Nihonkai Math. J.
Vol.9 (1998), 17–25

DUALITY FOR MULTIOBJECTIVE FRACTIONAL VARIATIONAL PROBLEMS WITH GENERALIZED INVEXITY*

DO SANG KIM, GUE MYUNG LEE AND HUN KUK

ABSTRACT. A multiobjective fractional variational problem (FVP) is considered. By establishing the multiobjective nonfractional variational problem $(NFVP)$ equivalent to (FVP), we formulate the Mond-Weir type dual problem (FVD) of (FVP) and prove some duality theorems for (FVP) under generalized invexity assumptions.

KEYWORDS. Multiobjective fractional variational problems, Mond- Weir dual, efficient solutions, pseudo-invexity, quasi-invexity.

1. Introduction

Duality theorems for fractional minimization problems have been of much interest in the past ([1],[4],[5],[8]). Recently there has been of growing interest in studying duality for multiobjective (fractional) variational and control problems ([2], [7], [10]). Using the parametric equivalence, Bector et al. [1] formulated a dual program for a multiobjective fractional program having continuously differentiable convex functions.

In this paper, a multiobjective fractional variational problem (FVP) is considered. By establishing the multiobjective nonfractional variational problem $(NFVP)$ equivalent to (FVP), we formulate the Mond-Weir type dual problem (FVD) of (FVP), and prove weak, strong and converse duality theorems for (FVP) under generalized invexity assumptions.

2. Notations and Preliminaries

The following conventions for vectors in R^n will be used:

*This research was supported by KOSEF 971-0106-113-1 .

$$
\begin{aligned}
&x \leqq y \iff x_i \leqq y_i, \; i = 1, \cdots, n; \\
&x < y \iff x_i < y_i, \; i = 1, \cdots, n; \\
&x \leq y \iff x_i \leqq y_i, \; i = 1, \cdots, n \text{ but } x \neq y; \\
&x \not\leq y \text{ is the negation of } x \leq y.
\end{aligned}
$$

Let $I = [a, b]$ be a real interval and $f : I \times R^n \times R^n \to R^p$, $g : I \times R^n \times R^n \to R^p$ and $h : I \times R^n \times R^n \to R^m$ be continuously differentiable functions.

Let $C(I, R^n)$ denote the space of piecewise smooth functions x with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiable operator D is given by

$$u = Dx \iff x(t) = \alpha + \int_a^t u(s)ds,$$

where is α is a given boundary value.

Consider the following multiobjective fractional variational problem:

$$
\begin{aligned}
(FVP) \quad \text{Minimize} \quad & \frac{\int_a^b f(t, x(t), \dot{x}(t))dt}{\int_a^b g(t, x(t), \dot{x}(t))dt} := \left(\frac{\int_a^b f^1 dt}{\int_a^b g^1 dt}, \cdots, \frac{\int_a^b f^p dt}{\int_a^b g^p dt} \right) \\
\text{subject to} \quad & x(a) = \alpha, x(b) = \beta, \\
& h(t, x(t), \dot{x}(t)) \leqq 0.
\end{aligned}
$$

Assume that $g^i(t, x, \dot{x}) > 0$ and $f^i(t, x, \dot{x}) \geqq 0$ for all $i = 1, \cdots, p$.

Let X denote the set of all feasible solutions of (FVP).

Definition 1. A point $x^* \in X$ is said to be an efficient solution of (FVP) if for all $x \in X$,

$$\frac{\int_a^b f(t, x, \dot{x})dt}{\int_a^b g(t, x, \dot{x})dt} \not\leq \frac{\int_a^b f(t, x^*, \dot{x}^*)dt}{\int_a^b g(t, x^*, \dot{x}^*)dt}$$

Now we define the pseudo-invex and the quasi-invex functionals as follows :

Definition 2. The functional $\int_a^b f$ is (strictly) pseudo-invex at $(u, \dot{u})$ w.r.t. η if there exists $\eta(t, x, u)$ with $\eta(t, x, x) = 0$ such that

$$
\begin{aligned}
&\int_a^b [\eta(t, x, u) f_x(t, u, \dot{u}) + (D\eta(t, x, u)) f_{\dot{x}}(t, u, \dot{u})]dt \geqq 0 \\
&\Rightarrow \int_a^b f(t, x, \dot{x})dt \geqq (>) \int_a^b f(t, u, \dot{u})dt.
\end{aligned}
$$

Definition 3. The functional $\int_a^b f$ is (strictly) quasi-invex at $(u, \dot{u})$ w.r.t. η if there exists $\eta(t, x, u)$ with $\eta(t, x, x) = 0$ such that

$$\int_a^b f(t, x, \dot{x})dt \leqq \int_a^b f(t, u, \dot{u})dt$$
$$\Rightarrow \int_a^b [\eta(t, x, u) f_x(t, u, \dot{u}) + (D\eta(t, x, u)) f_{\dot{x}}(t, u, \dot{u})]dt \leqq (<) 0.$$

Also we consider the following multiobjective nonfractional variational problem:

$$\begin{aligned} (NFVP) \quad \text{Minimize} \quad & v = (v_1, \cdots, v_p) \\ \text{subject to} \quad & x(a) = \alpha, \ \ x(b) = \beta, \\ & \int_a^b [f(t, x, \dot{x}) - vg(t, x, \dot{x})]dt \leqq 0, \quad h(t, x, \dot{x}) \leqq 0, \end{aligned}$$

where $f - vg := (f^1 - v_1 g^1, \cdots, f^p - v_p g^p)$.

We establish an equivalent relationship between (FVP) and $(NFVP)$.

Lemma 1. If x^* is an efficient solution of (FVP), then (x^*, v^*) is an efficient solution of $(NFVP)$, where $v^* = \frac{\int_a^b f(t,x^*,\dot{x}^*)dt}{\int_a^b g(t,x^*,\dot{x}^*)dt}$.

Proof. Suppose that (x^*, v^*) is not efficient for $(NFVP)$. Then there exists (x, v) such that

$$v \leq \frac{\int_a^b f(t, x^*, \dot{x}^*)dt}{\int_a^b g(t, x^*, \dot{x}^*)dt},$$
$$\int_a^b [f(t, x, \dot{x}) - vg(t, x, \dot{x})]dt \leqq 0, \quad h(t, x, \dot{x}) \leqq 0.$$

Thus $\frac{\int_a^b f(t,x,\dot{x})dt}{\int_a^b g(t,x,\dot{x})dt} \leq \frac{\int_a^b f(t,x^*,\dot{x}^*)dt}{\int_a^b g(t,x^*,\dot{x}^*)dt}$. Hence x^* is not efficient for (FVP).

Lemma 2. If (x^*, v^*) is an efficient solution of $(NFVP)$, then x^* is an efficient solution of (FVP).

Proof. Suppose that x^* is not efficient for (FVP). Then there exists x such that

$$\frac{\int_a^b f(t,x,\dot{x})dt}{\int_a^b g(t,x,\dot{x})dt} \leq \frac{\int_a^b f(t,x^*,\dot{x}^*)dt}{\int_a^b g(t,x^*,\dot{x}^*)dt}, \quad h(t,x,\dot{x}) \leqq 0.$$

By the feasibility of (x^*, v^*), we obtain

$$\frac{\int_a^b f(t,x,\dot{x})dt}{\int_a^b g(t,x,\dot{x})dt} \leq v^*. \tag{1}$$

Let $v = \frac{\int_a^b f(t,x,\dot{x})dt}{\int_a^b g(t,x,\dot{x})dt}$. Then (x, v) is a feasible solution of $(NFVP)$. Thus, from (1), (x^*, v^*) is not efficient for $(NFVP)$. □

Remark 1. I. By Lemma 1 and Lemma 2, $(NFVP)$ is equivalent to (FVP).
II. If (x^*, v^*) is an efficient solution of $(NFVP)$, then by the definition of efficiency,

$$v^* = \frac{\int_a^b f(t,x^*,\dot{x}^*)dt}{\int_a^b g(t,x^*,\dot{x}^*)dt}.$$

Now, taking the Mond-Weir [11] type dual of $(NFVP)$, we formulate our dual problem of (FVP) as follows:

$$\begin{aligned}
(FVD) \quad \text{Maximize} \quad & v = (v_1, \cdots, v_p) \\
\text{subject to} \quad & u(a) = \alpha, u(b) = \beta, \\
& \tau^T\{f_x - vg_x\} + \mu^T h_x = D[\tau^T\{f_{\dot{x}} - vg_{\dot{x}}\} + \mu^T h_{\dot{x}}], \\
& \int_a^b \tau^T(f - vg)dt \geqq 0, \\
& \mu^T h \geqq 0, \\
& \tau > 0, \ \ \mu \geqq 0,
\end{aligned}$$

where $\tau \in R^p$ and $\mu : I \to R^m$ is a piecewise smooth function.

Let Y denote the set of all feasible solutions of (FVD).

3. Duality Theorems

In this section, we establish the weak, strong and converse duality theorems for (FVP).

Lemma 3 ([3]). x^* is an efficient solution of (FVP) if and only if for all $k = 1, \cdots, p$, x^* solves (FVP_k),where (FVP_k) is the following problem:

$$(FVP_k) \quad \text{Minimize} \quad \frac{\int_a^b f_k(t,x,\dot{x})dt}{\int_a^b g_k(t,x,\dot{x})dt}$$

$$\text{subject to} \quad x(a) = \alpha, \ x(b) = \beta,$$

$$\frac{\int_a^b f_i(t,x,\dot{x})dt}{\int_a^b g_i(t,x,\dot{x})dt} \leqq \frac{\int_a^b f_i(t,x^*,\dot{x}^*)dt}{\int_a^b g_i(t,x^*,\dot{x}^*)dt}$$

$$\text{for all } i \neq k,$$

$$h(t,x,\dot{x}) \leqq 0, \quad k = 1, \cdots, p.$$

From Lemma 3, we can prove the following Kuhn-Tucker type necessary optimality theorem for (FVP) by the method similar to the proof in Theorem 3.4 of [6].

Theorem 1. Let x^* be an efficient solution of (FVP). Assume that x^* satisfies the Slater's constraint qualification [9] for (FVP_k), $k = 1, \cdots, p$. Then there exist $\tau^* \in R^p$, $v^* \in R^p$ and a piecewise smooth function $\mu^* : I \to R^m$ such that

$$\tau^{*T}(f_x^* - v^* g_x^*) + \mu^{*T} h_x^* = D[\tau^{*T}(f_{\dot{x}}^* - v^* g_{\dot{x}}^*) + \mu^{*T} h_{\dot{x}}^*],$$

$$\int_a^b (f^* - v^* g^*)dt = 0, \quad \mu^{*T} h^* = 0, \quad \tau^* > 0, \quad \mu^* \geqq 0.$$

Theorem 2 (Weak Duality). Let $x \in X$ and $(u, \tau, \mu, v) \in Y$. Assume that
I. $\int_a^b \tau^T(f - vg)$ is quasi-invex and $\int_a^b \mu^T h$ is strictly pseudo-invex, or
II. $\int_a^b \tau^T(f - vg)$ is pseudo-invex and $\int_a^b \mu^T h$ is strictly quasi-invex at $(u, \dot{u})$ w.r.t. η. Then

$$\frac{\int_a^b f(t,x,\dot{x})dt}{\int_a^b g(t,x,\dot{x})dt} \nleq (v_1, \cdots, v_p)$$

Proof. I. Suppose to the contrary that

$$\frac{\int_a^b f(t,x,\dot{x})dt}{\int_a^b g(t,x,\dot{x})dt} \leq (v_1, \cdots, v_p).$$

Then for all $\tau > 0$,

$$\int_a^b \tau^T \{f(t,x,\dot{x}) - vg(t,x,\dot{x})\}dt < 0$$

and from the feasible condition, we have

$$\int_a^b \tau^T \{f(t,x,\dot{x}) - vg(t,x,\dot{x})\}dt < \int_a^b \tau^T \{f(t,u,\dot{u}) - vg(t,u,\dot{u})\}dt$$

By the quasi-invexity of $\int_a^b \tau^T (f - v^T g)$,

$$\int_a^b \left[\eta(t,x,u)\{\tau^T (f_x - vg_x)\} + (D\eta(t,x,u))\{\tau^T (f_{\dot{x}} - vg_{\dot{x}})\}\right] dt \leqq 0.$$

By the feasibility of (u, τ, μ, v) and integration by parts, the above inequality becomes

$$-\int_a^b \eta(t,x,u)\{\mu^T h_x - D\mu^T h_{\dot{x}}\}dt \leqq 0. \tag{2}$$

Since $\int_a^b \mu^T h(t,x,\dot{x})dt \leqq \int_a^b \mu^T h(t,u,\dot{u})dt$, by the strict pseudo-invexity of $\int_a^b \mu^T h$ and integration by parts,

$$\int_a^b \eta(t,x,u)\{\mu^T h_x - D\mu^T h_{\dot{x}}\}dt < 0,$$

which is contradiction to (2).
II. By the method similar to the proof in I, the result holds. □

Theorem 3 (Strong Duality). Let x^* be an efficient solution of (FVP). Assume that x^* satisfies a constraint qualification [9] for (FVP_k), $k =$

$1, \cdots, p$. If assumptions of Theorem 2 hold, then there exist $\tau^* \in R^p$, $v^* \in R^p$ and a piecewise smooth function $\mu^* : I \to R^m$ such that $(x^*, \tau^*, \mu^*, v^*)$ is an efficient solution of (FVD)

Proof. By Theorem 1, there exist $\tau^* \in R^p$, $v^* \in R^p$ and a piecewise smooth function $\mu^* : I \to R^m$ such that $(x^*, \tau^*, \mu^*, v^*)$ is an feasible solution of (FVD) and $v^* = \frac{\int_a^b f(t,x^*,\dot{x}^*)dt}{\int_a^b g(t,x^*,\dot{x}^*)dt}$. By Theorem 2, $(x^*, \tau^*, \mu^*, v^*)$ is an efficient solution of (FVD). □

For the converse duality, we make the assumption that Z denotes the space of the piecewise differentiable function $x : I \to R^n$ for which $x(a) = 0 = x(b)$ equipped with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty + \|D^2x\|_\infty$.

(FVD) may be rewritten in the following form :

$$\begin{aligned} \text{Minimize} \quad & -v \\ \text{subject to} \quad & u(a) = \alpha, \quad u(b) = \beta, \\ & \theta(t, u, \dot{u}, \ddot{u}, \mu, \tau, v) = 0, \\ & \int_a^b \tau^T (f - vg) dt \geqq 0, \\ & \mu^T h \geqq 0, \qquad \tau > 0, \qquad \mu \geqq 0, \end{aligned}$$

where $\theta = \tau^T (f_x - vg_x) + \mu^T h_x - D\left[\tau^T (f_{\dot{x}} - vg_{\dot{x}}) + \mu^T h_{\dot{x}}\right]$ with $\ddot{u} = D^2u(t)$.

Consider $\theta(\cdot, u(\cdot), \dot{u}(\cdot), \ddot{u}(\cdot), \mu(\cdot), \tau, v)$ as defining a map $\psi : Z \times W \times R^p \times R^p \to A$, where W is the space of piecewise differentiable function $\mu : I \to R^m$ and A is a Banach space.

Theorem 4 (Converse Duality). Let $(u^*, \tau^*, \mu^*, v^*)$ be an efficient solution of (FVD). Assume that
I. the Fréchet derivative ψ' have a (weak*) closed range,
II. f, g and h are twice continuously differentiable,
III. $f_{\dot{x}}^i - v_i g_{\dot{x}}^i - D\left(f_{\dot{x}}^i - v_i g_{\dot{x}}^i\right)$, $i = 1, \cdots, p$, is linearly independent, and
IV. $\left(\beta(t)^T\theta_x - D\beta(t)^T\theta_{\dot{x}} + D^2\beta(t)^T\theta_{\ddot{x}}\right)\beta(t) = 0$
$\Rightarrow \beta(t) = 0, \quad t \in I.$
Then u^* is an efficient solution of (FVP).

Proof. Since $(u^*, \tau^*, \mu^*, v^*)$, with $u^* \in Z$ and ψ' having a (weak*) closed range, is an efficient solution, there exist $\alpha \in R^p$, $\gamma \in R$, $\delta \in R$, $\epsilon \in R^p$, and piecewise smooth functions $\omega : I \to R^m$ and $\beta : I \to R^n$ satisfying the following Fritz John conditions

$$\left(\beta^T\theta_x - D\beta^T\theta_{\dot{x}} + D^2\beta^T\theta_{\ddot{x}}\right) + \delta\left(\mu^T h_x - D\mu^T h_{\ddot{x}}\right) + \gamma\tau^T\{(f_x - vg_x) - D\left(f_{\dot{x}} - vg_{\ddot{x}}\right)\} = 0 \tag{3}$$

$$\beta^T\{(f_x - vg_x) - D\left(f_{\dot{x}} - vg_{\dot{x}}\right)\} + \gamma\left(f - vg\right) + \epsilon = 0 \tag{4}$$

$$\beta^T\left(h_x - Dh_{\dot{x}}\right) + D\beta^T h_{\dot{x}} + \delta h + \omega = 0 \tag{5}$$

$$\alpha_i - \beta^T\left(\tau_i g_x^i - D\tau_i g_{\dot{x}}^i\right) - \gamma\tau_i g^i = 0, \quad i = 1, \cdots, p \tag{6}$$

$$\gamma\tau^T\left(f - v^T g\right) = 0 \tag{7}$$

$$\delta\mu^T h = 0$$

$$\epsilon^T\tau = 0 \tag{8}$$

$$\omega^T\mu = 0$$

$$(\alpha, \beta, \gamma, \delta, \epsilon, \omega) \geq 0 \tag{9}$$

By feasibility of $(u^*, \tau^*, \mu^*, v^*)$, from (3), we get

$$(\gamma - \delta)\tau^T\{(f_x - vg_x) - D\left(f_{\dot{x}} - vg_{\dot{x}}\right)\} + \left(\beta^T\theta_x - D\beta^T\theta_{\dot{x}} + D^2\beta^T\theta_{\ddot{x}}\right) = 0. \tag{10}$$

Multiplying (4) by τ and using (7) and (8), we have

$$\left[\tau^T\{(f_x - vg_x) - D\left(f_{\dot{x}} - vg_{\dot{x}}\right)\}\right]\beta = 0.$$

Multiplying (10) by β and using the above equation, (10) becomes

$$\left(\beta^T\theta_x - D\beta^T\theta_{\dot{x}} + D^2\beta^T\theta_{\ddot{x}}\right)\beta = 0,$$

which along with hypothesis IV gives

$$\beta = 0. \tag{11}$$

Equations (10) and (11) now yield

$$(\gamma - \delta)\,\tau^T\{(f_x - vg_x) - D\left(f_{\dot{x}} - vg_{\dot{x}}\right)\} = 0,$$

which along with hypothesis III and $\tau > 0$ yields

$$\gamma = \delta.$$

We claim that $\gamma = \delta > 0$. If $\gamma = \delta = 0$, then from (4), (5) and (6), we have $\alpha = \epsilon = \omega = 0$. Thus $(\alpha, \beta, \gamma, \delta, \epsilon, \omega) = 0$, which contradicts (9). Therefore from (5) and (7), u^* is feasible for (FVP) and by Theorem 2, u^* is efficient for (FVP). □

References

1. C.R. Bector, S. Chandra and C. Singh, "Duality in Multiobjective Fractional Programming" Int. Workshop on Generalized Concavity, Fractional Programming and Economic Applications, 1988, 1-15.
2. C.R. Bector and I. Husain, "Duality for multiobjective variational problems" J. Math. Anal. Appl. 166(1992), 214-229.
3. V. Chankong and Y.Y. Haimes, "Multiobjective Decision Making: Theory and Methodology", North-Holland, New York, 1983.
4. B.D. Craven, "Duality for generalized convex fractional programs" in Generalized Concavity in Optimization and Economics, (S. Schaible and W.T. Ziemba, Eds), Academic Press, 1981, 473-489.
5. C.L. Jo , D.S. Kim and G.M. Lee, Duality for multiobjective fractional programming involving-set functions, Optimization, 29(1994), 205-213.
6. P. Kanniapan, "Necessary conditions for optimality of nondifferentiable convex multiobjective program" J. Optim. Th. Appl., 40(1983), 167-174.
7. D.S. Kim, G.M. Lee, J.Y. Park and K.H. Son, "Control problems with generalized invexity" Mathematica Japonica, 38(1993), 1-7.
8. G.M. Lee and D.S. Kim, "Duality theorems for fractional multiobjective minimization problems" Proceeding of the 1st Workshop on Applied Mathematics, PoHang University of Technology, 1993, 245-256.
9. O.L. Mangasarian, "Nonlinear Programming" McGraw-Hill, New York, 1969.
10. S.K. Mishra and R.N. Mukherjee, "Duality for multiobjective fractional variational problems" J. Math. Anal. Appl. 186(1994), 711-725.
11. B. Mond and T. Weir, "Generalized concavity and duality" in Generalized Concavity in Optimization and Economics, (S. Schaible and W.T. Ziemba, Eds), Academic Press, 1981, 263-279.

Department of Applied Mathematics
Pukyong National University
Pusan 608-737, KOREA

Received July 14, 1997 **Revised September 24, 1997**