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ON WEYL SPECTRUM AND A CLASS OF OPERATORS

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ABSTRACT. In this paper we show that the set $\mathcal{W}$ of all operators satisfying the equality of the Weyl and essential spectra is norm closed in $B(H)$, invariant under compact perturbation, and closed under approximate similarity. But $\mathcal{W}$ is not closed under addition. Also we show that the Weyl spectrum of an operator in $\mathcal{W}$ satisfies the spectral mapping theorem for analytic functions and give properties of an operator in $\mathcal{W}$.

0. Introduction. Let H be an infinite dimensional Hilbert space and we write $B(H)$ for the set of all bounded linear operators on H and $\mathcal{K}$ for the set of all compact operators on H. If $T \in B(H)$, we write $\sigma(T)$ for the spectrum of T and $\pi_{00}(T)$ for the isolated points of $\sigma(T)$ which are eigenvalues of finite multiplicity. An operator $T \in B(H)$ is said to be *Fredholm* if its range ran T is closed and both the null space ker T and ker T^* are finite dimensional. The *index* of a Fredholm operator T, denoted by $i(T)$, is defined by

$$i(T) = \dim\ker T - \dim\ker T^*.$$

It was well-known ([4]) that $i : \mathcal{F} \to \mathbb{Z}$ is a continuous function where the set $\mathcal{F}$ of Fredholm operators has the norm topology and $\mathbb{Z}$ has the discrete topology. The *essential spectrum* of T, denoted by $\sigma_e(T)$, is defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\}.$$

A Fredholm operator of index zero is called *Weyl.* The *Weyl spectrum* of T, denoted by $\omega(T)$, is defined by

$$\omega(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}.$$

It was shown ([1]) that for any operator T, $\sigma_e(T) \subset \omega(T) \subset \sigma(T)$ and equalities do not hold in general. Also

$$\omega(T) = \cap_{K \in \mathcal{K}} \sigma(T + K)$$

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and $\omega(T)$ is a nonempty compact subset of $\mathbb{C}$.

We write $\mathcal{W}$ for the set of all operators T satisfying $\sigma_e(T) = \omega(T)$. For example, every normal(compact, and quasinilpotent) operator is in $\mathcal{W}$. However, consider the unilateral shift U on l_2 given by

$$U(x_1, x_2, \cdots) = (0, x_1, x_2, x_3, \cdots).$$

Then U is hyponormal, $\omega(U) = \sigma(U) = D(=$ the closed unit disc) and $\sigma_e(U) = C(=$ the unit circle)(See [1, Example 1.2]). Hence U is not in $\mathcal{W}$.

It was also known that the mapping $T \to \omega(T)$ is upper semi-continuous, but not continuous at T([9]). However if $T_n \to T$ with $T_nT = TT_n$ for all $n \in \mathbb{N}$ then

$$\lim \omega(T_n) = \omega(T). \tag{0.1}$$

It was known that $\omega(T)$ satisfies the one-way spectral mapping theorem for analytic funcions: if f is analytic on a neighborhood of $\sigma(T)$, then

$$\omega(f(T)) \subset f(\omega(T)). \tag{0.2}$$

The inclusion (0.2) may be proper(see [1, Example 3.3]). If T is normal then $\sigma_e(T)$ and $\omega(T)$ coincide. Thus if T is normal since $f(T)$ is also normal, it follows that $\omega(T)$ satisfies the spectral mapping theorem for analytic functions.

In this paper we show that the set $\mathcal{W}$ of operators T satisfying the equality $\sigma_e(T) = \omega(T)$ of the Weyl and essential spectra is norm closed in $B(H)$, invariant under compact perturbation, and closed under approximate similarity. But $\mathcal{W}$ is not closed under addition. Also we show that the Weyl spectrum of an operator in $\mathcal{W}$ satisfies the spectral mapping theorem for analytic functions and give properties of an operator in $\mathcal{W}$.

1. Equality of the Weyl and essential spectra. By [1, Example 2.12], every compact operator K is in $\mathcal{W}$. Also it is easy to show that if T is in $\mathcal{W}$ and $\alpha \in \mathbb{C}$, then T^* and αT are in $\mathcal{W}$.

The following lemma shows that the Weyl spectrum of an operator is the disjoint union of the essential spectrum and a particular open set.

LEMMA 1. *([1],[4]) For any operator T in $B(H)$,*

$$\omega(T) = \sigma_e(T) \cup \theta(T) \qquad \text{(disjoint union)},$$

where $\theta(T) = \{\lambda : T - \lambda I$ is Fredholm and $i(T - \lambda I) \neq 0\}$.

For example, if U is the simple unilateral shift, then $\sigma_e(U) = \{\lambda : |\lambda| = 1\}$, and $\theta(U) = \{\lambda : |\lambda| < 1\}$. From Lemma 1, we note that $\sigma_e(T) = \omega(T)$ if and only if the open set $\theta(T)$ is empty. Also the following corollary gives some simple criteria for equality of the Weyl and essential spectra:

COROLLARY 2. *If any of the following conditions holds for T in $B(H)$, then T is in $\mathcal{W}$:*

(1) *T is normal,*
(2) *the point spectra of T and T^* are countably infinite.*

Proof. For any T in $B(H)$, λ in $\theta(T)$ implies that

$$\dim\ker(T - \lambda I) \neq \dim\ker(T^* - \overline{\lambda} I).$$

If T is normal, it was well-known that $\ker(T - \lambda I) = \ker(T^* - \overline{\lambda} I)$ for every λ. Therefore $\theta(T)$ is empty.

If λ is in $\theta(T)$, then either λ is an eigenvalue of T or $\overline{\lambda}$ is an eigenvalue of T^*. Hence if the point spectra of T and T^* are countably infinite, then $\theta(T)$ is countable. Since $\theta(T)$ is also open, $\theta(T)$ is empty.

Our class $\mathcal{W}$ is strictly larger than the class of normal operators. For an example of a nonnormal operator in $\mathcal{W}$, let T be a non-normal compact operator or an operator such as $\sigma(T) = \{0\}$. Then $\sigma_e(T) = \omega(T) = \sigma(T)$.

THEOREM 3. *The set $\mathcal{W}$ is norm closed in $B(H)$ and invariant under compact perturbations.*

Proof. Suppose T_n is in $\mathcal{W}$ for each n and $T_n \to T$ in norm topology. If $\sigma_e(T) \neq \omega(T)$, then by Lemma 1 there exists $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is Fredholm of nonzero index. By [7, Theorem IV.5.17], there exists an $\epsilon > 0$ such that if $\|T - \lambda I - S\| < \epsilon$, then S is a Fredholm operator. Also there exists an integer N_1 such that for $n \geq N_1$ we have

$$\|(T - \lambda I) - (T_n - \lambda I)\| < \frac{\epsilon}{2}.$$

Thus $T_n - \lambda I$ is Fredholm for $n \geq N_1$. Since the index i is continuous, there exists an integer N_2 such that for $n \geq N_2$, $i(T_n - \lambda I) \neq 0$. Hence for $n \geq N = \max(N_1, N_2)$, $T_n - \lambda I$ is Fredholm of nonzero index and so $\sigma_e(T_n) \neq \omega(T_n)$ by Lemma 1. This is a contradiction. Thus $\sigma_e(T) = \omega(T)$ and so T is in $\mathcal{W}$. Therefore the set of operators in $\mathcal{W}$ is closed in $B(H)$

If $T \in \mathcal{W}$ and if K is compact, then $\omega(T+K) = \omega(T)$ by [1, Corollary 2.7] and, clearly, $\sigma_e(T) = \sigma_e(T+K)$. Hence $T + K \in \mathcal{W}$ and so the set of operators in $\mathcal{W}$ is invariant under compact perturbations.

THEOREM 4. *The set $\mathcal{W}$ is not closed under addition.*

Proof. If it were, then every operator A would be in $\mathcal{W}$ from the symmetric decomposition $A = B + iC$, B, C selfadjoint. This is a contraction.

THEOREM 5. *If A is in $\mathcal{W}$ and if A is invertible, then A^{-1} is also in $\mathcal{W}$.*

Proof. By the spectral mapping theorem, $\sigma_e(A^{-1}) = 1/\sigma_e(A)$.

Claim: $\omega(A) = 1/\omega(A^{-1})$. Suppose $0 \neq z \notin \omega(A)$. Then $A - zI$ is Weyl and so $A - zI + K$ is invertible in $B(H)/\mathcal{K}$. Thus $z \notin \sigma(A + \mathcal{K})$ and so $1/z \notin \sigma(A+\mathcal{K})^{-1} = \sigma(A^{-1} + \mathcal{K})$. Hence $(A^{-1} - (1/z)I) + \mathcal{K}$ is invertible $B(H)/\mathcal{K}$ and $A^{-1} - (1/z)I$ is Fredholm. Also $\dim\ker(A - zI) = \dim\ker(A - zI)^* < \infty$, and so $\dim\ker(A^{-1} - (1/z)I) = \dim\ker(A^{-1} - (1/z)I)^* < \infty$. Hence $A^{-1} - (1/z)I$ is Weyl and so $1/z \notin \omega(A^{-1})$. (If $z = 0$, the claim is obvious.) Thus $1/\omega(A^{-1}) \subset \omega(A)$, which implies that $\omega(A^{-1}) \subseteq 1/\omega(A)$ and hence, replacing A by A^{-1}, $\omega(A) \subseteq 1/\omega(A^{-1})$. Therefore $\omega(A) = 1/\omega(A^{-1})$ as claimed. And then, we have $\sigma_e(A^{-1}) = 1/\sigma_e(A) = 1/\omega(A) = \omega(A^{-1})$, and so A^{-1} is in $\mathcal{W}$.

Two operators S and T in $B(H)$ are said to be *approximately equivalent* if there is a sequence $\{U_n\}$ of unitary operators such that $\|U_n^* S U_n - T\| \to 0$. They are *approximately similar* if there is a sequence $\{X_n\}$ of invertible operators such that

$$\sup\{\|X_n\|, \|X_n^{-1}\|\} < \infty \quad \text{and} \quad \|X_n^{-1} S X_n - T\| \to 0.$$

THEOREM 6. *The set $\mathcal{W}$ is closed under approximate similarity.*

Proof. Let $S \in \mathcal{W}$ and let T be approximately similar to S. Then there exists a sequence $\{X_n\}$ of invertible operators such that

$$\sup\{\|X_n\|, \|X_n^{-1}\|\} < \infty \quad \text{and} \quad \|X_n^{-1} S X_n - T\| \to 0.$$

Note that S is of the form invertible + compact if and only if $P^{-1}SP$ is of that form where P is invertible. Thus $\sigma_e(X_n^{-1}SX_n) = \sigma_e(S)$. And since $\dim\ker X_n^{-1}SX_n = \dim\ker S$, $i(X_n^{-1}SX_n) = i(S)$ and hence $\omega(X_n^{-1}SX_n) = \omega(S)$. Since $S \in \mathcal{W}$, for each n,

$$\omega(X_n^{-1}SX_n) = \omega(S) = \sigma_e(S) = \sigma_e(X_n^{-1}SX_n)$$

and so $X_n^{-1}SX_n \in \mathcal{W}$. By Theorem 3, $T \in \mathcal{W}$.

COROLLARY 7. *The set $\mathcal{W}$ is closed under similarity.*

LEMMA 8. *For $T, S \in B(H)$, we have*

$$\omega(T \oplus S) \subseteq \omega(T) \cup \omega(S). \tag{1.1}$$

If either $T \in \mathcal{W}$ or $S \in \mathcal{W}$, then the equality holds and $T \oplus S \in \mathcal{W}$.

Proof. It follows from the fact that

$$\sigma_e(T \oplus S) = \sigma_e(T) \cup \sigma_e(S)$$

and that the index of a direct sum is the sum of indices.

THEOREM 9. *If $\sigma(S) \cap \sigma(T) = \emptyset$ and if either $S \in \mathcal{W}$ or $T \in \mathcal{W}$, then $\begin{pmatrix} S & U \\ 0 & T \end{pmatrix}$ is in $\mathcal{W}$.*

Proof. By [11, Corollary 0.15], the operator $\begin{pmatrix} S & U \\ 0 & T \end{pmatrix}$ is similar to $S \oplus T$. By Lemma 8, $S \oplus T$ is in $\mathcal{W}$. By Corollary 7, $\begin{pmatrix} S & U \\ 0 & T \end{pmatrix}$ is in $\mathcal{W}$.

LEMMA 10. *([5]) If T is Weyl and if K is compact in $B(H)$, then $T + K$ is Weyl.*

THEOREM 11. *If T in $B(H)$ is of the form normal + compact, then T is in $\mathcal{W}$.*

Proof. Let $T = N + K$, where N is normal and K is compact. If T is not in $\mathcal{W}$, then by Lemma 1, there exists $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is Fredholm of nonzero index. Since $i(T - \lambda I - K) = i(N - \lambda I) = 0$, $T - \lambda I - K$ is Weyl and, by Lemma 10, $T - \lambda I$ is Weyl. This is a contradiction.

From this theorem we know that the unilateral shift U is not of the form normal + compact.

THEOREM 12. *T is in $\mathcal{W}$ if and only if there exists a compact operator K such that $\sigma(T+K) = \sigma_e(T)$.*

Proof. If $\sigma(T+K) = \sigma_e(T)$ for some compact operator K, then

$$\omega(T) = \cap_{K \in \mathcal{K}} \sigma(T+K) \subseteq \sigma_e(T).$$

Hence T is in $\mathcal{W}$.

Conversely if T is in $\mathcal{W}$, then $\sigma_e(T) = \omega(T)$, and so by [12, Theorem 4], there exists a compact operator K such that $\sigma(T+K) = \omega(T)$. Hence $\sigma(T+K) = \omega(T) = \sigma_e(T)$ for some compact operator K.

It was well-known([2]) that every Riesz operator T is in $\mathcal{W}$ since $\omega(T) = \{0\} = \sigma_e(T)$. Also we note that if T is a normal operator and f is any continuous complex-valued function on $\sigma(T)$, then $\omega(f(T)) = f(\omega(T))$ and so $f(T)$ is in $\mathcal{W}$([1, Theorem 3.1]).

THEOREM 13. *If T is in $\mathcal{W}$ and f is analytic on a neighborhood of $\sigma(T)$, then $\omega(f(T)) = f(\omega(T))$.*

Proof. Suppose that p is any polynomial. Then by the spectral mapping theorem,

$$p(\omega(T)) = p(\sigma_e(T)) = \sigma_e(p(T)) \subseteq \omega(p(T)).$$

But for any operator $T \in B(H)$, $\omega(p(T)) \subseteq p(\omega(T))$([1, Theorem 3.2]). Therefore

$$\omega(p(T)) = p(\omega(T)) \tag{1.2}$$

for any polynomial p.

Next suppose r is any rational function with no poles in $\sigma(T)$. Write $r = p/q$, where p and q are polynomials and q has no zeros in $\sigma(T)$. Then

$$r(T) - \lambda I = (p - \lambda q)(T)(q(T))^{-1}.$$

By (1.2),

$$(p-\lambda q)(T) \text{ Weyl} \Longleftrightarrow p - \lambda q \text{ has no zeros in } \omega(T).$$

Thus we have

$$\begin{aligned}\lambda \notin \omega(r(T)) &\Longleftrightarrow (p-\lambda q)(T) = \text{Weyl}\\ &\Longleftrightarrow p-\lambda q \ \text{ has no zeros in } \omega(T)\\ &\Longleftrightarrow ((p-\lambda q)(x))q(x)^{-1} \neq 0 \ \text{ for any } \ x \in \omega(T)\\ &\Longleftrightarrow \lambda \notin r(\omega(T))\end{aligned}$$

which says that $\omega(r(T)) = r(\omega(T))$. If f is analytic on a neighborhood of $\sigma(T)$, then by Runge's theorem([4]), there is a sequence $\{r_n\}$ of rational functions with no poles in $\sigma(T)$ such that $\{r_n\}$ converges to f uniformly on a neighborhood of $\sigma(T)$. Since $\{r_n(T)\}$ converges to $f(T)$ and each $r_n(T)$ commutes with $f(T)$, by [9],

$$\omega(f(T)) = \lim \omega(r_n(T)) = \lim r_n(\omega(T)) = f(\omega(T)).$$

COROLLARY 14. *If T is in $\mathcal{W}$ and f is analytic on a neighborhood of $\sigma(T)$, then $f(T)$ is in $\mathcal{W}$.*

Proof. By Theorem 12 and by the spectral mapping theorem, $\omega(f(T)) = f(\omega(T)) = f(\sigma_e(T)) = \sigma_e(f(T))$ and so $f(T)$ is in $\mathcal{W}$.

We say that *Weyl's theorem holds for T* if

$$\omega(T) = \sigma(T) - \pi_{00}(T).$$

There are several classes of operators including normal and hyponormal operators on a Hilbert space for which Weyl's theorem holds. Recall ([10]) that $T \in B(H)$ is said to be *isoloid* if isolated points of $\sigma(T)$ are eigenvalues of T.

REMARK 1. We note that every operator in $\mathcal{W}$ is not isoloid. For example, let V be a Volterra operator. Then V is a compact operator and so in $\mathcal{W}$. Since $\sigma(V) = \{0\}$ and V has no eigenvalues, 0 is an isolated point of $\sigma(V)$, but 0 is not an eigenvalue of $\sigma(V)$. Hence V is not isoloid.

REMARK 2. In general, Weyl's theorem does not hold for an operator in $\mathcal{W}$. For example, let T be an operator on l_2 defined by

$$T(x_1, x_2, x_3, \cdots) = (x_2, \frac{1}{2}x_3, \frac{1}{3}x_4, \cdots).$$

Then T is a compact operator and so in $\mathcal{W}$. Since $\sigma(T) = \{0\} = \omega(T)$ and also $\pi_{00}(T) = \{0\}$,

$$\sigma(T) - \omega(T) = \emptyset \neq \{0\} = \pi_{00}(T).$$

Hence Weyl's theorem does not hold for T.

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