leq \exp(\omega_\gamma(t-s))|v-w|$$

for $v,\ w \in D_\alpha$ and $t \in [s,\tau]$,

(II.b) *for $s \in [0,\tau)$, $v \in D$ and $t \in [s,\tau]$, $U(\cdot,s)v$ satisfies the growth condition*

$$\varphi(U(t,s)v) \leq e^{a(t-s)}[\varphi(v) + b(t-s)].$$

Moreover, if the modulus θ of time dependence belongs to $\mathcal{F}_{BV}$, then for $s \in [0,\tau)$ and $v \in D(\mathfrak{A}(s))$ $U(\cdot,s)v$ is Lipschitz continuous over $[s,\tau]$.

2. Quasilinear diffusion operators $\beta_i(\cdot,\nabla v)\Delta v$

In this section we formulate two quasilinear diffusion operators which contain terms representing the anomalous viscosities. In what follows, the coefficients $\beta_i(\cdot,\cdot)$ and $a_i(\cdot)$, $i = 1,2$, are assumed to satisfy the following conditions:

(C1) $\beta_i \in C(\overline{\Omega}\times\mathbb{R}^n)$ and there exist $q > n$ and a positive measurable function β_0 on Ω such that $1/\beta_0 \in L^q(\Omega)\cap L^\infty_{loc}(\Omega)$ and

$$\beta_i(x,\xi) \geq \beta_0(x) > 0 \quad \text{for } i = 1,2,\ x \in \Omega \text{ and } \xi \in \mathbb{R}^n.$$

(C2) $a_i(\cdot) \in C^1(\partial\Omega)$ and $a_i(x) > 0$ for $i = 1,2$ and $x \in \partial\Omega$.

In order to describe the nonlinear diffusion terms, we define quasilinear operators A_i in $L^\infty(\Omega)$ by

$$D(A_i) = \Big\{v \in L^\infty(\Omega) : v \in W^{2,q}(\Omega)\cap W^{2,p}_{loc}(\Omega) \quad \text{for } p > n,$$
$$\beta(\cdot,\nabla v)\Delta v \in L^\infty(\Omega),\ \frac{\partial v}{\partial \nu} + a_i(x)v = 0 \quad \text{on } \partial\Omega\Big\},$$

$$A_i v = \beta_i(\cdot,\nabla v)\Delta v, \qquad i = 1,2,$$

where q is the real number given in (C1). Since Ω is bounded and $W^{2,q}(\Omega)$ is continuously embedded in $C^1(\overline{\Omega})$ for $q > n$, $\partial u_i/\partial\nu$ belong to $C(\partial\Omega)$, $i = 1,2$, and the boundary conditions make sense.

For the quasilinear diffusion operators A_i defined above, the following results are basic:

THEOREM 5. *For $i = 1,2$, the operator A_i is m-dissipative in $L^\infty(\Omega)$ in the sense that*

$$|u - v|_\infty \le |(I - \lambda A_i)u - (I - \lambda A_i)v|_\infty$$

and

$$Ran(I - \lambda A_i) = L^\infty(\Omega) \quad \textit{for } \lambda > 0 \textit{ and } u, v \in D(A_i).$$

PROOF. We first show that the operators A_i are dissipative. For simplicity in notation we omit the subscript i. Let $u, v \in D(A)$. Since u and v are continuously differentiable over $\overline{\Omega}$, there exists $a \in \overline{\Omega}$ such that $|u(a) - v(a)| = \sup_{x\in\overline{\Omega}} |u(x) - v(x)|$. In view of the boundary condition (BC), we may assume that $a \in \Omega$, and that without loss of generality we may assume $u(a) - v(a) \ge 0$. For the point a we have $\nabla u(a) = \nabla v(a)$. By Proposition 2 there exists a 0-1 measure $\mu_a \in F_0(u - v)$ with the essential support concentrated at a such that

$$\langle \Delta(u - v), \mu_a\rangle \le 0. \tag{2.1}$$

This implies that

$$\begin{aligned}\langle Au - Av, \mu_a\rangle &= \langle \beta(\cdot, \nabla u)\Delta u, \mu_a\rangle - \langle \beta(\cdot, \nabla v)\Delta v, \mu_a\rangle \\ &= \beta(a, \nabla u(a))\langle \Delta u, \mu_a\rangle - \beta(a, \nabla v(a))\langle \Delta v, \mu_a\rangle \\ &= \beta(a, \nabla u(a))\langle \Delta(u - v), \mu_a\rangle \le 0,\end{aligned}$$

and hence A is dissipative. We next demonstrate that $Ran(I - \lambda A) = L^\infty(\Omega)$. A linear operator L in $L^q(\Omega)$ is defined by

$$\begin{aligned}&D(L) = \left\{v \in W^{2,q}(\Omega) : \frac{\partial v}{\partial \nu} + a(x)v = 0\right\}, \\ &Lv = \Delta v,\end{aligned}$$

where q is a positive number given in (C1). It is well-known that L generates a compact analytic semigroup of contractions in $L^q(\Omega)$. Let $\alpha \in (1/2 + n/(2q), 1)$. We then define a Banach space Y by

$$Y \equiv D((-L)^\alpha), \quad \|v\|_Y \equiv \max\{|v|_q, |(-L)^\alpha v|_q\} \quad \text{for } v \in Y.$$

From a result mentioned in Henry [9, Theorem 1.6.1], Y is continuously embedded in $C^1(\overline{\Omega})$. The proof is divided into two steps.

STEP 1. We first assume that there exists $\delta > 0$ such that

$$\beta(x, \xi) \ge \beta_0(x) > \delta. \tag{2.2}$$

It suffices to show that $Ran(I-\lambda A)=L^\infty(\Omega)$ for some $\lambda>0$. Let $f\in L^\infty(\Omega)$ and put

$$C\equiv\{v\in Y:\|v\|_Y\le|f|_q\}.$$

As easily seen, C is a bounded closed convex subset of Y. We then define an operator Γ from C into Y by

$$\Gamma v\equiv\left(I-\sqrt{\lambda}L\right)^{-1}\left(v+\frac{f-v}{\sqrt{\lambda}\beta(\cdot,\nabla v)}\right)\qquad\text{for } v\in C.$$

We prove that Γ has a fixed point in C by using Schauder's fixed point theorem. Let $v\in C$ and let $\lambda>\delta^{-2}$. Then we have from (2.2)

$$\begin{aligned}|\Gamma v|_q&\le\left|v+\frac{f-v}{\sqrt{\lambda}\beta(\cdot,\nabla v)}\right|_q\\&\le\left(1-\frac{1}{\sqrt{\lambda}\delta}\right)|v|_q+\frac{1}{\sqrt{\lambda}\delta}|f|_q\le|f|_q\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}|(-L)^\alpha\Gamma v|_q&\le M_\alpha\lambda^{-\alpha/2}\left|v+\frac{f-v}{\sqrt{\lambda}\beta(\cdot,\nabla v)}\right|_q\\&\le M_\alpha\lambda^{-\alpha/2}|f|_q,\end{aligned}\tag{2.4}$$

where M_α is a constant that depends only on α. If we choose λ large enough to satisfy $\lambda>\max\{\delta^{-2},M_\alpha^{2/\alpha}\}$, it follows from (2.3) and (2.4) that $\|\Gamma v\|_Y\le|f|_q$. This shows that Γ maps C into itself. We next show that Γ is continuous. Let v, $v_n\in C$, $n\ge1$, and assume that v_n converges to v in Y as $n\to\infty$. Since $v_n\to v$ in $C^1(\overline{\Omega})$ as $n\to\infty$, it is seen that Γ is continuous in Y. Finally, we prove that ΓC is totally bounded in $L^q(\Omega)$. Since the set

$$\left\{v+\frac{f-v}{\sqrt{\lambda}\beta(\cdot,\nabla v)}:v\in C\right\}$$

is bounded in $L^q(\Omega)$, it suffices to show that the resolvent $\left(I-\sqrt{\lambda}L\right)^{-1}$ maps the unit ball $B\equiv\{v\in L^q(\Omega):|v|_q\le1\}$ into a totally bounded set of $L^q(\Omega)$. Let $v_n\in B$ and $w_n=\left(I-\sqrt{\lambda}L\right)^{-1}v_n$ for $n\ge1$. Then we have

$$|Lw_n|_q\le M_1\lambda^{-1/2}|v_n|_q\le M_1\lambda^{-1/2}.$$

Since the resolvent $\left(I-\sqrt{\lambda}L\right)^{-1}$ is a compact operator, one can choose a subsequence w_{n_k} converging in $L^q(\Omega)$. By the moments inequality

$$|(-L)^\alpha w|_q\le N_\alpha|w|_q^{1-\alpha}|Lw|_q^\alpha\quad\text{for } w\in D(L),\tag{2.5}$$

we obtain

$$\begin{aligned}|(-L)^\alpha(w_{n_k}-w_{n_l})|_q&\le N_\alpha|w_{n_k}-w_{n_l}|_q^{1-\alpha}|L(w_{n_k}-w_{n_l})|_q^\alpha\\&\le 2N_\alpha M_1^\alpha\lambda^{-\alpha/2}|w_{n_k}-w_{n_l}|_q^{1-\alpha}.\end{aligned}$$

This means that $\{w_{n_k}\}$ is a Cauchy sequence in Y, and hence $\left(I-\sqrt{\lambda}L\right)^{-1}B$ is totally bounded in $L^q(\Omega)$. By the Schauder's fixed point theorem, one finds $v \in C$ such that $\Gamma v = v$. This implies that $v \in D(L)$ and

$$\left(I-\sqrt{\lambda}L\right)v = v + \frac{f-v}{\sqrt{\lambda}\beta(\cdot,\nabla v)}. \tag{2.6}$$

Since the right-hand side of (2.6) belongs to $L^\infty(\Omega)$, v must lie in $W^{2,p}(\Omega)$ for all $p > n$. This together with (2.6) implies that $v \in D(A)$ and $(I-\lambda A)v = f$. Therefore, it follows that $Ran(I-\lambda A) = L^\infty(\Omega)$ for $\lambda > 0$.

STEP 2. Let $f \in L^\infty(\Omega)$ and let $\lambda > 0$. Put $\beta_n(x,\xi) \equiv \beta(x,\xi) + 1/n$ for $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Then by the result of Step 1 it follows that for each n there exists $v_n \in D(A)$ such that $v_n - \lambda\beta_n(\cdot,\nabla v_n)Lv_n = f$. This equality can be rewritten as

$$v_n = (I-\lambda L)^{-1}\left(v_n + \frac{f-v_n}{\beta_n(\cdot,\nabla v_n)}\right). \tag{2.7}$$

We now apply the dissipativity of A to get $|v_n|_\infty \le |f|_\infty$. We also have

$$\begin{aligned}|v_n|_q &\le \left|v_n + \frac{f-v_n}{\beta_n(\cdot,\nabla v_n)}\right|_q \\ &\le |v_n|_\infty + |f-v_n|_\infty|\beta_0^{-1}|_q \\ &\le \left(1+2|\beta_0^{-1}|_q\right)|f|_\infty,\end{aligned}$$

and

$$\begin{aligned}|Lv_n|_q &\le M_1\lambda^{-1}\left|v_n + \frac{f-v_n}{\beta_n(\cdot,\nabla v_n)}\right|_q \\ &\le M_1\lambda^{-1}\left(1+2|\beta_0^{-1}|_q\right)|f|_\infty.\end{aligned}$$

Hence $\{v_n\}$ is bounded in the Banach space $D(L)$ equipped with the graph norm. Since the resolvent $(I-\lambda L)^{-1}$ is a compact operator, the sequence $\{v_n\}$ is relatively compact in Y and also relatively compact in $C^1(\overline{\Omega})$. Hence one can find a subsequence $\{v_{n_k}\}$ such that v_{n_k} converges to some v in $C^1(\overline{\Omega})$ as $n \to \infty$. This implies that

$$\frac{f-v_{n_k}}{\beta_{n_k}(\cdot,\nabla v_{n_k})} \to \frac{f-v}{\beta(\cdot,\nabla v)} \qquad \text{in } L^q(\Omega).$$

Replacing v_n in (2.7) by v_{n_k} and letting n_k go to the infinity, we obtain

$$v = (I-\lambda L)^{-1}\left(v + \frac{f-v}{\beta(\cdot,\nabla v)}\right).$$

This means that $v \in D(A)$ and $(I-\lambda A)v = f$. Thus the proof is complete.

COROLLARY 6. *The resolvent* $(I-\lambda A)^{-1}$ *is order-preserving for* $\lambda > 0$.

PROOF. Let $\lambda > 0$ and $u, v \in D(A)$. Put $f = (I-\lambda A)u$, $g = (I-\lambda A)v$ and assume that $f(x) \le g(x)$ for almost all $x \in \Omega$. We wish to show that $u(x) \le v(x)$ in $\overline{\Omega}$ by contradiction. Suppose that this is not valid. Then there must exist $a \in \overline{\Omega}$ such that $u(a) - v(a) = |u-v| = \sup_{x\in\overline{\Omega}} |u(x) - v(x)| > 0$. In the same way as in the proof of Theorem 5, we may assume that $a \in \Omega$ and that there exists a 0-1 measure $\mu_a \in F_0(u-v)$ with the essential support concentrated at a such that $\langle Au - Av, \mu_a\rangle \le 0$. It then follows that

$$\begin{aligned}\langle f-g, \mu_a\rangle &= \langle (I-\lambda A)u - (I-\lambda A)v, \mu_a\rangle \\ &= |u-v| - \lambda\langle Au - Av, \mu_a\rangle > 0.\end{aligned}$$

This contradicts that $f(x) \le g(x)$ a.e. in Ω. Therefore, we obtain that $u(x) \le v(x)$ in $\overline{\Omega}$. This completes the proof.

3. An evolution operators associated with (RDS)

In this section we convert the problem (RDS)-(BC)-(IC) to an abstract quasilinear problem in the product L^∞ space

$$\mathscr{X} = (L^\infty(\Omega))^4$$

which is equipped with the norm defined by $\|\boldsymbol{v}\| = \max_{1\le i\le 4} |v_i|_\infty$ for $\boldsymbol{v} = (v_i)_{i=1}^4 \in \mathscr{X}$. In what follows, the coefficients $d_i(\cdot,\cdot)$ are assumed to satisfy the following condition:

(C3) $d_i(\cdot) \in C([0,\tau]; L^\infty(\Omega))$ and $d_i(t,x) \ge 0$ for $i = 1,2,3$, $t \in [0,\tau]$, and a.e. x in Ω.

Under conditions (C1)–(C3) we define a quasilinear operator $\mathscr{A}$ from $\mathscr{X}$ into $\mathscr{X}$ by

$$D(\mathscr{A}) \equiv \Big\{ \boldsymbol{v} = (v_i) \in \mathscr{X} : v_j \in W^{2,q}(\Omega) \cap W^{2,p}_{loc}(\Omega) \quad \text{for } p > n,$$

$$\beta_j(\cdot, \nabla v_j)\Delta v_j \in L^\infty(\Omega),\ \frac{\partial v_j}{\partial \nu} + a_j(x)v_j = 0 \text{ on } \partial\Omega,\ j = 1,2 \Big\},$$

$$\mathscr{A}\boldsymbol{v} = [\beta_1(\cdot,\nabla v_1)\Delta v_1, \beta_2(\cdot,\nabla v_2)\Delta v_2, 0, 0].$$

The following lemma for the operator $\mathscr{A}$ is a direct consequence of Theorem 5.

LEMMA 7. $\mathscr{A}$ *is m-dissipative in* $\mathscr{X}$.

Next we define nonlinear operators $\widetilde{\mathscr{B}}(t)$, $t \in [0,\tau]$, from $\mathscr{X}$ into $\mathscr{X}$ by

$$\widetilde{D} = D(\widetilde{\mathscr{B}}(t)) \equiv C(\overline{\Omega}) \times C(\overline{\Omega}) \times L^\infty(\Omega) \times L^\infty(\Omega),$$

$$\widetilde{\mathscr{B}}(t)\boldsymbol{v} = (B_1(t)\boldsymbol{v}, B_2(t)\boldsymbol{v}, B_3(t)\boldsymbol{v}, B_4(t)\boldsymbol{v}),$$

$$[B_1(t)\boldsymbol{v}](x) = -d_1(t,x)v_1(x)v_4(x) - d_2(t,x)v_1(x)v_3(x)$$

$$[B_2(t)\boldsymbol{v}](x) = -d_3(t,x)v_2(x)v_4(x) + d_2(t,x)v_1(x)v_3(x)$$

$$[B_3(t)\boldsymbol{v}](x) = \quad d_3(t,x)v_2(x)v_4(x) - d_2(t,x)v_1(x)v_3(x)$$

$$[B_4(t)\boldsymbol{v}](x) = -d_1(t,x)v_1(x)v_4(x) - d_3(t,x)v_2(x)v_4(x).$$

We also employ nonlinear operators $\mathscr{B}(t)$ by taking the restriction of $\widetilde{\mathscr{B}}(t)$ to the set

$$D = D(\mathscr{B}(t)) \equiv C(\overline{\Omega})^+ \times C(\overline{\Omega})^+ \times L^\infty(\Omega)^+ \times L^\infty(\Omega)^+$$

for $t \in [0,\tau]$, where $C(\overline{\Omega})^+$ and $L^\infty(\Omega)^+$ denote the positive cones of $C(\overline{\Omega})$ and $L^\infty(\Omega)$, respectively. Then the reaction-diffusion system (RDS)-(BC)-(IC) is converted to an abstract quasilinear problem in $\mathscr{X}$

(QP)
$$\frac{d}{dt}\boldsymbol{v}(t) = \mathscr{A}\boldsymbol{v}(t) + \mathscr{B}(t)\boldsymbol{v}(t), \quad t \in (0,\tau)$$
$$\boldsymbol{v}(0) = \boldsymbol{w}.$$

Since D is defined as the product of positive cones of $C(\overline{\Omega})$ and $L^\infty(\Omega)$, our problem is to seek the solutions to (QP) in the cylindrical domain $\mathscr{D} = [0,\tau] \times D$.

In order to define Lipschitz continuity in a local sense of $\widetilde{\mathscr{B}}(t)$, we employ a continuous functional $\varphi : \mathscr{X} \to [0,\infty]$ defined by

$$\varphi(\boldsymbol{v}) = |v_1|_\infty + |v_2|_\infty + |v_4|_\infty + |v_3 + v_4|_\infty + |v_1|_\infty |v_3 + v_4|_\infty.$$

Let $\widetilde{D}_\alpha = \{\boldsymbol{v} \in \widetilde{D} : \varphi(\boldsymbol{v}) \le \alpha\}$ and $D_\alpha = \widetilde{D}_\alpha \cap D$ for $\alpha > 0$. The next lemma shows that $\widetilde{\mathscr{B}}(t)\boldsymbol{v}$ is Lipschitz continuous on bounded sets in $\widetilde{D}$ with respect to $\boldsymbol{v}$ and continuous in t in a local sense.

LEMMA 8. *Let $d_0 = \max_{1\le i\le 3}\max_{t\in[0,\tau]} |d_i(t,\cdot)|_\infty$. Then we have :*

(a) *For $\alpha > 0$, $t \in [0,\tau]$ and $\boldsymbol{v}$, $\boldsymbol{w} \in \widetilde{D}_\alpha$*

$$\|\widetilde{\mathscr{B}}(t)\boldsymbol{v} - \widetilde{\mathscr{B}}(t)\boldsymbol{w}\| \le 4d_0\alpha\|\boldsymbol{v} - \boldsymbol{w}\|.$$

(b) *For $\alpha > 0$, s, $t \in [0,\tau]$ and $\boldsymbol{v} \in \widetilde{D}_\alpha$*

$$\|\widetilde{\mathscr{B}}(s)\boldsymbol{v} - \widetilde{\mathscr{B}}(t)\boldsymbol{v}\| \le \sum_{i=1}^{3} |d_i(s) - d_i(t)|_\infty \alpha^2.$$

PROOF. We first show that (a) is valid. Let $\alpha > 0$, $t \in [0,\tau]$ and let $\boldsymbol{v} = (v_i)$, $\boldsymbol{w} = (w_i) \in \widetilde{D}_\alpha$. In view of the fact that $v_i(x)$, $w_i(x) \ge 0$ for $i = 1,2,3,4$ and $x \in \Omega$, we have

$$|B_1(t)\boldsymbol{v} - B_1(t)\boldsymbol{w}|_\infty = |-d_1(t,\cdot)v_1v_4 - d_2(t,\cdot)v_1v_3 + d_1(t,\cdot)w_1w_4 + d_2(t,\cdot)w_1w_3|_\infty$$
$$\le d_0((|w_3|_\infty + |w_4|_\infty)|v_1 - w_1|_\infty + |v_1|_\infty|v_3 - w_3|_\infty + |v_1|_\infty|v_4 - w_4|_\infty).$$

Similarly, we have

$$|B_2(t)\boldsymbol{v} - B_2(t)\boldsymbol{w}|_\infty \le d_0(|w_3|_\infty|v_1 - w_1|_\infty + |w_4|_\infty|v_2 - w_2|_\infty + |v_1|_\infty|v_3 - w_3|_\infty + |v_2|_\infty|v_4 - w_4|_\infty),$$
$$|B_3(t)\boldsymbol{v} - B_3(t)\boldsymbol{w}|_\infty \le d_0(|w_3|_\infty|v_1 - w_1|_\infty + |w_4|_\infty|v_2 - w_2|_\infty + |v_1|_\infty|v_3 - w_3|_\infty + |v_2|_\infty|v_4 - w_4|_\infty),$$
$$|B_4(t)\boldsymbol{v} - B_4(t)\boldsymbol{w}|_\infty \le d_0(|w_4|_\infty|v_1 - w_1|_\infty + |w_4|_\infty|v_2 - w_2|_\infty + (|v_1|_\infty + |v_2|_\infty)|v_4 - w_4|_\infty).$$

Combining these estimates with the trivial relation $|w_3|_\infty \le |w_3+w_4|_\infty+|w_4|_\infty$, we see that (a) is valid. Let $\alpha > 0$, $t \in [0,\tau]$ and let $\boldsymbol{v} = (v_i) \in \widetilde{D}_\alpha$. Noting that $|v_1v_4|, |v_1v_3|, |v_2v_4| \le \alpha^2$, we have

$$\|\widetilde{\mathscr{B}}(s)\boldsymbol{v} - \widetilde{\mathscr{B}}(t)\boldsymbol{v}\| \le \sum_{i=1}^{3} |d_i(s) - d_i(t)|\alpha^2.$$

This shows that (b) holds. Thus the proof is complete.

The next lemma shows that the family $\{\mathscr{A} + \mathscr{B}(t)\}$ of quasilinear operators satisfies the range conditions and growth condition:

Lemma 9. *Let $s \in [0,\tau)$, $\alpha > 0$ and let $\lambda(s,\alpha) = \min\{(8d_0(5\alpha + 4\alpha^2))^{-1}, \tau - s\}$. Then, for $\boldsymbol{f} = (f_i) \in D_\alpha$ and $\lambda \in (0, \lambda(s,\alpha))$, there exists $\boldsymbol{v}_\lambda = (v_{i,\lambda}) \in D$ such that*

$$\boldsymbol{v}_\lambda - \lambda(\mathscr{A} + \mathscr{B}(s+\lambda))\boldsymbol{v}_\lambda = \boldsymbol{f}$$

and

$$\varphi(\boldsymbol{v}_\lambda) \le (1 - d_0\lambda)^{-1}\varphi(\boldsymbol{f}).$$

Proof. Let $s \in [0,\tau)$, $\alpha \ge 0$. Let $\boldsymbol{f} = (f_i) \in D_\alpha$ and $\lambda \in (0, \lambda(s,\alpha))$. We define an operator $\mathscr{F}$ from $\mathscr{C} \equiv \{\boldsymbol{v} \in \widetilde{D} : \|\boldsymbol{v}\| \le 2\|\boldsymbol{f}\|\}$ into $\widetilde{D}$ by

$$\mathscr{F}\boldsymbol{v} = (I - \lambda\mathscr{A})^{-1}(\boldsymbol{f} + \lambda\widetilde{\mathscr{B}}(s+\lambda)\boldsymbol{v}) \qquad \text{for } \boldsymbol{v} \in \mathscr{C}.$$

Since $\|\boldsymbol{v}\| \le \varphi(\boldsymbol{v}) \le 5\|\boldsymbol{v}\| + 2\|\boldsymbol{v}\|^2$ for $\boldsymbol{v} \in \mathscr{X}$, it follows from Lemmas 7 and 8 that

$$\|\mathscr{F}\boldsymbol{v}\| \le \|\boldsymbol{f}\| + 4\lambda d_0(10\|\boldsymbol{f}\| + 8\|\boldsymbol{f}\|^2)\|\boldsymbol{v}\| \le 2\|\boldsymbol{f}\|$$

and

$$\|\mathscr{F}v - \mathscr{F}w\| \leq 8\lambda d_0(5\alpha + 4\alpha^2)\|v - w\|.$$

Thus, the Contraction Mapping Principle implies that $\mathscr{F}$ has a fixed point $v_\lambda = (v_{i,\lambda}) \in \mathscr{C}$, and v_λ satisfies

$$v_\lambda - \lambda(\mathscr{A} + \widetilde{\mathscr{B}}(s+\lambda))v_\lambda = f. \tag{3.1}$$

We next show that $v_\lambda \in D$. In view of the definitions of the operators $\mathscr{A}$ and $\widetilde{\mathscr{B}}(s+\lambda)$, equation (3.1) can be written as

$$v_{1,\lambda} - \lambda A_1 v_{1,\lambda} = f_1 - \lambda(d_1(t,\cdot)v_{1,\lambda}v_{4,\lambda} + d_2(t,\cdot)v_{1,\lambda}v_{3,\lambda}) \tag{3.2}$$

$$v_{2,\lambda} - \lambda A_2 v_{2,\lambda} = f_2 - \lambda(d_3(t,\cdot)v_{2,\lambda}v_{4,\lambda} - d_2(t,\cdot)v_{1,\lambda}v_{3,\lambda}) \tag{3.3}$$

$$v_{3,\lambda} = f_3 + \lambda(d_3(t,\cdot)v_{2,\lambda}v_{4,\lambda} - d_2(t,\cdot)v_{1,\lambda}v_{3,\lambda}) \tag{3.4}$$

$$v_{4,\lambda} = f_4 - \lambda(d_1(t,\cdot)v_{1,\lambda}v_{4,\lambda} + d_2(t,\cdot)v_{2,\lambda}v_{4,\lambda}) \tag{3.5}$$

The identity (3.5) yields

$$v_{4,\lambda}(x)(1 + \lambda(d_1(t,x)v_{1,\lambda}(x) + d_2(t,x)v_{2,\lambda}(x))) = f_4(x) \geq 0 \quad a.e. \text{ in } \Omega.$$

Since $|d_1(t,x)v_{1,\lambda}(x) + d_2(t,x)v_{2,\lambda}(x)| \leq d_0(|v_{1,\lambda}|_\infty + |v_{2,\lambda}|_\infty) \leq 2d_0\alpha$ for almost all $x \in \Omega$, we have $v_{4,\lambda}(x) \geq 0$ a.e. in Ω. Suppose that $v_{1,\lambda}$ attains its minimum at $a \in \overline{\Omega}$ and $v_{1,\lambda}(a) < 0$. As in the proof of Theorem 5, we may assume that $a \in \Omega$. Then,by Proposition 2, there exists a 0-1 measure μ_a with the essential support concentrated at a such that $\langle \Delta v_{1,\lambda}, \mu_a \rangle \geq 0$. We infer from (3.2) that

$$v_{1,\lambda}(a)\langle 1 + \lambda(d_1(t,\cdot)v_{4,\lambda} + d_2(t,\cdot)v_{3,\lambda}), \mu_a\rangle = \langle f_1, \mu_a\rangle + \lambda\beta_1(a, \nabla v_{1,\lambda}(a))\langle \Delta v_{1,\lambda}, \mu_a\rangle \geq 0.$$

Since $\langle 1 + \lambda(d_1(t,\cdot)v_{4,\lambda} + d_2(t,\cdot)v_{3,\lambda}, \mu_a\rangle \geq 1 - 2\lambda d_0\alpha > 0$, it holds that $v_{1,\lambda}(a) \geq 0$. This is a contradiction, and hence $v_{1,\lambda}(x) \geq 0$ for $x \in \Omega$. Adding (3.3) and (3.4), we have

$$v_{2,\lambda} + v_{3,\lambda} - \lambda\beta_2(\cdot, \nabla v_{2,\lambda})\Delta v_{2,\lambda} = f_2 + f_3. \tag{3.6}$$

Let $b \in \Omega$ be such that $v_{2,\lambda}(b) = \min_{x\in\overline{\Omega}} v_{2,\lambda}(x)$, and suppose that $v_{2,\lambda}(b) < 0$. Then there exists a 0-1 measure μ_b with the essential support concentrated at b such that $\langle \Delta v_{2,\lambda}, \mu_b\rangle \geq 0$. Then (3.4) and (3.6) together imply

$$v_{2,\lambda}(b) + \langle v_{3,\lambda}, \mu_b\rangle \geq 0,$$

$$\langle v_{3,\lambda}, \mu_b\rangle(1 + \lambda\langle d_2(t,\cdot)v_{1,\lambda}, \mu_b\rangle) \geq \lambda\langle d_3(t,\cdot)v_{4,\lambda}, \mu_b\rangle v_{2,\lambda}(b)$$

Combining these two inequalities, we obtain

$$\langle v_{3,\lambda}, \mu_b\rangle(1 + \lambda(\langle d_2(t,\cdot)v_{1,\lambda}, \mu_b\rangle + \langle d_3(t,\cdot)v_{4,\lambda}, \mu_b\rangle)) \geq 0,$$

and hence $\langle v_{3,\lambda}, \mu_b\rangle \geq 0$. This together with (3.3)and the relation $v_{1,\lambda}(x) \geq 0$ implies that

$$v_{2,\lambda}(b)\langle 1 - \lambda d_3(t,\cdot)v_{4,\lambda}, \mu_b\rangle \geq \lambda\langle A_2 v_{2,\lambda}, \mu_b\rangle + \lambda v_{1,\lambda}(b)\langle d_2(t,\cdot), \mu_b\rangle\langle v_{3,\lambda}, \mu_b\rangle \geq 0.$$

This contradicts $v_{2,\lambda}(b) < 0$. Thus we have $v_{2,\lambda}(x) \geq 0$ for $x \in \Omega$. It follows from (3.4) that

$$v_{3,\lambda}(x)(1 - \lambda d_2(t,x)v_{1,\lambda}(x)) = f_3(x) + \lambda d_3(t,x)v_{2,\lambda}(x)v_{4,\lambda}(x) \geq 0.$$

This yields that $v_{3,\lambda}(x) \geq 0$ a.e. in Ω. Therefore, we have proved that $\boldsymbol{v}_\lambda \in D$.

Finally, we demonstrate that $\varphi(\boldsymbol{v}_\lambda) \leq (1 - d_0\lambda)^{-1}\varphi(\boldsymbol{f})$. Since $v_{1,\lambda}(x) \geq 0$ for $x \in \Omega$ and $d_j(t,x) \geq 0$ for $j = 1,2,3$ and almost all $x \in \Omega$, we have

$$0 \leq f_1(x) + \lambda[B_1(s+\lambda)\boldsymbol{v}_\lambda](x) \leq f_1(x)$$

for almost all $x \in \Omega$. This implies that $|f_1 + \lambda B_1(s+\lambda)\boldsymbol{v}_\lambda|_\infty \leq |f_1|_\infty$. Hence we have $|(I - \lambda A_1)^{-1}(f_1 + \lambda B_1(s+\lambda)\boldsymbol{v}_\lambda|_\infty \leq |f_1|_\infty$. Similarly, we can show

$$|(I - \lambda A_2)^{-1}(f_2 + \lambda B_2(s+\lambda)\boldsymbol{v}_\lambda)|_\infty \leq |f_2|_\infty + \lambda d_0 |v_{1,\lambda}|_\infty |v_{3,\lambda} + v_{4,\lambda}|_\infty,$$

$$|f_3 + f_4 + \lambda B_3(s+\lambda)\boldsymbol{v}_\lambda + \lambda B_4(s+\lambda)\boldsymbol{v}_\lambda|_\infty \leq |f_3 + f_4|_\infty,$$

$$|f_4 + \lambda B_4(s+\lambda)\boldsymbol{v}_\lambda|_\infty \leq |f_4|_\infty.$$

Combining these estimates gives $\varphi(\boldsymbol{v}_n) \leq (1 - d_0\lambda)^{-1}\varphi(\boldsymbol{v})$. This completes the proof.

We are now in a position to state our main result:

THEOREM 10. *Suppose that* (C1) *through* (C3) *hold. Then the family of nonlinear operators* $\{\mathscr{A} + \mathscr{B}(t)\}$ *generates a nonlinear evolution operator* $\mathscr{U} \equiv \{U(t,s) : 0 \leq s \leq t \leq \tau\}$ *on* D *such that*

$$\varphi(U(t,0)\boldsymbol{v}) \leq e^{d_0 t}\varphi(\boldsymbol{v}), \qquad \text{for } \boldsymbol{v} \in D \text{ and } t \in [0,\tau].$$

If the coefficients $d_i(\cdot)$, $i = 1,2,3$, *belong to* $BV([0,\tau]; L^\infty(\Omega))$, *then for* $\boldsymbol{v} \in D(\mathscr{A}) \cap D$ *the* $\mathscr{X}$*-valued function* $\boldsymbol{u}(t) \equiv U(t,0)\boldsymbol{v}$ *gives a weakly-star continuously differentiable solution to* (QP) *in* $\mathscr{X}$.

PROOF. Let $D(\mathfrak{A}(t)) \equiv D(\mathscr{A}) \cap D$ and $\mathfrak{A}(t) \equiv \mathscr{A} + \mathscr{B}(t)$ for $t \in [0,\tau]$. By Lemmas 7, 8 and 9, it is seen that $\{\mathfrak{A}(t)\}$ satisfies all the assumptions of Theorem 4 for $a = d_0$ and $b = 0$. Hence it follows that there exists a nonlinear evolution operator $\mathscr{U} \equiv \{U(t,s) : 0 \leq s \leq t \leq \tau\}$ satisfying the growth condition

$$\varphi(U(t,0)\boldsymbol{v}) \leq e^{d_0 t}\varphi(\boldsymbol{v}) \quad \text{for } \boldsymbol{v} \in D \text{ and } t \in [0,\tau].$$

Next we assume that $d_i(\cdot) \in BV([0,\tau]; L^\infty(\Omega))$ for $i = 1,2,3$. Let $\alpha > 0$ and put $\gamma = e^{2d_0\tau}\alpha$, $\omega_\gamma = 4d_0\alpha$ and $\theta_\gamma(s,t) = \sum_{i=1}^{3} |d_i(s) - d_i(t)|_\infty \gamma^2$. Let $\boldsymbol{v} \in D(\mathscr{A}) \cap D_\alpha$. We then take a natural number n satisfying $n > 8d_0(5\gamma + 4\gamma^2)$, set $t_k^n = k\tau n^{-1}$ for $k = 0,1,\ldots,n$, and define

$$\begin{aligned} &\boldsymbol{v}_0 = \boldsymbol{v}, \\ (3.7) \qquad &\boldsymbol{v}_k = \prod_{i=1}^{k}\left(I - \frac{1}{n}\mathfrak{A}(t_k^n)\right)^{-1}\boldsymbol{v} \quad \text{for } k = 1,2,\ldots,n, \\ &\boldsymbol{u}_n(t) = \boldsymbol{v}_k \quad \text{for } t \in (t_{k-1}^n, t_k^n] \text{ and } \boldsymbol{u}_n(0) = \boldsymbol{v}. \end{aligned}$$

Then applying Lemma 9 and [10, Theorem 3.5], one can assert that $\varphi(u_n(t)) \le e^{2d_0\tau}\alpha = \gamma$ for $t \in [0,\tau]$, and that $\boldsymbol{u}_n(t)$ converges to $\boldsymbol{u}(t) \equiv U(t,0)\boldsymbol{v}$ for $t \in [0,\tau]$ in $\mathscr{X}$ as n tends to the infinity. It is easily seen from (H2) that the nonlinear operator $\mathfrak{A}(t)$ satisfies

$$(1-\lambda\omega_\gamma)\|\boldsymbol{v}-\boldsymbol{w}\| \le \|(I-\lambda\mathfrak{A}(s))\boldsymbol{v}-(I-\lambda\mathfrak{A}(t))\boldsymbol{w}\| + \lambda\theta_\gamma(s,t)$$

for $\lambda \in (0,\omega_\gamma^{-1})$, $s,t \in [0,\tau]$, $\boldsymbol{v} \in D(\mathfrak{A}(s)) \cap D_\gamma$, and $\boldsymbol{w} \in D(\mathfrak{A}(t)) \cap D_\gamma$. Replacing $\boldsymbol{v}$ and $\boldsymbol{w}$ by $\boldsymbol{v}_k$ and $\boldsymbol{v}_{k-1}$, respectively, and using the relation $(I-\tau/n\mathfrak{A}(t_k^n))\,\boldsymbol{v}_k = \boldsymbol{v}_{k-1}$, we have

$$\left(1-\frac{\tau}{n}\omega_\gamma\right)\|\mathfrak{A}(t_k^n)\boldsymbol{v}_k\| \le \left\|\mathfrak{A}(t_{k-1}^n)\boldsymbol{v}_{k-1}\right\| + \theta_\gamma(t_{k-1}^n,t_k^n) \qquad \text{for } k=1,2,\ldots,n.$$

This implies that

$$\text{(3.8)}\quad \|\mathscr{A}\boldsymbol{v}_k\| \le \|\mathfrak{A}(t_k^n)\boldsymbol{v}_k\| + \|\mathscr{B}(t_k^n)\boldsymbol{v}_k\| \le e^{2\omega_\gamma\tau}\left(\|\mathfrak{A}(0)\boldsymbol{v}_0\| + \sum_{i=1}^{3} TV(d_i)\gamma^2\right) + 4d_0\gamma^2$$

for $k=1,2,\ldots,n$, where $TV(d_i)$ denotes the total variation of d_i for $i=1,2,3$. If we write $\boldsymbol{u}_n(t) = (u_{i,n}(t))$, then we have

$$|Lu_{i,n}(t)|_q \le |\beta_0^{-1}|_q\left(e^{2\omega_\gamma\tau}\left(\|\mathfrak{A}(0)\boldsymbol{v}_0\| + \sum_{i=1}^{3} TV(d_i)\right) + 4d_0\gamma^2\right) \qquad \text{for } t\in[0,\tau] \text{ and } i=1,2.$$

5 Hence, the set $\{Lu_{i,n}; n > 8d_0(5\gamma+4\gamma^2)\}$ is bounded in $L^\infty(0,\tau;L^q(\Omega))$ for $i=1,2$. Since $L^\infty(0,\tau;L^q(\Omega))$ is the dual space of $L^1(0,\tau;L^r(\Omega))$, where r is the Hölder conjugate to q, it is seen that $Lu_{i,n}(t)$ converges weakly to $Lu_i(t)$ in $L^q(\Omega)$ for each $t\in[0,\tau]$. Let $\delta \in (1/2+n/(2q),1)$. Then the moments inequality (2.5) implies that $u_{i,n}(t)$ converges uniformly to $u_i(t)$ on $[0,\tau]$ in the Banach space $D((-L)^\delta)$ equipped with the graph norm. Since $D((-L)^\delta)$ is continuously imbedded in $C^1(\overline{\Omega})$, it follows that $u_{i,n}(t)$ converges uniformly to $u_i(t)$ on $[0,\tau]$ in $C^1(\overline{\Omega})$. Combining these facts with the strong convergence of the sequences $u_{j,n}(t)$, $j=3,4$, we assert that $\mathscr{A}\boldsymbol{u}_n(t)$ converges to $\mathscr{A}\boldsymbol{u}(t)$ for each $t\in[0,\tau]$ in the weak-star topology of $\mathscr{X}$. It is also shown that $\mathscr{A}\boldsymbol{u}(\cdot)$ is weakly-star continuous on $[0,\tau]$. Since, for each $t\in[0,\tau]$, $\mathscr{B}(t_k^n)\boldsymbol{u}_n(t)$ converges to $\mathscr{B}(t)\boldsymbol{u}(t)$ and $\mathscr{B}(\cdot)\boldsymbol{u}(\cdot)$ is continuous in $\mathscr{X}$, it follows that $\mathfrak{A}(t)\boldsymbol{u}_n(t)$ converges to $\mathfrak{A}(t)\boldsymbol{u}(t)$ for $t\in[0,\tau]$ in the weak-star topology of $\mathscr{X}$ and that $\mathfrak{A}(\cdot)\boldsymbol{u}(\cdot)$ is weakly-star continuous. For simplicity in notation we define

$$\sigma_n(t) = \begin{cases} 0 & t=0 \\ t_k & t\in(t_{k-1}^n,t_k^n] \quad \text{for } k=1,2,\ldots,n. \end{cases}$$

Then by the definition of $\boldsymbol{u}_n(t)$ we have

$$\boldsymbol{u}_n(t) - \boldsymbol{v} = \int_0^{\sigma_n(t)} (\mathscr{A} + \mathscr{B}(\sigma_n(s)))\boldsymbol{u}_n(s)\,ds \qquad \text{for } t\in[0,\tau].$$

Letting n go to the infinity, we obtain

$$\boldsymbol{u}(t) - \boldsymbol{v} = w^*\text{-}\int_0^t (\mathscr{A} + \mathscr{B}(s))\boldsymbol{u}(s)\,ds \qquad \text{for } t\in[0,\tau],$$

where the integral is taken in the sense of Riemann integral with respect to the weak-star topology in $\mathscr{X}$. Since the integrand of the above equality is weakly-star continuous, we conclude that $\boldsymbol{u}(t)$ is weakly-star continuously differentiable and satisfies

$$w^*\text{-}\frac{d}{dt}\boldsymbol{u}(t) = (\mathscr{A} + \mathscr{B}(t))\boldsymbol{u}(t), \qquad \text{for } t \in (0, \tau). \tag{3.9}$$

Thus the proof is complete.

COROLLARY 11. *Let $\boldsymbol{v} \in D(\mathscr{A}) \cap D$. Let $\boldsymbol{u}(\cdot) = (u_i(t))$ be a weakly-star differentiable solution to* (QP) *obtained by Theorem 10. Then $u_j \in C^1([0,\tau]; L^\infty(\Omega))$ for $j = 3,4$ and $u_i \in Lip([0,\tau]; L^\infty(\Omega)) \cap L^\infty(0,\tau; D(L)) \cap C^\nu([0,\tau]; C^1(\overline{\Omega}))$ for $\nu \in (0, 1/2 - n/(2q))$ and $i = 1,2$.*

PROOF. Since $\mathscr{B}(\cdot)\boldsymbol{u}(\cdot)$ is strongly continuous in $\mathscr{X}$, it follows from the definition of the operator $\mathscr{A}$ and (3.9) that $u_j \in C^1([0,\tau]; L^\infty(\Omega))$ for $j = 3,4$. It is a direct consequence of Theorem 4 that $u_i \in Lip([0,\tau]; L^\infty(\Omega))$ for $i = 1,2$. From the proof of Theorem 10 we see that $u_i \in L^\infty(0,\tau; D(L))$ for $i = 1,2$. It now remains to show that $u_i \in C^\nu([0,\tau]; C^1(\overline{\Omega}))$. Let $\delta \in (1/2 + n/(2q), 1)$. Since $u_i \in Lip([0,\tau]; L^\infty(\Omega)) \cap L^\infty(0,\tau; D(L))$, one can apply the moments inequality to assert that

$$\begin{aligned} |u(t) - u(s)|_{C^1(\overline{\Omega})} &\le |(-L)^\delta(u(t) - u(s))|_q \\ &\le N_\delta |u(t) - u(s)|_q^{1-\delta} |Lu(t) - Lu(s)|_q^\delta \le C|t - s|^{1-\delta}, \end{aligned}$$

where C is a positive constant. Thus we obtain the desired result.

ACKNOWLEDGEMENT. The authors express their gratitude to Professor S. Oharu for his valuable suggestions and comments.

References

[1] B. C. Burch and J. A. Goldstein, Nonlinear semigroups and a problem in heat equilibrium, Houston J. Math. **4** (1978), 311–328.

[2] J. R. Dorroh and G. R. Rieder, A singular quasilinear parabolic problem in one space dimension, J. Differential Equations **91** (1991), 1–23.

[3] P. Fife, Mathematical aspects of reacting and diffusion systems, Lecture Notes in Biomath., **28**, Springer-Verlag, Berlin, 1979.

[4] J. A. Goldstein, K. Hashimoto and S. Oharu, The duality mapping of $L^\infty[0,1]$ and its application to nonlinear degenerate diffusion operators, preprint.

[5] J. A. Goldstein and C.-Y. Lin, Singular nonlinear parabolic boundary value problems in one space dimensions, J. Differential Equations **68** (1987), 429–443.

[6] J. A. Goldstein and C.-Y. Lin, Highly degenerate parabolic boundary value problems, Differential Integral Equations **2** (1989), 216–227.

[7] J. A. Goldstein and C.-Y. Lin, Degenerate parabolic problems and the Wentzel boundary condition, Semigroup Theory and Applications (Trieste, 1987), Lecture Notes in Pure and Appl. Math., 116, Dekker, New York, 1989, 189–199,

[8] J. A. Goldstein, C.-Y. Lin and K. Wang, Degenerate nonlinear parabolic problems, The Influence of Probability Theory, Stochastic Processes and Functional Analysis (Riverside, CA, 1994), Lecture Notes in Pure and Appl. Math., 186, Dekker, New York, 1997, 101–111.

[9] D. Henry, Geometric theory of semilinear parabolic equations, Lecture Notes in Mathematics, **840**, Springer-Verlag, Berlin, Heidelberg 1981.

[10] K. Kobayasi, Y. Kobayashi and S. Oharu, Nonlinear evolution operators in Banach spaces, Osaka J. Math., **21** (1984), 281–310.

[11] K. Kobayasi and S. Oharu, On nonlinear evolution operators associated with certain nonlinear equations of evolution, Mathematical Analysis on Structures in Nonlinear Phenomena, Lecture Notes in Num. Appl. Anal., **2** (1980), 139–210.

[12] C.-Y. Lin, Degenerate nonlinear parabolic boundary value problems, Nonlinear Anal., **13** (1989), 1303–1315.

[13] M. Mimura and A. Nakaoka, On some degenerate diffusion system related with a certain reaction system, J. Math. Kyoto Univ. (JMKYAZ), **12-1**, (1972), 95–121.

[14] T. Matsumoto, Time-dependent nonlinear perturbations of integrated semigroups, Nihonkai Math. J., **7** (1996), 1–28.

[15] J. J. Peiris, On the duality mappings of L^∞ spaces, to appear in Hiroshima Math. J.

[16] H. Serizawa, M-Browder-accretiveness of a quasi-linear differential operator, Houston J. Math. **10** (1984), 147–152.

Department of Mathematics
Faculty of Science
Hiroshima University
Higashi-Hiroshima 739-8526, Japan

Received December 24, 1997