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ON EXISTENCE OF SOLUTIONS OF NONDEGENERATE WAVE EQUATIONS WITH NONLINEAR DAMPING TERMS

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Abstract In this paper, we consider the existence and asymptotic behavior of solutions of the following problems;

$$\begin{aligned}
&u_{tt}(t,x) - M(\|\nabla u(t,x)\|_2^2 + \|\nabla v(t,x)\|_2^2)\Delta u(t,x) + \delta|u_t(t,x)|^{p-1}u_t(t,x) \\
&\quad = \mu|u(t,x)|^{q-1}u(t,x), \quad x \in \Omega, \quad t \geq 0, \\
&v_{tt}(t,x) - M(\|\nabla u(t,x)\|_2^2 + \|\nabla v(t,x)\|_2^2)\Delta v(t,x) + \delta|v_t(t,x)|^{p-1}v_t(t,x) \\
&\quad = \mu|v(t,x)|^{q-1}v(t,x), \quad x \in \Omega, \quad t \geq 0, \\
&u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega, \\
&v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), \quad x \in \Omega
\end{aligned}$$

where $q > 1, p \geq 1, \delta > 0, \mu \in R, \Delta$ the Laplacian in R^N, $M(s) = a + bs^\gamma, a + b \geq 0, b \geq 0$ and $\gamma \geq 1$.

Keywords and Phrases Existence and uniqueness, asymptotic behavior, degenerate wave equation, Galerkin method.

1. Introduction

Let Ω be a bounded domain in R^N with smooth boundary $\partial\Omega$. In this paper, we consider the existence of solutions of the following problems;

$$\begin{aligned}
&u_{tt}(t,x) - M(\|\nabla u(t,x)\|_2^2 + \|\nabla v(t,x)\|_2^2)\Delta u(t,x) + \delta|u_t(t,x)|^{p-1}u_t(t,x) \\
&\quad = \mu|u(t,x)|^{q-1}u(t,x), \quad x \in \Omega, \quad t \geq 0, \\
&v_{tt}(t,x) - M(\|\nabla u(t,x)\|_2^2 + \|\nabla v(t,x)\|_2^2)\Delta v(t,x) + \delta|v_t(t,x)|^{p-1}v_t(t,x) \\
&\quad = \mu|v(t,x)|^{q-1}v(t,x), \quad x \in \Omega, \quad t \geq 0, \\
&u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega, \\
&v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), \quad x \in \Omega
\end{aligned} \tag{1.1}$$

where $q > 1, p \geq 1, \delta > 0, \mu \in R, \Delta$ the Laplacian in R^N, $M(s) = a + bs^\gamma, a + b \geq 0, b \geq 0$ and $\gamma \geq 1$.

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Here

$$\|\nabla u\|_2^2 = \sum_{i=1}^{N} \int_\Omega |\frac{\partial u}{\partial x_i}(t,x)|^2 dx, \quad u_t = \frac{\partial u}{\partial t} \quad \text{and} \quad \Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}.$$

Equation (1.1) has its origin in the nonlinear vibrations of an elastic string (cf. R. Narasimha [6]). Many authors have studied the existence and uniqueness of solutions of (1.1) by using various methods.

When $\delta > 0$ and $\mu = 0$, for degenerate case, Nishihara and Yamada [7] have proved the global existence of a unique solution under the the assumptions that the initial data $\{u_0, u_1\}$ are sufficiently small and $u_0 \neq 0$. For the problem with linear damping δu_t, there are the works of Brito [1], Ikehata [3], K. Ono [8] and the references therein. In the case of $\gamma = 1$, M. D. Silva Aleves ([9]) has proved the existence of weak solutions of the unilateral problem using the Galerkin method. In the present paper we will study the existence and uniqueness of solutions of unilateral problem (1.1) with $\gamma > 1$ by using Galerkin method and will also investigate its asymptotic behavior.

The contents of this paper are as follows; In section 2, we present the preliminaries and some lemmas. In section 3, we give the statement of the main theorem. In section 4, we deal with a priori estimates for solutions of (1.1) and prove our main Theorem and section 5 deals with the asymptotic behavior of the solutions obtained in section 4.

2. Preliminaries

We first prepare the following well known lemmas which will be needed later.

Lemma 2.1. *(Sobolev-Poincaré [4]) If either* $1 \leq q < +\infty$ $(N = 1, 2)$ *or* $1 \leq q \leq \frac{N+2}{N-2}$ $(N \geq 3)$, *then there is a positive constant* $C(\Omega, q+1)$ *such that*

$$\|u\|_{q+1} \leq C(\Omega, q+1)\|\nabla u\|_2 \quad \textit{for} \quad u \in H_0^1(\Omega).$$

In other words,

$$C(\Omega, q+1) = \sup\{\frac{\|u\|_{q+1}}{\|\nabla u\|_2} | \quad u \in H_0^1(\Omega), \quad u \neq 0\}$$

is positive and finite.

Lemma 2.2. *(Gagliardo-Nirenberg [4]) Let* $1 \leq r < q \leq +\infty$ *and* $p \leq q$. *Then the inequality*

$$\|u\|_{W^{k,q}} \leq C\|u\|_{W^{m,p}}^{\theta}\|u\|_r^{1-\theta} \quad \textit{for} \quad u \in W^{m,p}(\Omega) \cap L^r(\Omega)$$

holds with some $C > 0$ *and*

$$\theta = \left(\frac{k}{N} + \frac{1}{r} - \frac{1}{q}\right)\left(\frac{m}{N} + \frac{1}{r} - \frac{1}{p}\right)^{-1}$$

provided that $0 < \theta \leq 1$ (*we assume* $0 < \theta < 1$ *if* $q = +\infty$).

We conclude this section by stating a lemma concerning a difference inequality, which will be used later.

Lemma 2.3. *(Nakao [5]) Let* $\phi(t)$ *be a nonincreasing and nonnegative function on* $[0,T], T > 1$, *such that*

$$\phi(t)^{1+r} \leq k_0(\phi(t) - \phi(t+1)) \quad on \quad [0,T]$$

where k_0 *is a positive constant and* r *a nonnegative constant. Then we have*

(*i*) *if* $r > 0$, *then*

$$\phi(t) \leq (\phi(0)^{-r} + k_0^{-1} r[t-1]^+)^{-\frac{1}{r}}, \quad where \quad [t-1]^+ = \max\{t-1, 0\},$$

(*ii*) *if* $r = 0$, *then*

$$\phi(t) \leq \phi(0) e^{-k_1[t-1]^+} \quad on \quad [0,T], \quad where \quad k_1 = \log(\frac{k_0}{k_0 - 1}).$$

3. Statement of the result

We consider the following initial value problems;

$$(3.1) \qquad \begin{aligned} & u_{tt}(t) - (a + b(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)^\gamma)\Delta u(t) + \delta |u_t(t)|^{p-1} u_t(t) \\ & \quad = \mu |u(t)|^{q-1} u(t), \quad t \geq 0, \\ & v_{tt}(t) - (a + b(\|\nabla u(t)\|_2^2 + \beta \|\nabla v(t)\|_2^2)^\gamma)\Delta v(t) + \delta |v_t(t)|^{p-1} v_t(t) \\ & \quad = \mu |v(t)|^{q-1} v(t), \quad t \geq 0, \\ & u(0) = u_0, \quad u_t(0) = u_1, \\ & v(0) = v_0, \quad v_t(0) = v_1, \end{aligned}$$

where $\gamma \geq 1$.

Now we set

$$\begin{aligned} J(u,v) &= \frac{a}{2}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) + \frac{b}{2(\gamma+1)}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)^{\gamma+1} \\ &\quad - \frac{\mu}{q+1}(\|u\|_{q+1}^{q+1} + \|v\|_{q+1}^{q+1}), \\ \bar{I}(u,v) &= a(\|\nabla u\|_2^2 + \|\nabla v(t)\|_2^2) + b(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)^{\gamma+1} - \mu(\|u\|_{q+1}^{q+1} + \|v\|_{q+1}^{q+1}), \\ I(u,v) &= a(\|\nabla u\|_2^2 + \|\nabla v(t)\|_2^2) - \mu(\|u\|_{q+1}^{q+1} + \|v\|_{q+1}^{q+1}) \end{aligned}$$

and define the potential as

$$W = \{(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega) | I(u,v) > 0\} \cup \{0\}.$$

Next, by setting

$$E(u(t),v(t)) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|v_t(t)\|_2^2 + J(u(t),v(t)),$$

we can state our main Theorem.

Theorem 3.1. *Let N be a positive integer. Suppose that $\delta > 0$ and $\mu > 0$. Assume that $p < \min\{q, \frac{N+4q-Nq}{2}\}$ is such that*

$$(i) 1 \le p < +\infty (N = 1,2)$$
$$(ii) 1 \le p \le 3, 1 < q \le 5 (N = 3)$$
$$(iii) 1 \le p \le \frac{N}{N-2}, \frac{N}{N-2} \le q \le \min\{\frac{N+2}{N-2}, \frac{N-2}{[N-4]^+}\} (N \ge 4).$$

If $u_0, v_0 \in W \cap H^2(\Omega), u_1, v_1 \in H_0^1(\Omega)$ and

$$\frac{\mu}{a}[C(\Omega, q+1)]^{q+1}\left(\frac{2(q+1)}{a(q-1)}E(u_0,v_0)\right)^{\frac{q-1}{2}} < 1,$$

then the problem (3.1) has solutions $u = u(t,x)$ and $v = v(t,x)$ satisfying

$$u, v \in L^\infty(0,\infty; H_0^1(\Omega) \cap H^2(\Omega)),$$
$$u', v' \in L^\infty(0,\infty; H_0^1(\Omega)),$$
$$u'', v'' \in L^\infty(0,\infty; L^2(\Omega)).$$

4. Proof of Theorem 3.1

Throughout this section we assume that $u_0, v_0 \in W \cap H^2(\Omega)$ and $u_1, v_1 \in H_0^1(\Omega)$. We employ the Galerkin method to construct a global solution. Let $\{\lambda_j\}_{j=1}^\infty$ be a sequence of eigenvalues for $-\Delta w = \lambda w$ in Ω. Let $w_j \in H_0^1(\Omega) \cap H^2(\Omega)$ be the corresponding eigenfunction to λ_j and take $\{w_j\}_{j=1}^\infty$ as a complete orthonormal system in $L^2(\Omega)$. We construct approximate solutions u_m, v_m $(m = 1, 2, \cdots)$ in the form

$$u_m(t) = \sum_{j=1}^m g_{jm}(t) w_j, v_m(t) = \sum_{j=1}^m h_{jm}(t) w_j$$

which are determined by the following ordinary differential equations

$$\begin{aligned}(u_m''(t), w) - ((a + b(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^\gamma)\Delta u_m(t), w) \\ + \delta|u_m'(t)|^{p-1}(u_m'(t), w) = \mu|u_m(t)|^{q-1}(u_m(t), w),\end{aligned} \tag{4.1}$$

$$\begin{aligned}(v_m''(t), w) - ((a + b(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^\gamma)\Delta v_m(t), w) \\ + \delta|v_m'(t)|^{p-1}(v_m'(t), w) = \mu|v_m(t)|^{q-1}(v_m(t), w)\end{aligned} \tag{4.2}$$

($' = \frac{\partial}{\partial t}$ and $'' = \frac{\partial^2}{\partial t^2}$) with the initial conditions,

$$(4.3)\qquad \begin{aligned} u_m(0) = u_{0m} &= \sum_{j=1}^{m}(u_0, w_j)w_j \to u_0 \quad \text{in } H_0^1(\Omega)\cap H^2(\Omega),\\ v_m(0) = v_{0m} &= \sum_{j=1}^{m}(v_0, w_j)w_j \to v_0 \quad \text{in } H_0^1(\Omega)\cap H^2(\Omega),\end{aligned}$$

$$(4.4)\qquad \begin{aligned} u_m'(0) = u_{1m} &= \sum_{j=1}^{m}(u_1, w_j)w_j \to u_1 \quad \text{strongly in } H_0^1(\Omega),\\ v_m'(0) = v_{1m} &= \sum_{j=1}^{m}(v_1, w_j)w_j \to v_1 \quad \text{strongly in } H_0^1(\Omega).\end{aligned}$$

Therefore we can solve the system (4.1)-(4.4) by Picard's iteration method. Hence the system (4.1)-(4.4) have a unique solution on some interval $[0, T_m)$ with $0 < T_m \leq +\infty$. Note that $u_m(t)$ is in the C^2-class. We shall see that $u_m(t)$ can be extended to $[0, \infty)$. We can utilize a standard compactness argument for the limiting procedure and it suffices to derive some a priori estimates for u_m. But this procedure allows us to employ the energy method for a smooth solution $u(t)$ to the problem (4.1)-(4.4) (the results should be in fact applied to the approximated solutions).

A Priori Estimates I

Multiplying the equation in (4.1) by $u_m'(t)$ and multiplying the equation in (4.2) by $v_m'(t)$ yield

$$(4.5)\qquad \begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|u_m'(t)\|_2^2 + \frac{a}{2}\|\nabla u_m(t)\|_2^2 - \frac{\mu}{q+1}\|u_m(t)\|_{q+1}^{q+1}\right)\\ &+ \frac{b}{2}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^\gamma \frac{d}{dt}\|\nabla u_m(t)\|_2^2\\ &+ \delta\|u_m'(t)\|_{p+1}^{p+1} = 0\end{aligned}$$

and

$$(4.6)\qquad \begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|v_m'(t)\|_2^2 + \frac{a}{2}\|\nabla v_m(t)\|_2^2 - \frac{\mu}{q+1}\|v_m(t)\|_{q+1}^{q+1}\right)\\ &+ \frac{b}{2}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^\gamma \frac{d}{dt}\|\nabla v_m(t)\|_2^2\\ &+ \delta\|v_m'(t)\|_{p+1}^{p+1} = 0.\end{aligned}$$

Adding (4.5) and (4.6) and then integrating from 0 to t yield

$$(4.7)\qquad E(u_m(t), v_m(t)) + \delta\int_0^t (\|u_m'(s)\|_{p+1}^{p+1} + \|v_m'(s)\|_{p+1}^{p+1})ds = E(u_0, v_0)$$

where

$$E(u_m(t), v_m(t)) = \frac{1}{2}\Big(\|u'_m(t)\|_2^2 + \|v'_m(t)\|_2^2 + a\|\nabla u_m(t)\|_2^2 + a\|\nabla v_m(t)\|_2^2 \Big)$$
$$+ \frac{b}{2(\gamma+1)}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma+1}$$
$$- \frac{\mu}{q+1}\|u_m(t)\|_{q+1}^{q+1} - \frac{\mu}{q+1}\|v_m(t)\|_{q+1}^{q+1}.$$

In particular, $E(u_m(t), v_m(t))$ is nonincreasing and

(4.8) $$E(u_m(t), v_m(t)) \le E(u_0, v_0).$$

Now, to obtain a priori estimates, we need the following result.

Lemma4.1. *Assume that either*

$$1 \le q < +\infty (N = 1, 2), \quad \text{or} \quad 1 \le q \le \frac{N+3}{N-2} (N \ge 3).$$

Let $(u_m(t), v_m(t))$ be the solution of (4.1) with $(u_0, v_0) \in W \cap H_0^1(\Omega) \times H_0^1(\Omega)$ and $u_1, v_1 \in H_0^1(\Omega)$. If

(4.9) $$\frac{\mu}{a}[C(\Omega, q+1)]^{q+1}\left(\frac{2(q+1)}{a(q-1)}E(u_0, v_0)\right)^{\frac{q-1}{2}} < 1,$$

then $(u_m(t), v_m(t)) \in W$ on $[0, +\infty)$, that is,

$$a(\|\nabla u_m\|_2^2 + \|\nabla v_m\|_2^2) - \mu(\|u_m\|_{q+1}^{q+1} + \|v_m\|_{q+1}^{q+1}) > 0 \quad \text{on} \quad [0, +\infty).$$

Proof. Since $I(u_0, v_0) > 0$, it follows from the continuity of $u_m(t)$ and $v_m(t)$ that

(4.10) $$I(u_m(t), v_m(t)) \ge 0 \quad \text{for some interval near} \quad t = 0.$$

Let t_{max} be a maximal time (possibly $t_{max} = T_m$) when (4.10) holds on $[0, t_{max})$. Note that

(4.11)
$$J(u_m(t), v_m(t)) = \frac{a}{2}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2) - \frac{\mu}{q+1}(\|u_m(t)\|_{q+1}^{q+1} + \|v_m(t)\|_{q+1}^{q+1})$$
$$+ \frac{b}{2(\gamma+1)}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma+1}$$
$$= \frac{1}{q+1}I(u_m(t), v_m(t)) + \frac{a(q-1)}{2(q+1)}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)$$
$$+ \frac{b}{2(\gamma+1)}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma+1}$$
$$\ge \frac{a(q-1)}{2(q+1)}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2) \quad \text{on} \quad [0, t_{max}).$$

By the energy identity (4.7), (4.8) and (4.11), we have

$$
\begin{aligned}
\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2 &\le \frac{2(q+1)}{a(q-1)} J(u_m(t), v_m(t)) \\
&\le \frac{2(q+1)}{a(q-1)} E(u_m(t), v_m(t)) \\
&\le \frac{2(q+1)}{a(q-1)} E(u_0, v_0) \\
&\quad \text{on} \quad [0, t_{max}).
\end{aligned}
\tag{4.12}
$$

It follows from the Sobolev- Poincaré inequality and (4.12) that

$$
\begin{aligned}
\mu\|u_m(t)\|_{q+1}^{q+1} &\le \mu C(\Omega, q+1)^{q+1} \|\nabla u_m(t)\|_2^{q+1} \\
&= \frac{\mu}{a} C(\Omega, q+1)^{q+1} \|\nabla u_m(t)\|_2^{q-1} \cdot a\|\nabla u_m(t)\|_2^2 \\
&\le \frac{\mu}{a} C(\Omega, q+1)^{q+1} \left(\frac{2(q+1)}{a(q-1)} E(u_0, v_0) \right)^{\frac{q-1}{2}} \\
&\quad \times a\|\nabla u_m(t)\|_2^2 \quad \text{on} \quad [0, t_{max}).
\end{aligned}
\tag{4.13}
$$

Similarly,

$$
\begin{aligned}
\mu\|v_m(t)\|_{q+1}^{q+1} &\le \frac{\mu}{a} C(\Omega, q+1)^{q+1} \left(\frac{2(q+1)}{a(q-1)} E(u_0, v_0) \right)^{\frac{q-1}{2}} \\
&\quad \times a\|\nabla v_m(t)\|_2^2 \quad \text{on} \quad [0, t_{max}).
\end{aligned}
\tag{4.14}
$$

Thus from (4.9), (4.13) and (4.14), we obtain

$$
\begin{aligned}
&\mu(\|u_m(t)\|_{q+1}^{q+1} + \|v_m(t)\|_{q+1}^{q+1}) \\
&\le \frac{\mu}{a} C(\Omega, q+1)^{q+1} \left(\frac{2(q+1)}{a(q-1)} E(u_0, v_0) \right)^{\frac{q-1}{2}} \\
&\quad \times a(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2) \\
&\le a(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2) \\
&\quad \text{on} \quad [0, t_{max}).
\end{aligned}
\tag{4.15}
$$

Therefore we get $I(u(t), v(t)) > 0$ on $[0, t_{max})$. This implies that we can take $t_{max} = T_m$. This completes the proof of Lemma 4.1.

Using Lemma 4.1, we can deduce a priori-bounded on u_m and v_m. Lemma 4.1

implies that

$$\begin{aligned}
E(u_m(t), v_m(t)) &= \frac{1}{2}\|u'_m(t)\|_2^2 + \frac{1}{2}\|v'_m(t)\|_2^2 + J(u_m(t), v_m(t)) \\
&= \frac{1}{2}\|u'_m(t)\|_2^2 + \frac{1}{2}\|v'_m(t)\|_2^2 + \frac{1}{q+1}I(u_m(t), v_m(t)) \\
&\quad + \frac{a(q-1)}{2(q+1)}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2) \\
&\quad + \frac{b}{2(\gamma+1)}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma+1} \\
&\geq \frac{1}{2}\|u'_m(t)\|_2^2 + \frac{1}{2}\|v'_m(t)\|_2^2 \\
&\quad + \frac{a(q-1)}{2(q+1)}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{2}(\|u'_m(t)\|_2^2 + \|v'_m(t)\|_2^2) + \frac{a(q-1)}{2(q+1)}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2) \\
&\quad + \delta \int_0^t (\|u'_m(s)\|_{p+1}^{p+1} + \|v'_m(s)\|_{p+1}^{p+1}) ds \\
&\leq E(u_0, v_0).
\end{aligned} \tag{4.16}$$

A Priori Estimates II

Multiplying the equation (4.1) by $-\Delta u'_m(t)$, multiplying the equation (4.2) by $-\Delta v'_m(t)$ and adding these two equations give

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\nabla u'_m(t)\|_2^2 + \|\nabla v'_m(t)\|_2^2 + a\|\nabla u_m(t)\|_2^2 + a\|\nabla v_m(t)\|_2^2\Big) \\
&\quad + \frac{b}{2}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma}\frac{d}{dt}(\|\Delta u_m(t)\|_2^2 + \|\Delta v_m(t)\|_2^2) \\
&\quad + p\delta(|u'_m(t)|^{p-1}\nabla u'_m(t), \nabla u'_m(t)) + p\delta(|v'_m(t)|^{p-1}\nabla v'_m(t), \nabla v'_m(t)) \\
&= \mu(\nabla[|u_m(t)|^{q-1}u_m(t)], \nabla u'_m(t)) + \mu(\nabla[|v_m(t)|^{q-1}v_m(t)], \nabla v'_m(t)).
\end{aligned} \tag{4.17}$$

We set

$$\begin{aligned}
H(t) &= H(u_m(t), v_m(t)) \\
&= \frac{\|\nabla u'_m(t)\|_2^2 + \|\nabla v'_m(t)\|_2^2 + a\|\Delta u_m(t)\|_2^2 + a\|\Delta v_m(t)\|_2^2}{(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma}} \\
&\quad + b(\|\Delta u_m(t)\|_2^2 + \|\Delta v_m(t)\|_2^2).
\end{aligned}$$

Then from (4.17), we have

$$
\begin{aligned}
&\frac{1}{2}H'(t) + \frac{p\delta(|u_m'(t)|^{p-1}\nabla u_m'(t), \nabla u_m'(t)) + p\delta(|v_m'(t)|^{p-1}\nabla v_m'(t), \nabla v_m'(t))}{(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^\gamma} \\
&= \frac{\gamma(\|\nabla u_m'(t)\|_2^2 + \|\nabla v_m'(t)\|_2^2 + a(\|\Delta u_m(t)\|_2^2 + \|\Delta v_m(t)\|_2^2))}{(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma+1}} \\
&\quad \times ((\nabla u_m(t), \nabla u_m'(t)) + (\nabla v_m(t), \nabla v_m'(t))) \\
&\quad + \frac{\mu(\nabla[|u_m(t)|^{q-1}u_m(t)], \nabla u_m'(t)) + \mu(\nabla[|v_m(t)|^{q-1}v_m(t)], \nabla v_m'(t))}{(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^\gamma} \\
&\equiv I_1(t) + I_2(t).
\end{aligned} \tag{4.18}
$$

Now, (4.16) implies

$$
\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2 \le \frac{2(q+1)}{a(q-1)} E(u_0, v_0).
$$

Thus we have

$$
\begin{aligned}
I_1(t) &\le \frac{\gamma(\|\nabla u_m'(t)\|_2^2 + \|\nabla v_m'(t)\|_2^2 + a\|\Delta u_m(t)\|_2^2 + a\|\Delta v_m(t)\|_2^2)}{(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma+1}} \\
&\quad \times (\|\nabla u_m(t)\|_2\|\nabla u_m'(t)\|_2 + \|\nabla v_m(t)\|_2\|\nabla v_m'(t))\|_2) \\
&\le \frac{\gamma}{2}\frac{\|\nabla u_m'(t)\|_2^2 + \|\nabla v_m'(t)\|_2^2 + a\|\Delta u_m(t)\|_2^2 + a\|\Delta v_m(t)\|_2^2}{(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^\gamma} \\
&\quad + \frac{\gamma}{2}\frac{(\|\nabla u_m'(t)\|_2^2 + \|\nabla v_m'(t))\|_2^2 + a\|\Delta u_m(t)\|_2^2 + a\|\Delta v_m(t)\|_2^2)}{(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma+1}} \\
&\quad \times (\|\nabla u_m'(t)\|_2^2 + \|\nabla v_m'(t))\|_2^2) \\
&= \frac{\gamma}{2}\frac{\|\nabla u_m'(t)\|_2^2 + \|\nabla v_m'(t)\|_2^2 + a\|\Delta u_m(t)\|_2^2 + a\|\Delta v_m(t)\|_2^2}{(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^\gamma} \\
&\quad + \frac{\gamma}{2}\left(\frac{\|\nabla u_m'(t)\|_2^2 + \|\nabla v_m'(t))\|_2^2 + a\|\Delta u_m(t)\|_2^2 + a\|\Delta v_m(t)\|_2^2}{(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^\gamma}\right)^2 \\
&\quad \times (\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^{\gamma-1} \\
&\le \frac{\gamma}{2}H(t) + \frac{\gamma}{2}H(t)^2\left(\frac{2(q+1)}{a(q-1)}E(u_0, v_0)\right)^{\gamma-1}
\end{aligned} \tag{4.19}
$$

Now we shall compute the second term in the right hand side of (4.18). In the case $\frac{N}{N-2} \le q \le \min\{\frac{N+2}{N-2}, \frac{N-2}{[N-4]^+}\}(N \ge 3)$, we also see that

(4.20)

$$
\begin{aligned}
|(\nabla[|u_m(t)|^{q-1}u_m(t)], \nabla u_m'(t))| &\le q\||u_m(t)|^{q-1}\nabla u_m(t)\|_2\|\nabla u_m'(t)\|_2 \\
&\le q\|u_m(t)\|_{(q-1)N}^{q-1}\|\nabla u_m(t)\|_{\frac{2N}{N-2}}\|\nabla u_m'(t)\|_2 \\
&\le qC\|u_m(t)\|_{(q-1)N}^{q-1}\|\Delta u_m(t)\|_2\|\nabla u_m'(t)\|_2
\end{aligned}
$$

where we have used Hölder's inequality and Sobolev-Poincaré's inequality. We observe from Gagliardo-Nirenberg inequality and Sobolev-Pointcaré's inequality that

$$
\begin{aligned}
\|u_m(t)\|_{(q-1)N}^{q-1} &\le C\|u_m(t)\|_{\frac{2N}{N-2}}^{(q-1)(1-\theta)}\|\Delta u_m(t)\|_2^{(q-1)\theta} \\
&\le C\|\nabla u_m(t)\|_2^{(q-1)(1-\theta)}\|\Delta u_m(t)\|_2^{(q-1)\theta} \\
&\quad \text{with} \quad \theta = \frac{N-2}{2} - \frac{1}{q-1}(<1).
\end{aligned} \tag{4.21}
$$

Thus, (4.20) and (4.21) imply

$$
\begin{aligned}
&|\mu(\nabla[|u_m(t)|^{q-1}u_m(t)], \nabla u_m'(t))| \\
&\le q\mu C\|\nabla u_m(t)\|_2^{(q-1)(1-\theta)}\|\Delta u_m(t)\|_2^{1+(q-1)\theta}\|\nabla u_m'(t)\|_2 \\
&\le \frac{q\mu C}{2}\|\nabla u_m(t)\|_2^{2(q-1)(1-\theta)}\|\Delta u_m(t)\|_2^{2+2(q-1)\theta} \\
&\quad + \frac{q\mu C}{2}\|\nabla u_m'(t)\|_2^2.
\end{aligned} \tag{4.22}
$$

Similarly

$$
\begin{aligned}
&|\mu(\nabla[|v_m(t)|^{q-1}v_m(t)], \nabla v_m'(t))| \\
&\le \frac{q\mu C}{2}\|\nabla v_m(t)\|_2^{2(q-1)(1-\theta)}\|\Delta v_m(t)\|_2^{2+2(q-1)\theta} \\
&\quad + \frac{q\mu C}{2}\|\nabla v_m'(t)\|_2^2.
\end{aligned} \tag{4.23}
$$

Thus (4.18), (4.22) and (4.23) imply

$$
\begin{aligned}
I_2(t) &\le \frac{q\mu C}{2}\frac{\|\nabla u_m(t)\|_2^{2(q-1)(1-\theta)}\|\Delta u_m(t)\|_2^{2+2(q-1)\theta}}{(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^\gamma} \\
&\quad + \frac{q\mu C}{2}\frac{\|\nabla v_m(t)\|_2^{2(q-1)(1-\theta)}\|\Delta v_m(t)\|_2^{2+2(q-1)\theta}}{(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^\gamma} \\
&\quad + \frac{q\mu C}{2}\frac{\|\nabla u_m'(t)\|_2^2 + \|\nabla v_m'(t)\|_2^2}{(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^\gamma} \\
&\le \frac{q\mu C}{2}\Big(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2\Big)^{(q-1)(1-\theta)-\gamma} \\
&\quad \times (\|\Delta u_m(t)\|_2^{2+2(q-1)\theta} + \|\Delta v_m(t)\|_2^{2+2(q-1)\theta}) \\
&\quad + \frac{q\mu C}{2}\frac{\|\nabla u_m'(t)\|_2^2 + \|\nabla v_m'(t)\|_2^2}{(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^\gamma} \\
&\le \frac{q\mu C}{2}\Big(\frac{2(q+1)}{a(q-1)}E(u_0, v_0)\Big)^{(q-1)(1-\theta)-\gamma} \\
&\quad \times (\|\Delta u_m(t)\|_2^2 + \|\Delta v_m(t)\|_2^2)^{1+(q-1)\theta} \\
&\quad + \frac{q\mu C}{2}\frac{\|\nabla u_m'(t)\|_2^2 + \|\nabla v_m'(t)\|_2^2}{(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^\gamma}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{q\mu C}{2}\left(\frac{2(q+1)}{a(q-1)}E(u_0,v_0)\right)^{(q-1)(1-\theta)-\gamma} H(t)^{1+(q-1)\theta} \\
&+\frac{q\mu C}{2}H(t).
\end{aligned}
\tag{4.24}
$$

Hence (4.18), (4.19) and (4.24) imply

$$
\begin{aligned}
\frac{1}{2}H'(t) &\leq \frac{\gamma}{2}H(t)+\frac{\gamma}{2}H(t)^2\left(\frac{2(q+1)}{a(q-1)}E(u_0,v_0)\right)^{\gamma-1} \\
&+\frac{q\mu C}{2}\left(\frac{2(q+1)}{a(q-1)}E(u_0,v_0)\right)^{(q-1)(1-\theta)-\gamma} H(t)^{1+(q-1)\theta} \\
&+\frac{q\mu C}{2}H(t) \\
&\leq C_1(H(t)+H(t)^{1+(q-1)\theta}+H(t)^2)
\end{aligned}
\tag{4.25}
$$

where we have used the fact

$$
\frac{p\delta(|u'_m(t)|^{p-1}\nabla u'_m(t),\nabla u'_m(t))+p\delta(|v'_m(s)|^{p-1}\nabla v'_m(t),\nabla v'_m(t))}{(\|\nabla u_m(t)\|_2^2+\|\nabla v_m(t)\|_2^2)^\gamma}\geq 0.
$$

Integrating (4.25) from 0 to t give

$$
\frac{1}{2}H(t)\leq\frac{1}{2}H(0)+C_1\int_0^t(H(s)+H(s)^{1+(q-1)\theta}+H(s)^2)ds.
$$

We set $g(s)=s+s^{1+(q-1)\theta}+s^2$ on $s\geq 0$. Then we have

$$
\frac{1}{2}H(t)\leq\frac{1}{2}H(0)+C_1\int_0^t g(H(s))ds.
$$

Note that $g(s)$ is continuous and nondecreasing on $s\geq 0$. By applying Bihari-Langenhop's inequality (see [2]), then we get

$$
H(t)\leq M_1 \quad \text{for some constant} \quad M_1>0.
$$

Hence

$$
\|\Delta u_m(t)\|_2^2+\|\Delta v_m(t)\|_2^2\leq M_2 \tag{4.26}
$$

for some constant $M_2>0$.

A Priori Estimates III

Finally, by multiplying the equation (4.1) by $u''_m(t)$, we have

$$
\begin{aligned}
\|u''_m(t)\|_2^2 &\leq (a+b(\|\nabla u_m(t)\|_2^2+\|\nabla v_m(t)\|_2^2)^\gamma)\|\Delta u_m(t)\|_2\|u''_m(t)\|_2 \\
&+|\delta|u'_m(t)|^{p-1}(u'_m(t),u''_m(t))|+|\mu|u_m(t)|^{q-1}(u_m(t),u''_m(t))|.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\delta|u'_m(t)|^{p-1}(u'_m(t)), u''_m(t)) \\
&\leq \delta \int_\Omega |u'_m(t)|^p |u''_m(t)| dx \\
&\leq \delta \Big(\int_\Omega |u'_m(t)|^{2p} dx\Big)^{\frac{1}{2}} \Big(\int_\Omega |u''_m(t)|^2 dx\Big)^{\frac{1}{2}} \\
&= \delta \|u'_m(t)\|_{2p}^p \|u''_m(t)\|_2
\end{aligned}
$$

and similarly

$$
\mu|u_m(t)|^{q-1}(u_m(t)), u''_m(t)) \leq \mu \|u_m(t)\|_{2q}^q \|u''_m(t)\|_2.
$$

Now it follows from the Gagliardo-Nirenberg inequality that

$$
\begin{aligned}
\|u'_m(t)\|_{2p}^p &\leq C_2 \|\nabla u'_m(t)\|_2^{p\theta_1} \|u'_m(t)\|_2^{p(1-\theta_1)} \\
&\leq C_3 \|\nabla u'_m(t)\|_2^{p\theta_1}, \quad \text{with} \quad \theta_1 = \frac{(p-1)N}{2p}
\end{aligned}
$$

and

$$
\begin{aligned}
\|u_m(t)\|_{2q}^q &\leq C_4 \|\nabla u_m(t)\|_2^{q\theta_2} \|u_m(t)\|_2^{q(1-\theta_2)} \\
&\leq C_5 \|\nabla u_m(t)\|_2^{q\theta_2} \quad \text{with} \quad \theta_2 = \frac{(q-1)N}{2q}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|u''_m(t)\|_2 &\leq (a + b(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)^\gamma) \|\Delta u_m(t)\|_2 \\
&\quad + C_3 \|\nabla u'_m(t)\|_2^{p\theta_1} + C_5 \|\nabla u_m(t)\|_2^{q\theta_2} \\
&\leq M_3 \quad \text{for some constant} \quad M_3 > 0.
\end{aligned} \tag{4.27}
$$

By applying similar method as the one for u_m, we get

$$
\|v''_m(t)\|_2 \leq M_4 \quad \text{for some constant} \quad M_4 > 0. \tag{4.28}
$$

Limiting process

By the above estimates (4.16), (4.26), (4.27) and (4.28), $\{u_m\}, \{v_m\}$ have subsequences still denoted by $\{u_m\}, \{v_m\}$ such that

$$
u_m \to u, \quad v_m \to v \quad \text{in} \quad L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega)) \quad \text{weak}^*, \tag{4.29}
$$

$$
u'_m \to u', \quad v'_m \to v' \quad \text{in} \quad L^\infty(0,T; H_0^1(\Omega)) \quad \text{weak}^*, \tag{4.30}
$$

$$(4.31) \qquad u''_m \to u'', \quad v''_m \to v'' \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)) \quad \text{weak}^*,$$

$$(4.32) \qquad u'_m \to u', \quad v'_m \to v' \quad \text{in} \quad L^2(0,T;H^1_0(\Omega)) \quad \text{weak},$$

$$(4.33) \qquad -\Delta u_m \to -\Delta u, \quad -\Delta v_m \to -\Delta v \quad \text{in} \quad L^\infty(0,T;H^{-1}(\Omega)) \quad \text{weak}^*.$$

Using Aubin-Lions compactness lemma, we can extract from $\{u_m\}, \{v_m\}$ subsequences, still denoted by $\{u_m\}, \{v_m\}$ such that

$$(4.34) \qquad u_m \to u, \quad v_m \to v \quad \text{strongly in} \quad L^2(0,T;H^1_0(\Omega)).$$

It follows from (4.34) that for each $t \in [0,T]$,

$$(4.35) \qquad u_m(t) \to u(t), \quad v_m(t) \to v(t) \quad \text{strongly in} \quad L^2(\Omega).$$

By letting $m \to \infty$ in (4.1) and (4.2), we can find that u and v satisfy the equations;

$$(4.36) \qquad \begin{aligned} &(u''(t),w) - (a + b(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)^\gamma \Delta u(t), w) \\ &\quad + \delta|u'(t)|^{p-1}(u'(t),w) = \mu|u(t)|^{q-1}(u(t),w) \quad \text{for all} \quad w \in H^1_0(\Omega), \end{aligned}$$

$$(4.37) \qquad \begin{aligned} &(v''(t),w) - (a + b(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)^\gamma \Delta v(t), w) \\ &\quad + \delta|v'(t)|^{p-1}(v'(t),w) = \mu|v(t)|^{q-1}(v(t),w) \quad \text{for all} \quad w \in H^1_0(\Omega). \end{aligned}$$

Now, the above result (4.35) imply

$$(4.38) \qquad u_m(0) = u_{0m} \to u_0 \quad \text{strongly in} \quad H^1_0(\Omega).$$

Thus, from (4.3) and (4.38) $u(0) = u_0$. Also, from (4.35) we obtain

$$(4.39) \qquad (u'_m(0) - u'(0), w) \to 0 \quad \text{as} \quad m \to \infty \quad \text{for each} \quad w \in H^1_0(\Omega).$$

Thus, (4.4) and (4.39) imply

$$u'(0) = u_1.$$

Similarly, we obtain $v(0) = v_0$ and $v'(0) = v_1$. This completes the proof of Theorem 3.1.

In fact, a priori estimates imply that the (approximated) solution $u(t)$ exists in $[0,\infty)$.

5. Asymptotic behavior of solutions

Theorem 5.1. *Let $u(t), v(t)$ and q be the same as in Theorem 3.1. Assume that either $1 \le p < +\infty$ $(N = 1,2)$ or $1 \le p \le \frac{N}{N-2}$ $(N \ge 3)$ holds. Then we have the decay estimates if $p = 1$,*

$$E(u(t),v(t)) \le C_0 e^{-kt} \quad \textit{on} \quad [0,+\infty),$$

and if $p > 1$, then

$$E(u(t), v(t)) \le C_1(1+t)^{-\frac{2}{p-1}} \quad \text{on} \quad [0, +\infty)$$

where k, C_0 and C_1 are certain positive constants depending on $\|\nabla u_0\|_2, \|u_1\|_2$.

To prove our Theorem 5.1, we need the following Lemma.

Lemma 5.2. *Let $u(t)$ and q be the same as in Lemma 4.1. Then there is a certain number η_0 with $0 < \eta_0 < 1$ such that*

$$\mu(\|u(t)\|_{q+1}^{q+1} + \|v(t)\|_{q+1}^{q+1}) \le a(1-\eta_0)(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2) \quad \text{on} \quad [0, \infty)$$

where

$$\eta_0 \equiv 1 - \frac{\mu}{a} C(\Omega, q+1)^{q+1} \left(\frac{2(q+1)}{a(q-1)} E(u_0, v_0) \right)^{\frac{q-1}{2}}.$$

Proof. It follows from the Sobolev-Poincaré inequality and (4.16)

$$\begin{aligned}
\mu\|u(t)\|_{q+1}^{q+1} &\le \mu C(\Omega, q+1)^{q+1} \|\nabla u(t)\|_2^{q+1} \\
&= \frac{\mu}{a} C(\Omega, q+1)^{q+1} \|\nabla u(t)\|_2^{q-1} \cdot a\|\nabla u(t)\|_2^2 \\
&\le \frac{\mu}{a} C(\Omega, q+1)^{q+1} \left(\frac{2(q+1)}{a(q-1)} E(u_0, v_0) \right)^{\frac{q-1}{2}} a\|\nabla u(t)\|_2^2
\end{aligned}$$

and

$$\mu\|v(t)\|_{q+1}^{q+1} \le \frac{\mu}{a} C(\Omega, q+1)^{q+1} \left(\frac{2(q+1)}{a(q-1)} E(u_0, v_0) \right)^{\frac{q-1}{2}} a\|\nabla v(t)\|_2^2.$$

Thus we get

$$\begin{aligned}
\mu\|u(t)\|_{q+1}^{q+1} + \mu\|v(t)\|_{q+1}^{q+1} &\le \frac{\mu}{a} C(\Omega, q+1)^{q+1} \left(\frac{2(q+1)}{a(q-1)} E(u_0, v_0) \right)^{\frac{q-1}{2}} \\
&\qquad \times a(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2) \\
&\equiv a(1-\eta_0)(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2) \quad \text{on} \quad [0, \infty).
\end{aligned}$$

This completes the proof of Lemma 5.2.

Proof of Theorem 5.1. We denote $E(u(t), v(t))$ by $E(t)$ and $E(u_0, v_0)$ by $E(0)$. Let $u(t)$ and $v(t)$ be solutions of the following problems;

$$\begin{aligned}
&u''(t) - (a + b(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)^\gamma)\Delta u(t) + \delta|u'(t)|^{p-1}u'(t) \\
&= \mu|u(t)|^{q-1}u(t),
\end{aligned} \tag{5.1}$$

$$v''(t) - (a + b(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)^\gamma)\Delta v(t) + \delta|v'(t)|^{p-1}v'(t) \\ = \mu|v(t)|^{q-1}v(t), \tag{5.2}$$

$$u(0) = u_0, \quad u'(0) = u_1, \\ v(0) = v_0, \quad v'(0) = v_1. \tag{5.3}$$

Multiplying the equation (5.1) by $u'(t)$, multiplying the equation (5.2) by $v'(t)$, summation these two equations and then integrating over $[t, t+1] \times \Omega$, we get

$$\delta \int_t^{t+1} (\|u'(s)\|_{p+1}^{p+1} + \|v'(s)\|_{p+1}^{p+1}) ds = E(t) - E(t+1) \\ \equiv \delta F(t)^{p+1} \tag{5.4}$$

where

$$E(t) = \frac{1}{2}\Big(\|u'(t)\|_2^2 + \|v'(t)\|_2^2 + a\|\nabla u(t)\|_2^2 + a\|\nabla v(t)\|_2^2\Big) \\ + \frac{b}{2(\gamma+1)}(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)^{\gamma+1} - \frac{\mu}{q+1}(\|u(t)\|_{q+1}^{q+1} + \|v(t)\|_{q+1}^{q+1}).$$

It follows from Hölder's inequality and (5.4) that

$$\begin{aligned} \int_t^{t+1} \|u'(s)\|_2^2 ds &= \int_t^{t+1} \int_\Omega |u'(s)|^2 dx ds \\ &\le m(\Omega)^{\frac{p-1}{p+1}} \int_t^{t+1} \Big(\int_\Omega |u'(s)|^{p+1} dx\Big)^{\frac{2}{p+1}} ds \\ &\le m(\Omega)^{\frac{p-1}{p+1}} \int_t^{t+1} \|u'(s)\|_{p+1}^2 ds \\ &\le m(\Omega)^{\frac{p-1}{p+1}} \Big(\int_t^{t+1} \|u'(s)\|_{p+1}^{p+1} ds\Big)^{\frac{2}{p+1}} \Big(\int_t^{t+1} ds\Big)^{\frac{p-1}{p+1}} \\ &\le m(\Omega)^{\frac{p-1}{p+1}} F(t)^2. \end{aligned} \tag{5.5}$$

Similarly, we obtain

$$\int_t^{t+1} \|v'(s)\|_2^2 ds \le m(\Omega)^{\frac{p-1}{p+1}} F(t)^2. \tag{5.6}$$

Applying the mean value theorem to the left hand side of (5.5)-(5.6), there exist two points $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|u'(t_i)\|_2 \le 2m(\Omega)^{\frac{p-1}{2(p+1)}} F(t) \quad i = 1, 2, \tag{5.7}$$

and

$$\|v'(t_i)\|_2 \le 2m(\Omega)^{\frac{p-1}{2(p+1)}} F(t) \quad i = 1, 2. \tag{5.8}$$

Next, multiplying (5.1) by $u(t)$ and multiplying (5.2) by $v(t)$, adding these two equations and integrating over $[t_1, t_2] \times \Omega$ give (cf. (5.7) and (5.8))

$$
\begin{aligned}
&\int_{t_1}^{t_2} I(u(s), v(s))ds \\
&= \int_{t_1}^{t_2} (a\|\nabla u(s)\|_2^2 + a\|\nabla v(s)\|_2^2 - \mu\|u(s)\|_{q+1}^{q+1} - \mu\|v(s)\|_{q+1}^{q+1})ds \\
&\le \sum_{i=1}^{2}(\|u'(t_i)\|_2\|u(t_i)\|_2 + \|v'(t_i)\|_2\|v(t_i)\|_2) + \int_{t_1}^{t_2}(\|u'(s)\|_2^2 + \|v'(s)\|_2^2)ds \\
&\quad + \delta\left|\int_{t_1}^{t_2} |u'(s)|^{p-1}(u'(s), u(s))ds\right| + \delta\left|\int_{t_1}^{t_2} |v'(s)|^{p-1}(v'(s), v(s))ds\right| \qquad (5.9)\\
&\le 4m(\Omega)^{\frac{p-1}{2(p+1)}}F(t)(\max_{t_1\le s\le t_2}\|u(s)\|_2 + \max_{t_1\le s\le t_2}\|v(s)\|_2) + 2m(\Omega)^{\frac{p-1}{p+1}}F(t)^2 \\
&\quad + \delta\int_{t_1}^{t_2}\int_\Omega |u'(s)|^p|u(s)|dxds + \delta\int_{t_1}^{t_2}\int_\Omega |v'(s)|^p|v(s)|dxds \\
&\le 8m(\Omega)^{\frac{p-1}{2(p+1)}}F(t)\max_{t_1\le s\le t_2}E(s)^{\frac{1}{2}} + 2m(\Omega)^{\frac{p-1}{p+1}}F(t)^2 \\
&\quad + \delta\int_{t_1}^{t_2}\int_\Omega |u'(s)|^p|u(s)|dxds + \delta\int_{t_1}^{t_2}\int_\Omega |v'(s)|^p|v(s)|dxds.
\end{aligned}
$$

Here we note that

$$
\begin{aligned}
&\delta\int_{t_1}^{t_2}\int_\Omega |u'(s)|^p|u(s)|dxds \\
&\le \delta\int_{t_1}^{t_2}\left(\int_\Omega |u'(s)|^{p+1}dx\right)^{\frac{p}{p+1}}\left(\int_\Omega |u(s)|^{p+1}dx\right)^{\frac{1}{p+1}}ds \qquad (5.10)\\
&= \delta\int_{t_1}^{t_2}\|u'(s)\|_{p+1}^p\|u(s)\|_{p+1}ds \\
&\le \delta C(\Omega, p+1)\int_{t_1}^{t_2}\|u'(s)\|_{p+1}^p\|\nabla u(s)\|_2 ds
\end{aligned}
$$

where we have used Hölder's inequality and Sobolev-Poincaré's inequality. Since $I(u(t), v(t)) \ge 0$ on $[0, \infty)$, we see that

$$
\begin{aligned}
E(t) &\ge J(u(t), v(t)) \\
&= \frac{1}{q+1}I(u(t), v(t)) + \frac{a(q-1)}{2(q+1)}(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2) \\
&\quad + \frac{b}{2(\gamma+1)}(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)^{\gamma+1} \qquad (5.11)\\
&\ge \frac{a(q-1)}{2(q+1)}(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2).
\end{aligned}
$$

From (5.4), (5.10) and (5.11), we get

$$
\begin{aligned}
&\delta \int_{t_1}^{t_2} \int_{\Omega} |u'(s)|^p |u(s)| dx ds \\
&\le \delta C(\Omega, p+1) \Big(\int_{t_1}^{t_2} \|u'(s)\|_{p+1}^{p+1} ds \Big)^{\frac{p}{p+1}} \Big(\int_{t_1}^{t_2} ds \Big)^{\frac{1}{p+1}} \\
&\qquad \times \left(\frac{2(q+1)}{a(q-1)} \right)^{\frac{1}{2}} \sup_{t_1 \le s \le t_2} E(s)^{\frac{1}{2}} \\
&\le \delta C(\Omega, p+1) \Big(\frac{2(q+1)}{a(q-1)} \Big)^{\frac{1}{2}} F(t)^p \sup_{t_1 \le s \le t_2} E(s)^{\frac{1}{2}}.
\end{aligned} \tag{5.12}
$$

Similarly we have

$$
\begin{aligned}
&\delta \int_{t_1}^{t_2} \int_{\Omega} |v'(s)|^p |v(s)| dx ds \\
&\le \delta C(\Omega, p+1) \Big(\frac{2(q+1)}{a(q-1)} \Big)^{\frac{1}{2}} F(t)^p \sup_{t_1 \le s \le t_2} E(s)^{\frac{1}{2}}.
\end{aligned} \tag{5.13}
$$

From (5.9), (5.12) and (5.13), we have

$$
\begin{aligned}
&\int_{t_1}^{t_2} I(u(s), v(s)) ds \\
&\le 8m(\Omega)^{\frac{p-1}{2(p+1)}} F(t) \max_{t_1 \le s \le t_2} E(s)^{\frac{1}{2}} + 2m(\Omega)^{\frac{p-1}{p+1}} F(t)^2 \\
&\quad + 2\delta C(\Omega, p+1) \Big(\frac{2(q+1)}{a(q-1)} \Big)^{\frac{1}{2}} F(t)^p \sup_{t_1 \le s \le t_2} E(s)^{\frac{1}{2}}.
\end{aligned} \tag{5.14}
$$

On the other hand, from Lemma 5.2 and the definition of $I(u(t), v(t))$, we have

$$
a\eta_0 (\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2) \le I(u(t), v(t)). \tag{5.15}
$$

and

$$
\begin{aligned}
&(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)^{\gamma+1} \\
&= (\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)^{\gamma} \\
&\le \frac{1}{a\eta_0} \left(\frac{2(q+1)}{a(q-1)} \right)^{\gamma} I(u(t), v(t))
\end{aligned} \tag{5.16}
$$

Thus, from (5.14)-(5.16), we obtain

$$
\begin{aligned}
&\int_{t_1}^{t_2} E(s) ds \\
&= \frac{1}{2} \int_{t_1}^{t_2} (\|u'(s)\|_2^2 + \|v'(s)\|_2^2) ds + \int_{t_1}^{t_2} J(u(s), v(s)) ds
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{t_1}^{t_2}(\|u'(s)\|_2^2 + \|v'(s)\|_2^2)ds + \frac{1}{q+1}\int_{t_1}^{t_2} I(u(s),v(s))ds \\
&\quad + \frac{a(q-1)}{2(q+1)}\int_{t_1}^{t_2}(\|\nabla u(s)\|_2^2 + \|\nabla v(s)\|_2^2)ds \\
&\quad + \frac{b}{2(\gamma+1)}\int_{t_1}^{t_2}(\|\nabla u(s)\|_2^2 + \|\nabla v(s)\|_2^2)^{\gamma+1}ds \\
&\le m(\Omega)^{\frac{p-1}{p+1}}F(t)^2 \\
&\quad + \left(\frac{1}{q+1} + \frac{(q-1)}{2\eta_0(q+1)} + \frac{b}{2a\eta_0(\gamma+1)}\left(\frac{2(q+1)}{a(q-1)}\right)^{\gamma}\right)\int_{t_1}^{t_2} I(u(s),v(s))ds.
\end{aligned} \tag{5.17}
$$

Consequently, from (5.14), we get

$$
\begin{aligned}
\int_{t_1}^{t_2} E(s)ds &\le C_1\Big(F(t)\sup_{t_1\le s\le t_2} E(s)^{\frac{1}{2}} + F(t)^2 + F(t)^p \sup_{t_1\le s\le t_2} E(s)^{\frac{1}{2}}\Big) \\
&\le C_2(E(t)^{\frac{1}{2}}F(t) + F(t)^2 + E(t)^{\frac{1}{2}}F(t)^p).
\end{aligned} \tag{5.18}
$$

Again multiplying (5.1) by $u'(t)$, multiplying (5.2) by $v'(t)$, adding these two equations and integrating over $[t,t_2]\times\Omega$ give

$$
E(t) = E(t_2) + \delta\int_t^{t_2}(\|u'(s)\|_{p+1}^{p+1} + \|v'(s)\|_{p+1}^{p+1})ds.
$$

Since $t_2 - t_1 \ge \frac{1}{2}$, we get

$$
\begin{aligned}
\int_{t_1}^{t_2} E(s)ds &\ge \int_{t_1}^{t_2} E(t_2)ds \\
&= (t_2 - t_1)E(t_2) \\
&\ge \frac{1}{2}E(t_2)
\end{aligned}
$$

that is,

$$
E(t_2) \le 2\int_{t_1}^{t_2} E(s)ds.
$$

From (5.4) and (5.18), we have

$$
\begin{aligned}
E(t) &= E(t_2) + \delta\int_t^{t_2}(\|u'(s)\|_{p+1}^{p+1} + \|v'(s)\|_{p+1}^{p+1})ds \\
&\le 2\int_{t_1}^{t_2} E(s)ds + \delta\int_t^{t+1}(\|u'(s)\|_{p+1}^{p+1} + \|v'(s)\|_{p+1}^{p+1})ds \\
&\le 2C_2(E(t)^{\frac{1}{2}}F(t) + F(t)^2 + E(t)^{\frac{1}{2}}F(t)^p) + \delta F(t)^{p+1} \\
&\le C_3(E(t)^{\frac{1}{2}}F(t) + F(t)^2 + E(t)^{\frac{1}{2}}F(t)^p + F(t)^{p+1}) \\
&\quad \text{for some constant} \quad C_3 > 0.
\end{aligned}
$$

Thus, we get

$$E(t) \le C_4(F(t)^2 + F(t)^{2p} + F(t)^{p+1}) \quad \text{for some constant} \quad C_4 > 0. \tag{5.19}$$

When $p = 1$, we have

$$\begin{aligned} E(t) &\le C_4(F(t)^2) \\ &= C_4(E(t) - E(t+1)). \end{aligned} \tag{5.20}$$

Thus, applying Nakao's inequality to (5.20) yield

$$E(t) \le E(0)e^{-kt} \quad \text{where} \quad k = \log \frac{C_4}{C_4 - 1}.$$

Note that since $E(t)$ is decreasing and $E(t) \ge 0$ on $[0, \infty)$,

$$\begin{aligned} \delta F(t)^{p+1} &= E(t) - E(t+1) \\ &\le E(0). \end{aligned}$$

Thus

$$F(t) \le (\frac{1}{\delta} E(0))^{\frac{1}{p+1}}. \tag{5.21}$$

On the other hand, when $p > 1$, it follows from (5.19) that

$$\begin{aligned} E(t) &\le C_4(1 + F(t)^{2p-2} + F(t)^{p-1})F(t)^2 \\ &\le C_5\left(1 + E(0)^{\frac{2p-2}{p+1}} + E(0)^{\frac{p-1}{p+1}}\right)F(t)^2. \\ &\equiv C_6(E(0))F(t)^2 \end{aligned}$$

with $\lim_{E(0)\to 0} C_6(E(0)) = C_7 > 0$.

Hence

$$\begin{aligned} E(t)^{1+\frac{p-1}{2}} &\le C_6(E(0))^{\frac{p+1}{2}} F(t)^{p+1} \\ &\le \frac{1}{\delta} C_6(E(0))^{\frac{p+1}{2}} (E(t) - E(t+1)). \end{aligned} \tag{5.22}$$

Setting $C(E(0)) \equiv \delta C_6(E(0))^{-\frac{p+1}{2}}$, applying Nakao's inequality to (5.22) yield

$$E(t) \le \left(E(0)^{-\frac{p-1}{2}} + \frac{(p-1)C(E(0))}{2}[t-1]^+\right)^{-\frac{2}{p-1}}.$$

This completes the proof of Theorem 5.1.

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