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Von Neumann-Jordan constant and uniformly non-square Banach spaces

Yasuji TAKAHASHI* and Mikio KATO†

Abstract. A sequence of characterizations of uniform non-squareness is given, some of which are similar to the well-known homogeneous characterization of uniformly convex spaces. As corollaries: (i) Banach spaces with von Neumann-Jordan constant less than 2 are characterized as those uniformly non-square; (ii) it is presented that uniform non-squareness is inherited by dual spaces.

1. Introduction and preliminaries

According to Clarkson [5] the *von Neumann-Jordan (NJ-) constant* of a Banach space X, we denote it by $C_{NJ}(X)$, is the smallest constant C for which

$$\frac{1}{C} \leqq \frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)} \leqq C \qquad (1)$$

holds for all x, y $\in$ X with $\|x\|^2+\|y\|^2 \neq 0$. (Note that the first and second inequalities of (1) are equivalent; put $x+y=u$, $x-y=v$.) Classical results state that: (i) $1 \leqq C_{NJ}(X) \leqq 2$ for any Banach space

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X, and X is a Hilbert space if and only if $C_{NJ}(X) = 1$ (Jordan and von Neumann [13]). (ii) $C_{NJ}(L_p) = 2^{2/t-1}$, where $t = \min\{p, p'\}$, $1/p + 1/p' = 1$ (Clarkson [5]). For the NJ-constant of some other Banach spaces we refer the reader to [16, 15, 19]. Recently the authors [18] showed that the uniform convexity of a Banach space X is nearly characterized by the condition $C_{NJ}(X) < 2$: More precisely, if X is uniformly convex, then $C_{NJ}(X) < 2$, while conversely if $C_{NJ}(X) < 2$, X admits an equivalent uniformly convex norm. In other words, X is super-reflexive if and only if $\widetilde{C}_{NJ}(X) < 2$, where $\widetilde{C}_{NJ}(X)$ denotes the infimum of all NJ-constants of equivalent norms of X.

In this paper we first present a sequence of characterizations of uniformly non-square spaces, some of which are similar to the well-known homogeneous characterization of uniformly convex spaces. It is in particular observed that uniform non-squareness is characterized by behavior of norms of the Littlewood matrix as operators between $l_r^2(X)$-spaces. As direct consequences; (i) those Banach spaces with NJ-constant less than 2 are precisely characterized as uniformly non-square spaces, which improves the authors' result stated above; (ii) it is obtained that uniform non-squareness is inherited by dual spaces, which seems not to have appeared in literature. The same for super-reflexivity (James [12]) is immediately derived as well.

Let us recall some definitions and previous results (cf. [1, 7]).

DEFINITIONS. Let B_X denote the closed unit ball of a Banach space X. X is called *uniformly convex* if for any ε $(0 < \varepsilon < 2)$ there exists a $\delta > 0$ such that $\|(x + y)/2\| < 1 - \delta$ whenever $\|x - y\| \geqq \varepsilon$, x, y $\in B_X$. X is said to be *uniformly non-square* ([11]) if there exists a $\delta > 0$ such that $\|(x + y)/2\| \leqq 1 - \delta$ whenever $\|(x - y)/2\| > 1 - \delta$,

$x, y \in B_X$. A Banach space Y is said to be *finitely representable in* X provided for any $\lambda > 1$ and for any finite-dimensional subspace F of Y there is an isomorphism T of F into X for which

$$\lambda^{-1} \|x\| \leqq \|Tx\| \leqq \lambda \|x\| \qquad \text{for all } x \in F.$$

X is said to be *super-reflexive* ([12; cf. 1, 7, 27]) if no non-reflexive Banach space is finitely representable in X.

It is well known that uniformly convex spaces are uniformly non-square, and uniformly non-square spaces are super-reflexive; the converse is not true in each assertion ([11, 12]; cf. [1, 7, 27], see also [18]). Super-reflexive spaces are just those uniformly convexifiable ([8]; cf. [24]); these spaces are also characterized by means of the NJ-constant as follows:

THEOREM A (Kato and Takahashi [18]). A Banach space X is super-reflexive if and only if $\tilde{C}_{NJ}(X) < 2$, where $\tilde{C}_{NJ}(X)$ denotes the infimum of all NJ-constants of equivalent norms of X.

In the following let $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and let $l_r^2(X)$ denote the X-valued l_r^2-space.

2. Homogeneous characterizations of uniformly non-square spaces

We first recall the following characterization of uniformly convex spaces (see e.g., [22, 1]):

PROPOSITION A. Let $1 < p < \infty$. A Banach space X is uniformly convex if and only if for any $\varepsilon > 0$ there exists $\delta = \delta_p(\varepsilon) > 0$ such that $\|x - y\| \geqq \varepsilon$, $x, y \in B_X$ implies

$$\left\| \frac{x+y}{2} \right\|^p \leqq (1-\delta)\frac{\|x\|^p + \|y\|^p}{2}. \tag{2}$$

Now let us present some similar homogeneous characterizations for uniformly non-square spaces. We need the following lemma which is easily seen:

LEMMA 1. Let $1<p<\infty$. Then the function $\phi(t)=(1+t)^p/(1+t^p)$ $(0\leqq t\leqq 1)$ is strictly increasing.

THEOREM 1. Let $1 < p < \infty$. For a Banach space X the following are equivalent:

(i) X is uniformly non-square.

(ii) There exist some ε and δ $(0<\varepsilon, \delta<1)$ such that $\|x-y\| \geqq 2(1-\varepsilon)$, $x, y \in B_X$ implies

$$\left\| \frac{x+y}{2} \right\|^p \leqq (1-\delta)\frac{\|x\|^p + \|y\|^p}{2}. \tag{2}$$

(iii) There exists some δ $(0<\delta<1)$ such that $\|x-y\| \geqq 2(1-\delta)$, $x, y \in B_X$ implies the inequality (2).

(iv) There exists some δ $(0 < \delta < 2)$ such that for any x, y in X,

$$\left\| \frac{x+y}{2} \right\|^p + \left\| \frac{x-y}{2} \right\|^p \leqq (2-\delta)\frac{\|x\|^p + \|y\|^p}{2}. \tag{3}$$

(v) $\|A : l_p^2(X) \to l_p^2(X)\| < 2$.

Proof. (i) ⇒ (ii): Note first that the assertion (ii) is equivalent to

(ii') There exist some ε and δ $(0 < \varepsilon, \delta < 1)$ such that $\|x - y\| \geqq 2(1 - \varepsilon)$, $\|x\| = 1$, $\|y\| \leqq 1$ implies (2).

Now, assume (ii') fails to hold. Then for any positive integer n there exist x_n and y_n in X such that $\| x_n - y_n \| \geqq 2(1 - 1/n)$, $\| x_n \| = 1$, $\| y_n \| \leqq 1$, and

$$\left\| \frac{x_n + y_n}{2} \right\|^p > (1 - \frac{1}{n}) \frac{\| x_n \|^p + \| y_n \|^p}{2}.$$

Since

$$2(1 - \frac{1}{n}) \leqq \| x_n - y_n \| \leqq 1 + \| y_n \| \leqq 2,$$

we have $\| y_n \| \to 1$ and

$$\left\| \frac{x_n - y_n}{2} \right\| \to 1 \tag{4}$$

as $n \to \infty$. On the other hand, since

$$(1 - \frac{1}{n}) \frac{1 + \| y_n \|^p}{2} < \left\| \frac{x_n + y_n}{2} \right\|^p \leqq 1,$$

we have

$$\left\| \frac{x_n + y_n}{2} \right\| \to 1 \quad \text{as } n \to \infty. \tag{5}$$

From (4) and (5) it follows that X is not uniformly non-square.

(ii) ⇒ (iii): This is a direct consequence of the fact that in the assertion (ii) we may replace ε and δ with any $\varepsilon' < \varepsilon$ and $\delta' < \delta$.

The assertion (iii) ⇒ (i) is clear.

(i) ⇒ (iv): Assume (i). Suppose that (iv) fails to hold. Then for any positive integer n there exist x_n and y_n in X such that

$$\left\| \frac{x_n + y_n}{2} \right\|^p + \left\| \frac{x_n - y_n}{2} \right\|^p > (2 - \frac{1}{n}) \frac{\| x_n \|^p + \| y_n \|^p}{2}. \tag{6}$$

Here we may assume $\| y_n \| \leqq \| x_n \| = 1$ for all n without loss of generality. Further we may assume that $\{\|y_n\|\}$ converges to some α ($0 \leqq \alpha \leqq 1$); if necessary, take a suitable subsequence of $\{\|y_n\|\}$. By (6) we have

$$\left(2 - \frac{1}{n}\right)\frac{1 + \|y_n\|^p}{2} < \left\|\frac{x_n + y_n}{2}\right\|^p + \left\|\frac{x_n - y_n}{2}\right\|^p$$

$$\leqq 2\left(\frac{\|x_n\| + \|y_n\|}{2}\right)^p$$

$$\leqq 2\left(\frac{1 + \|y_n\|^p}{2}\right). \tag{7}$$

Letting $n \to \infty$ in (7), we have $(1 + \alpha)^p/(1 + \alpha^p) = 2^{p-1}$, which implies $\alpha = 1$ by Lemma 2. Hence by (7) again we have

$$\left\|\frac{x_n + y_n}{2}\right\|^p + \left\|\frac{x_n - y_n}{2}\right\|^p \longrightarrow 2 \quad \text{as } n \to \infty. \tag{8}$$

Therefore, $\|(x_n + y_n)/2\| \to 1$, $\|(x_n - y_n)/2\| \to 1$ as $n \to \infty$ (note that each term in (8) is not greater than one). This contradicts (i).

(iv) $\Rightarrow$ (i): Assume that there exists a δ ($0 < \delta < 2$) for which (4) holds for all x, y in X. Then if $\|x\| \leqq 1$, $\|y\| \leqq 1$, we have

$$\left\|\frac{x + y}{2}\right\|^p + \left\|\frac{x - y}{2}\right\|^p \leqq 2 - \delta.$$

Hence $\min\{\|(x + y)/2\|, \|(x - y)/2\|\} \leqq (1 - \delta/2)^{1/p} = 1 - \delta_0$, where $\delta_0 = 1 - (1 - \delta/2)^{1/p}$, that is, X is uniformly non-square.

(iv) $\Leftrightarrow$ (v): Note merely that the inequality (3) is rewritten as

$$\|A : l_p^2(X) \to l_p^2(X)\| \leqq 2(1 - \delta/2)^{1/p}.$$

This completes the proof.

Remarks. (i) The characterization of uniform convexity stated in Proposition A fails to hold for the case $p = 1$ ((2) is false in this case; put $y = 0$), while the corresponding characterizations of uniform non-squareness (ii) and (iii) in Theorem 1 hold for $p = 1$. Indeed the above proofs of the equivalence of (i)–(iii) remain valid for $p = 1$; the assertions (iv) and (v) are false in this case (the inequality (3) fails to hold with $y = 0$ ($x \neq 0$)).

(ii) For any Banach space X and for any $1 \leqq p \leqq \infty$ it holds that

$$\| A : l_p^2(X) \to l_p^2(X) \| \leqq 2$$

(see (15) in Remarks after Theorem 2). For $X = L_p$ we have

$$\| A : l_p^2(L_p) \to l_p^2(L_p) \| = 2^{1/\min(p, p')},$$

which is equivalent to the following Clarkson's inequality:

$$(\| f+g \|_p^p + \| f-g \|_p^p)^{1/p} \leqq 2^{1/\min(p, p')} (\| f \|_p^p + \| g \|_p^p)^{1/p} \quad (\forall f, \ g \in L_p)$$

(Clarkson [4]; cf. [14]).

(iii) A result of Smith and Turett [26, Lemma 14] concerning uniformly non-$l_1(n)$ spaces implies the equivalence of (i) and (iv) of Theorem 1 in the case $n = 2$ (our proof is different from theirs).

By Theorem 1 (use (iv)) we have

COROLLARY 1. Let $1 < p < \infty$. Then the Lebesgue-Bochner space $L_p(X)$ is uniformly non-square if and only if X is (Smith and Turett [26]; cf. [25, 9]).

3. Von Neumann-Jordan constant and uniform non-squareness

We now characterize Banach spaces with NJ-constant less than 2.

LEMMA 2. (i) $C_{NJ}(X) = 2^{2/t-1}$, $1 \leqq t \leqq 2$, if and only if

$$\| A : l_2^2(X) \to l_2^2(X) \| = 2^{1/t}. \tag{9}$$

(ii) $C_{NJ}(X') = C_{NJ}(X)$ (X' is the dual space of X).

Proof. (i) is easy to see (recall the note after the definition of the NJ-constant).

(ii) Since A is symmetric,

$$\| A : l_2^2(X') \to l_2^2(X') \| = \| A : l_2^2(X) \to l_2^2(X) \|,$$

from which the conclusion follows by (i).

THEOREM 2. For a Banach space X the following are equivalent:

(i) $C_{NJ}(X) < 2$.

(ii) X is uniformly non-square.

(iii) For any (resp. some) r and s with $1 < r \leqq \infty$, $1 \leqq s < \infty$

$$\| A : l_r^2(X) \to l_s^2(X) \| < 2^{1/r' + 1/s}, \tag{10}$$

where $1/r + 1/r' = 1$.

Proof. (i) ⇔ (ii): By Lemma 2 the assertion $C_{NJ}(X) < 2$ is equivalent to

$$\| A : l_2^2(X) \to l_2^2(X) \| < 2, \tag{11}$$

which is valid if and only if X is uniformly non-square by Theorem 1.

(ii) ⇒ (iii): Let X be uniformly non-square. We first see that for any p with $1 < p < 2$

$$\| A : l_p^2(X) \to l_{p'}^2(X) \| < 2^{2/p'}, \tag{12}$$

where $1/p + 1/p' = 1$. By (11) we can put

$$\| A : l_2^2(X) \to l_2^2(X) \| = 2^{1/t} \tag{9}$$

with some t, $1 < t \leqq 2$. On the other hand, we obviously have

$$\| A : l_1^2(X) \to l_\infty^2(X) \| = 1. \tag{13}$$

Put $\theta = 2/p'$ $(0 < \theta < 1)$. Then, since $(1-\theta)/1 + \theta/2 = 1/p$, $(1-\theta)/\infty + \theta/2 = 1/p'$, we have

$$\| A : l_p^2(X) \to l_{p'}^2(X) \| \leqq 1^{1-\theta} 2^{\theta/t} = 2^{2/p't} < 2^{2/p'}$$

by interpolation (cf. [2; Theorems 5.1.2, 4.2.1 and 4.1.2]) with (13) and (9). Now, let $1 < r \leqq \infty$, $1 \leqq s < \infty$. Take p as $1 < p < r$ and $s < p' < \infty$. Then by (12) we have

$$\begin{aligned} &\| A: l_r^2(X) \to l_s^2(X) \| \\ &\leqq \| I: l_r^2(X) \to l_p^2(X) \| \, \| A: l_p^2(X) \to l_{p'}^2(X) \| \, \| I: l_{p'}^2(X) \to l_s^2(X) \| \\ &< 2^{1/p-1/r} 2^{2/p'} 2^{1/s-1/p'} \\ &= 2^{1/r'+1/s}, \end{aligned} \tag{14}$$

where I's are identity operators.

(iii) ⇒ (ii): Assume the inequality (10) to be valid for some r and s with $1 < r \leqq \infty$, $1 \leqq s < \infty$. Then there exists a δ $(0 < \delta < 1)$ such that for any x and y in X

$$\left(\frac{\|x+y\|^s+\|x-y\|^s}{2}\right)^{1/s} \leqq 2(1-\delta)\left(\frac{\|x\|^r+\|y\|^r}{2}\right)^{1/r}$$

(usual modification is required for the right term if $r = \infty$). Let here $\|x\| \leqq 1$, $\|y\| \leqq 1$. Then we have $\min\{\|x+y\|, \|x-y\|\} \leqq 2(1-\delta)$, as is desired. This completes the proof.

Remarks. (i) For any Banach space X

$$\|A : l_r^2(X) \to l_s^2(X)\| \leqq 2^{1/r'+1/s} \quad \text{for all } 1 \leqq r, s \leqq \infty \tag{15}$$

and

$$\|A : l_r^2(X) \to l_s^2(X)\| = 2^{1/r'+1/s} \quad \text{for } r = 1 \text{ or } s = \infty. \tag{16}$$

Indeed to see (15), merely put $p = 1$ in the first inequality of (14) and use (13); if $r = 1$ resp. $s = \infty$, in the inequality

$$(\|x+y\|^s + \|x-y\|^s)^{1/s} \leqq 2^{1/r'+1/s}(\|x\|^r + \|y\|^r)^{1/r},$$

equality attains with $(x, 0)$ resp. (x, x) $(x \neq 0)$, which implies (16). For $X = L_p$ $(1 \leqq p \leqq \infty)$ we have

$$\|A : l_r^2(L_p) \to l_s^2(L_p)\| = 2^{c(r,s;p)} \quad \text{for all } 1 \leqq r, s \leqq \infty, \tag{17}$$

where $c(r,s;p) = \max\{1/r', 1/s, 1/r'+1/s-1/\max(p,p')\}$, which yields the following Clarkson-Boas-Koskela inequality:

$$(\|f+g\|_p^s + \|f-g\|_p^s)^{1/s} \leqq 2^{c(r,s;p)}(\|f\|_p^r + \|g\|_p^r)^{1/r} \quad \text{for } \forall f, g \in L_p \tag{18}$$

(Boas [3], Koskela [21], Kato [14]; see also [16, 23]).

(ii) As Boas [3] (cf. [4]) observed, the inequality (18), or (17), with $c(r,s;p) = 1/r'$ (for the case $s' \leqq r < \infty$, $\max(p, p') \leqq s < \infty$)

implies the uniform convexity of L_p. The same is clearly true for a general Banach space X, that is, if

$$\| A : l_r^2(X) \to l_s^2(X) \| \leqq 2^{1/r'} \tag{19}$$

for some $1 \leqq r, s < \infty$, then X is uniformly convex. (Note that if (19) is valid, then (19) is in fact reduced to identity; (x, x), $x \neq 0$, is norm-attaining.) This fails to be valid if the above norm of A is greater than $2^{1/r'}$ (see Example below). By a recent result of the authors [17, Theorem 2.4], if $2 \leqq s < \infty$, $s' \leqq r \leqq s$, (19) implies that X is of cotype s and 'cotype s constant' is 1, and vice versa; a similar result for type is also given in [ibid., Theorem 2.2].

(iii) The equivalence of (i) and (ii) in Theorem 2 is very recently proved in a generalized form by the authors and Hashimoto [20] by using a result of Smith and Turett [26; Lemma 14].

The following example explains difference between uniform convexity and uniform non-squareness via behavior of norms of the Littlewood matrix:

Example. Let $1 < p \leqq 2$ and $1 < \lambda < 2^{1/p'}$. Let $X_{p,\lambda}$ be the space $l_{p'}$ equipped with the norm $\| x \|_{p,\lambda} := \max\{ \| x \|_{p'}, \lambda \| x \|_\infty \}$, where $1/p + 1/p' = 1$. Then, in the same way as the proof of Proposition 1 in [18] we have for $p \leqq r < \infty$

$$2^{1/r'} < \| A: l_r^2(X_{p,\lambda}) \to l_{p'}^2(X_{p,\lambda}) \| = \lambda 2^{1/r'} < 2^{1/r' + 1/p'}. \tag{20}$$

By Proposition 1 of [18], $X_{p,\lambda}$ is not uniformly convex (nor strictly convex) for all $1 < \lambda < 2^{1/p'}$, whereas Theorem 2 asserts that $X_{p,\lambda}$ is uniformly non-square (compare (20) with (19)).

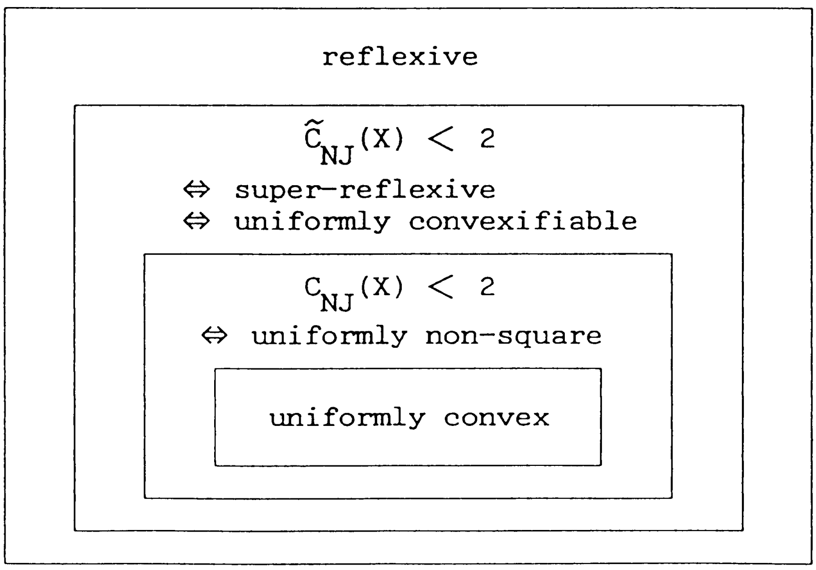
Now, Theorems 2 and A, combined with Lemma 2, immediately yields that uniform non-squareness and super-reflexivity are inherited by dual spaces (recall that uniform convexity is not so; cf. [1]):

COROLLARY 2. (i) The dual space X' is uniformly non-square if and only if X is.

(ii) The dual space X' is super-reflexive if and only if X is (James [12, Theorem 2]).

Remark. The above result (i) of Corollary 2 seems not to have appeared in literature. Giesy [9] showed that the bidual X'' is uniformly non-$l_1(n)$ if and only if X is; and the dual space X' is B-convex (uniformly non-$l_1(n)$ for some n) if and only if X is (see also [6; Corollary 13.7]). It is known that for some Orlicz spaces uniform non-squareness coincides with reflexivity and also with B-convexity ([10]).

Our results about relation between NJ-constant and some geometrical properties of Banach spaces are illustrated as follows:



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Department of System Engineering,
Okayama Prefectural University,
Soja 719-1197, *Japan*
e-mail: takahasi@cse.oka-pu.ac.jp

Department of Mathematics,
Kyushu Institute of Technology,
Tobata, Kitakyushu 804-8550, *Japan*
e-mail: katom@tobata.isc.kyutech.ac.jp

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