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Strong convergence theorems of iterations for a pair of nonexpansive mappings in Banach spaces

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1. Introduction

Let E be a real Banach space and let C be a nonempty closed convex subset of E. Then a mapping T of C into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping T of C into itself is called quasi-nonexpansive if the set $F(T)$ of fixed points of T is nonempty and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$. For two mappings S, T of C into itself, Takahashi and Tamura [15] considered the following three iteration schemes :

$$x_{n+1} = \alpha_n S[\beta_n S x_n + (1-\beta_n)x_n] + (1-\alpha_n)T[\beta_n T x_n + (1-\beta_n)x_n], \tag{1}$$

$$x_{n+1} = \alpha_n S[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)[\gamma_n S x_n + (1-\gamma_n)x_n], \tag{2}$$

and

$$x_{n+1} = \alpha_n S[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)[\gamma_n T x_n + (1-\gamma_n)x_n] \tag{3}$$

for all $n \geq 1$, where $x_1 \in C$ and $\alpha_n, \beta_n, \gamma_n \in [0,1]$. In the case when $S = T$ and $\gamma_n = 0$ in (2) or (3), such an iteration scheme was considered by Ishikawa [5]; see also Mann [6]. Das and Debata [2] studied the strong convergence of the iterates $\{x_n\}$ defined by (2) or (3) in the case when $\gamma_n = 0$ and S, T are quasi-nonexpansive mappings in a strictly convex Banach space; see also Rhoades [10]. Tan and Xu [16] also discussed the weak convergence of the iterates $\{x_n\}$ defined by (2) or (3) in the case when $\gamma_n = 0$ and S, T are nonexpansive mappings in a uniformly convex Banach space which satisfies Opial's condition or whose norm is Fréchet differentiable. Later Takahashi and Kim[13] obtained strong and weak convergence theorems which are different from Tan and Xu [16].

In this paper, we deal with the strong convergence of iterates $\{x_n\}$ defined by (1), (2) and (3) in a strictly convex Banach space. First we prove that if we choose their suitable coefficients, the sequence $\{x_n\}$ defined by (1) converges strongly to an element of $F(S), F(T)$ or $F(S) \cap F(T)$. Further we study the strong convergence of iterates $\{x_n\}$ defined by (2) and (3). We note that there are many differences concerning the strong convergence of iterates defined by (2) and (3).

2. Preliminaries

Throughout this paper, we denote by $\mathbf{N}$ the set of positive integers and by $\mathbf{R}$ the set of real numbers. Let E be a Banach space and let I be the identity operator on E. Let C be a nonempty subset of E. Then, a mapping T of C into itself is said to be *nonexpansive* on C if $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$. Let T be a mapping of C into itself. Then we denote by $F(T)$ the set of fixed points of T. For every ε with $0 \leq \varepsilon \leq 2$, we define the modulus $\delta(\varepsilon)$ of convexity of E by

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} \mid \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\right\}.$$

A Banach space E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If E is uniformly convex, then for each $r \geq \varepsilon > 0$,

$$\delta\left(\frac{\varepsilon}{r}\right) > 0 \text{ and } \left\|\frac{x + y}{2}\right\| \leq r\left(1 - \delta\left(\frac{\varepsilon}{r}\right)\right)$$

for every $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x - y\| \geq \varepsilon$. A Banach space E is also said to be strictly convex if

$$\left\|\frac{x + y}{2}\right\| < 1$$

for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A uniformly convex Banach space is reflexive and strictly convex. In a strictly convex Banach space, we have that if

$$\|x\| = \|y\| = \|(1 - \lambda)x + \lambda y\| \text{ for } x, y \in E \text{ and } \lambda \in (0, 1),$$

then $x = y$. When $\{x_n\}$ is a sequence in E, $x_n \to x$ and $x_n \rightharpoonup x$ will symbolize strong and weak convergence, respectively. We also denote by $\overline{co}A$ the closure of the convex hull of A. The following lemma which was proved by Mazur[7] is essential to prove the theorems in Sections 3 and 4; see also [13].

Lemma 2.1 (Mazur) *Let C be a relatively compact subset of a Banach space E. Then $\overline{co}C$ is also compact.*

3. Strong convergence of iterates defined by (1)

In this section, we consider the strong convergence of the iterates defined by (1).

Theorem 3.1 *Let C be a nonempty closed convex subset of a strictly convex Banach space E and let S, T be nonexpansive mappings of C into itself such that $S(C) \cup T(C)$ is contained in a compact subset of C and $F(S) \cap F(T)$ is nonempty. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and $x_{n+1} = \alpha_n S[\beta_n S x_n + (1 - \beta_n)x_n] + (1 - \alpha_n)T[\beta_n T x_n + (1 - \beta_n)x_n]$ for all $n \geq 1$, where $\alpha_n, \beta_n \in [0, 1]$. Then:*

- *(i) If there exists a cluster point (α_0, β_0) of $\{(\alpha_n, \beta_n)\}$ such that $(\alpha_0, \beta_0) \in [0, b] \times [a, b]$ for some $a, b \in (0, 1)$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to z \in F(T)$;*

(ii) *if there exists a cluster point* (α_0, β_0) *of* $\{(\alpha_n, \beta_n)\}$ *such that* $(\alpha_0, \beta_0) \in [a, 1] \times [a, b]$ *for some* $a, b \in (0, 1)$, *there exists a subsequence* $\{x_{n_i}\}$ *of* $\{x_n\}$ *such that* $x_{n_i} \to z \in F(S)$;

(iii) *if there exists a cluster point* (α_0, β_0) *of* $\{(\alpha_n, \beta_n)\}$ *such that* $(\alpha_0, \beta_0) \in [a, b] \times [a, b]$ *for some* $a, b \in (0, 1)$, *there exists a subsequence* $\{x_{n_i}\}$ *of* $\{x_n\}$ *such that* $x_{n_i} \to z \in F(S) \cap F(T)$.

Proof By Mazur's theorem, $D = \overline{co}\{S(C) \cup T(C) \cup \{x_1\}\}$ is a compact subset of C containing $\{x_n\}$. Let w be a common fixed point of S and T. Then we have

$$\begin{aligned} \|x_{n+1} - w\| &= \|\alpha_n S[\beta_n S x_n + (1-\beta_n)x_n] + (1-\alpha_n)T[\beta_n T x_n + (1-\beta_n)x_n] - w\| \\ &\le \alpha_n \|S[\beta_n S x_n + (1-\beta_n)x_n] - w\| \\ &\quad + (1-\alpha_n)\|\beta_n T x_n + (1-\beta_n)x_n - w\| \\ &\le \alpha_n \|\beta_n S x_n + (1-\beta_n)x_n - w\| \\ &\quad + (1-\alpha_n)\|x_n - w\| \\ &\le \|x_n - w\| \end{aligned}$$

for each $n \ge 1$. Since $\{\|x_n - w\|\}$ is nonincreasing, we get that $\lim_{n\to\infty}\|x_n - w\|$ exists. To prove (i), let $\{(\alpha_{n_k}, \beta_{n_k})\}$ be a subsequence of $\{(\alpha_n, \beta_n)\}$ such that $(\alpha_{n_k}, \beta_{n_k}) \to (\alpha_0, \beta_0)$. Without loss of generality, we assume that $(\alpha_{n_k}, \beta_{n_k}) \in [0, b] \times [a, b]$ with some $a, b \in (0, 1)$ for all $k \ge 1$. By the compactness of D, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}} \to z$ for some $z \in D$. Assume $Tz \ne z$. Let $e = \lim_{n\to\infty}\|x_n - w\|$. Since $x_{n_{k_l}} \to z$, we have $\|z - w\| = e$. From $Tz \ne z$, we have $e > 0$. We also know that

$$\|Tz - w\| \le \|z - w\| = e.$$

So, we have that for any $\alpha \in [0, b]$ and $\beta \in [a, b]$,

$$\begin{aligned} &\|\alpha S[\beta S z + (1-\beta)z] + (1-\alpha)T[\beta T z + (1-\beta)z] - w\| \\ \le\ &\alpha\|S[\beta S z + (1-\beta)z] - w\| + (1-\alpha)\|T[\beta T z + (1-\beta)z] - w\| \\ \le\ &\alpha\|z - w\| + (1-\alpha)\|\beta T z + (1-\beta)z - w\| \\ <\ &e \end{aligned}$$

using strict convexity of E. Further, consider a two variable real valued function g on $[0, 1] \times [0, 1]$ given by

$$g(\alpha, \beta) = \|\alpha S[\beta S z + (1-\beta)z] + (1-\alpha)T[\beta T z + (1-\beta)z] - w\|$$

for $\alpha, \beta \in [0, 1] \times [0, 1]$. Then g is continuous. From compactness of $[0, b] \times [a, b]$, we have

$$max\{g(\alpha, \beta) : (\alpha, \beta) \in [0, b] \times [a, b]\} < e.$$

Choose a positive number r such that

$$max\{g(\alpha,\beta) : (\alpha,\beta) \in [0,b] \times [a,b]\} < e - r.$$

Then from $x_{n_{k_l}} \to z$, we obtain an integer $m \geq 1$ such that $\|x_m - z\| < r$. Hence we have

$$\begin{aligned}
e &\leq \|x_{m+1} - w\| \\
&\leq \|x_{m+1} - \alpha_m S[\beta_m Sz + (1-\beta_m)z] - (1-\alpha_m)T[\beta_m Tz + (1-\beta_m)z]\| \\
&\quad + \|\alpha_m S[\beta_m Sz + (1-\beta_m)z] + (1-\alpha_m)T[\beta_m Tz + (1-\beta_m)z] - w\| \\
&\leq \alpha_m \|S[\beta_m Sx_m + (1-\beta_m)x_m] - S[\beta_m Sz + (1-\beta_m)z]\| \\
&\quad + (1-\alpha_m)\|T[\beta_m Tx_m + (1-\beta_m)x_m] - T[\beta_m Tz + (1-\beta_m)z]\| \\
&\quad + \|\alpha_m S[\beta_m Sz + (1-\beta_m)z] + (1-\alpha_m)T[\beta_m Tz + (1-\beta_m)z] - w\| \\
&< \alpha_m \|\beta_m(Sx_m - Sz) + (1-\beta_m)(x_m - z)\| + \\
&\quad (1-\alpha_m)\|\beta_m(Tx_m - Tz) + (1-\beta_m)(x_m - z)\| + e - r \\
&\leq \alpha_m(\beta_m\|x_m - z\| + (1-\beta_m)\|x_m - z\|) \\
&\quad + (1-\alpha_m)(\beta_m\|x_m - z\| + (1-\beta_m)\|x_m - z\|) + e - r \\
&= \|x_m - z\| + e - r \\
&< e.
\end{aligned}$$

This is a contradiction. So, we obtain $z = Tz$. This completes the proof of (i). To prove (ii), let $\{(\alpha_{n_k}, \beta_{n_k})\}$ be a subsequence of $\{(\alpha_n, \beta_n)\}$ such that $(\alpha_{n_k}, \beta_{n_k}) \to (\alpha_0, \beta_0)$. Without loss of generality, we assume that $(\alpha_{n_k}, \beta_{n_k}) \in [a,1] \times [a,b]$ with some $a, b \in (0,1)$ for all $k \geq 1$. By the compactness of D, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}} \to z$ for some $z \in D$. Assume $Sz \neq z$. As in the proof of (i), we have that $\lim_{n\to\infty} \|x_n - w\|$ exists. Let $e = \lim_{n\to\infty} \|x_n - w\|$. Then we know $\|z - w\| = e > 0$ and

$$\|Sz - w\| \leq \|z - w\| = e.$$

Further we have that for any $\alpha \in [a,1]$ and $\beta \in (0,1)$,

$$\begin{aligned}
&\|\alpha S[\beta Sz + (1-\beta)z] + (1-\alpha)T[\beta Tz + (1-\beta)z] - w\| \\
\leq\ &\alpha \|\beta Sz + (1-\beta)z - w\| + (1-\alpha)\|z - w\| \\
<\ &e
\end{aligned}$$

using strict convexity of E. As in the proof of (i), we also have

$$max\{g(\alpha,\beta) : (\alpha,\beta) \in [a,1] \times [a,b]\} < e.$$

As in the proof of (i), choose a positive number r such that

$$max\{g(\alpha,\beta) : (\alpha,\beta) \in [a,1] \times [a,b]\} < e - r.$$

Then we obtain $e \leq \|x_{m+1} - w\| < e$. This is a contradiction. So, we obtain $z = Sz$. This completes the proof of (ii). To prove (iii), let $\{(\alpha_{n_k}, \beta_{n_k})\}$ be a subsequence of $\{(\alpha_n, \beta_n)\}$ such that $(\alpha_{n_k}, \beta_{n_k}) \to (\alpha_0, \beta_0)$. Without loss of generality, we assume that $(\alpha_{n_k}, \beta_{n_k}) \in [a, b] \times [a, b]$ with some $a, b \in (0, 1)$ for all $k \geq 1$. By the compactness of D, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}} \to z$ for some $z \in D$. As in the proof of (i) and (ii), we have $z \in F(S) \cap F(T)$. This completes the proof of (iii). □

The following lemma was proved by Tan and Xu[16].

Lemma 3.2 (Tan and Xu) *Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences of non-negative numbers such that $a_{n+1} \leq a_n + b_n$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.*

Using Lemma 3.2, we proved the following strong convergence theorem of iterates defined by (1).

Theorem 3.3 *Let C be a nonempty closed convex subset of a strictly convex Banach space E and let S, T be nonexpansive mappings of C into itself such that $S(C) \cup T(C)$ is contained in a compact subset of C and $F(S) \cap F(T)$ is nonempty. Suppose $\{x_n\}$ is given by $x_1 \in C$ and $x_{n+1} = \alpha_n S[\beta_n S x_n + (1-\beta_n)x_n] + (1-\alpha_n)T[\beta_n T x_n + (1-\beta_n)x_n]$ for all $n \geq 1$, where $\alpha_n, \beta_n \in [0, 1]$. Then:*

- *(i) If $\sum_{n=1}^{\infty} \alpha_n < \infty$ and there exists a cluster point β_0 of $\{\beta_n\}$ with $\beta_0 \in [a, b]$ for some $a, b \in (0, 1)$, $\{x_n\}$ converges strongly to a fixed point of T;*
- *(ii) if $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$ and there exists a cluster point β_0 of $\{\beta_n\}$ with $\beta_0 \in [a, b]$ for some $a, b \in (0, 1)$, $\{x_n\}$ converges strongly to a fixed point of S;*
- *(iii) if there exists a cluster point (α_0, β_0) of $\{(\alpha_n, \beta_n)\}$ such that $(\alpha_0, \beta_0) \in [a, b] \times [a, b]$ for some $a, b \in (0, 1)$, $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof By Mazur's theorem, $D = \overline{co}\{S(C) \cup T(C) \cup \{x_1\}\}$ is a compact subset of C containing $\{x_n\}$. We shall prove (i). Let w be a fixed point of T. Then, we have

$$\begin{aligned} \|x_{n+1} - w\| &= \|\alpha_n S[\beta_n S x_n + (1-\beta_n)x_n] + (1-\alpha_n)T[\beta_n T x_n + (1-\beta_n)x_n] - w\| \\ &\leq \alpha_n \|S[\beta_n S x_n + (1-\beta_n)x_n] - w\| + (1-\alpha_n)\|x_n - w\| \\ &\leq \alpha_n \|S[\beta_n S x_n + (1-\beta_n)x_n] - w\| + \|x_n - w\| \end{aligned}$$

for each $n \geq 1$. By Lemma 3.2 and $\sum_{n=1}^{\infty} \alpha_n < \infty$, we get that $\lim_{n\to\infty} \|x_n - w\|$ exists. Since $(0, \beta_0)$ is a cluster point of $\{(\alpha_n, \beta_n)\}$ such that $(0, \beta_0) \in [0, b] \times [a, b]$ for some $a, b \in (0, 1)$, we have that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to z$ for some $z \in F(T)$ by (i) of Theorem 3.1. Then, we have

$$\lim_{n\to\infty} \|x_n - z\| = \lim_{i\to\infty} \|x_{n_i} - z\| = 0.$$

Hence $\{x_n\}$ converges strongly to $z \in F(T)$. This completes the proof of (i). We shall prove (ii). Let w be a fixed point of S. Then, we have

$$\begin{aligned} \|x_{n+1} - w\| &= \|\alpha_n S[\beta_n S x_n + (1-\beta_n)x_n] + (1-\alpha_n)T[\beta_n T x_n + (1-\beta_n)x_n] - w\| \\ &\leq \alpha_n \|x_n - w\| + (1-\alpha_n)\|T[\beta_n T x_n + (1-\beta_n)x_n] - w\| \\ &\leq \|x_n - w\| + (1-\alpha_n)\|T[\beta_n T x_n + (1-\beta_n)x_n] - w\| \end{aligned}$$

for each $n \geq 1$. By Lemma 3.2 and $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$, we get that $\lim_{n\to\infty} \|x_n - w\|$ exists. Since $(1, \beta_0)$ is a cluster point of $\{(\alpha_n, \beta_n)\}$ such that $(1, \beta_0) \in [a, 1] \times [a, b]$ for some $a, b \in (0, 1)$, we have that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to z$ for some $z \in F(S)$ by *(ii)* of Theorem 3.1. Then we have

$$\lim_{n\to\infty} \|x_n - z\| = \lim_{i\to\infty} \|x_{n_i} - z\| = 0.$$

Hence $\{x_n\}$ converges strongly to $z \in F(S)$. This completes the proof of *(ii)*. We shall prove *(iii)*. Let w be a common fixed point of S and T. Then, as in the proof of Theorem 3.1, we have that $\lim_{n\to\infty} \|x_n - w\|$ exists. Since (α_0, β_0) is a cluster point of $\{(\alpha_n, \beta_n)\}$ such that $(\alpha_0, \beta_0) \in [a, b] \times [a, b]$ for some $a, b \in (0, 1)$, we have that there exists a subsequence $\{x_{n_i}\}$ of the sequence $\{x_n\}$ such that $x_{n_i} \to z$ for some $z \in F(S) \cap F(T)$ by *(iii)* of Theorem 3.1. Then we have

$$\lim_{n\to\infty} \|x_n - z\| = \lim_{i\to\infty} \|x_{n_i} - z\| = 0.$$

Hence $\{x_n\}$ converges strongly to $z \in F(S) \cap F(T)$. □

Remark 3.4 *(i) and (ii) in Theorem 3.3 are proved in the case that $F(S) \cap F(T)$ is empty.*

4. Strong convergence of iterates defined by (2) and (3)

In this section, we consider the strong convergence of the iterates defined by (2) and (3). First, we discuss the strong convergence of iterates defined by (2).

Theorem 4.1 *Let C be a nonempty closed convex subset of a strictly convex Banach space E and let S, T be nonexpansive mappings of C into itself such that $S(C) \cup T(C)$ is contained in a compact subset of C and $F(S) \cap F(T)$ is nonempty. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and $x_{n+1} = \alpha_n S[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)[\gamma_n S x_n + (1-\gamma_n)x_n]$ for all $n \geq 1$, where $\alpha_n, \beta_n, \gamma_n \in [0, 1]$. Then:*

(i) If there exists a cluster point $(\alpha_0, \beta_0, \gamma_0)$ of $\{(\alpha_n, \beta_n, \gamma_n)\}$ such that $(\alpha_0, \beta_0, \gamma_0) \in [0, b] \times [0, 1] \times [a, b]$ for some $a, b \in (0, 1)$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to z \in F(S)$;

(ii) if there exists a cluster point $(\alpha_0, \beta_0, \gamma_0)$ of $\{(\alpha_n, \beta_n, \gamma_n)\}$ such that $(\alpha_0, \beta_0, \gamma_0) \in [a, 1] \times [a, b] \times [0, 1]$ for some $a, b \in (0, 1)$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to z \in F(T)$;

(iii) if there exists a cluster point $(\alpha_0, \beta_0, \gamma_0)$ of $\{(\alpha_n, \beta_n, \gamma_n)\}$ such that $(\alpha_0, \beta_0, \gamma_0) \in [a, b] \times [0, b] \times [0, b]$ for some $a, b \in (0, 1)$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to z \in F(S)$;

(iv) if there exists a cluster point $(\alpha_0, \beta_0, \gamma_0)$ of $\{(\alpha_n, \beta_n, \gamma_n)\}$ such that $(\alpha_0, \beta_0, \gamma_0) \in [a, b] \times [a, b] \times [0, b]$ for some $a, b \in (0, 1)$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to z \in F(S) \cap F(T)$;

(v) *if there exists a cluster point* $(\alpha_0, \beta_0, \gamma_0)$ *of* $\{(\alpha_n, \beta_n, \gamma_n)\}$ *such that* $(\alpha_0, \beta_0, \gamma_0) \in [a, b] \times [a, 1] \times [0, b]$ *for some* $a, b \in (0, 1)$, *then there exists a subsequence* $\{x_{n_i}\}$ *of* $\{x_n\}$ *such that* $x_{n_i} \to z \in F(ST)$;

Further we assume that S *and* T *commute.*

(vi) *If there exists a cluster point* $(\alpha_0, \beta_0, \gamma_0)$ *of* $\{(\alpha_n, \beta_n, \gamma_n)\}$ *such that* $(\alpha_0, \beta_0, \gamma_0) \in [a, b] \times [a, 1] \times [a, b]$ *for some* $a, b \in (0, 1)$, *then there exists a subsequence* $\{x_{n_i}\}$ *of* $\{x_n\}$ *such that* $x_{n_i} \to z \in F(S) \cap F(T)$.

Proof By Mazur's theorem, $D = \overline{co}\{S(C) \cup T(C) \cup \{x_1\}\}$ is a compact subset of C containing $\{x_n\}$. Let w be a common fixed point of S and T. Also we have

$$\begin{aligned}
& \|\alpha S[\beta Tz + (1-\beta)z] + (1-\alpha)[\gamma Sz + (1-\gamma)z] - w\| \\
\leq\ & \alpha \|S[\beta Tz + (1-\beta)z] - w\| + (1-\alpha) \|\gamma Sz + (1-\gamma)z - w\| \\
\leq\ & \alpha \|\beta Tz + (1-\beta)z - w\| + (1-\alpha) \|\gamma Sz + (1-\gamma)z - w\| \\
\leq\ & \|z - w\| \qquad (4)
\end{aligned}$$

for any $\alpha, \beta, \gamma \in [0, 1]$ and $z \in C$. Since $\{\|x_n - w\|\}$ is nonincreasing, we get that $\lim_{n\to\infty} \|x_n - w\|$ exists. Let $h = \lim_{n\to\infty} \|x_n - w\|$. To prove (i), let $\{(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k})\}$ be a subsequence of $\{(\alpha_n, \beta_n, \gamma_n)\}$ such that $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \to (\alpha_0, \beta_0, \gamma_0)$. Without loss of generality, we assume that $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \in [0, b] \times [0, 1] \times [a, b]$ with some $a, b \in (0, 1)$ for all $k \geq 1$. By the compactness of D, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}} \to z$ for some $z \in D$. Assume $Sz \neq z$. Since $x_{n_{k_l}} \to z$, we have $\|z - w\| = h$. From $Sz \neq z$, we have $h > 0$. We also know that

$$\|Sz - w\| \leq \|z - w\| = h.$$

So, we have that for any $\alpha \in [0, b], \beta \in [0, 1]$ and $\gamma \in (0, 1)$,

$$\|\alpha S[\beta Tz + (1-\beta)z] + (1-\alpha)[\gamma Sz + (1-\gamma)z] - w\| < h$$

using strict convexity of E and (4). Further, consider a three variable real valued function g on $[0, 1] \times [0, 1] \times [0, 1]$ given by

$$g(\alpha, \beta, \gamma) = \|\alpha S[\beta Tz + (1-\beta)z] + (1-\alpha)[\gamma Sz + (1-\gamma)z] - w\|$$

for $(\alpha, \beta, \gamma) \in [0, 1] \times [0, 1] \times [0, 1]$. Then g is continuous. From compactness of $[0, b] \times [0, 1] \times [a, b]$, we have

$$max\{g(\alpha, \beta, \gamma) : (\alpha, \beta, \gamma) \in [0, b] \times [0, 1] \times [a, b]\} < h.$$

Choose a positive number r such that

$$max\{g(\alpha, \beta, \gamma) : (\alpha, \beta, \gamma) \in [0, b] \times [0, 1] \times [a, b]\} < h - r.$$

Then from $x_{n_{k_l}} \to z$, we obtain an integer $m \geq 1$ such that $\|x_m - z\| < r$. Hence we have

$$\begin{aligned} h &\leq \|x_{m+1} - w\| \\ &\leq \|x_{m+1} - \alpha_m S[\beta_m Tz + (1-\beta_m)z] - (1-\alpha_m)[\gamma_m Sz + (1-\gamma_m)z]\| \\ &\quad + \|\alpha_m S[\beta_m Tz + (1-\beta_m)z] + (1-\alpha_m)[\gamma_m Sz + (1-\gamma_m)z] - w\| \\ &\leq \alpha_m \|S[\beta_m Tx_m + (1-\beta_m)x_m] - S[\beta_m Tz + (1-\beta_m)z]\| \\ &\quad + (1-\alpha_m)\|[\gamma_m Sx_m + (1-\gamma_m)x_m] - [\gamma_m Sz + (1-\gamma_m)z]\| \\ &\quad + \|\alpha_m S[\beta_m Tz + (1-\beta_m)z] + (1-\alpha_m)[\gamma_m Sz + (1-\gamma_m)z] - w\| \\ &\leq \alpha_m \|\beta_m (Tx_m - Tz) + (1-\beta_m)(x_m - z)\| + \\ &\quad (1-\alpha_m)\|\gamma_m (Sx_m - Sz) + (1-\gamma_m)(x_m - z)\| + h - r \\ &\leq \alpha_m (\beta_m \|x_m - z\| + (1-\beta_m)\|x_m - z\|) \\ &\quad + (1-\alpha_m)[\gamma_m \|x_m - z\| + (1-\gamma_m)\|x_m - z\|] + h - r \\ &= \|x_m - z\| + h - r \\ &< h. \end{aligned}$$

This is a contradiction. So, we obtain $z = Sz$. This completes the proof of (i). To prove (ii), let $\{(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k})\}$ be a subsequence of $\{(\alpha_n, \beta_n, \gamma_n)\}$ such that $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \to (\alpha_0, \beta_0, \gamma_0)$. Without loss of generality, we assume that $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \in [a, 1] \times [a, b] \times [0, 1]$ with some $a, b \in (0, 1)$ for all $k \geq 1$. By the compactness of D, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}} \to z$ for some $z \in D$. Assume $Tz \neq z$. Then, as in the proof of (i), we have that $\|z - w\| = h$,

$$\|Sz - w\| \leq \|z - w\| = h$$

and

$$\|Tz - w\| \leq \|z - w\| = h.$$

From $Tz \neq z$, we have $h > 0$. Further we have that for any $\alpha \in [a, 1]$ and $\beta \in (0, 1)$,

$$\|\alpha S[\beta Tz + (1-\beta)z] + (1-\alpha)[\gamma Sz + (1-\gamma)z] - w\| < h$$

using strict convexity of E and (4). As in the proof of (i), we have

$$max\{g(\alpha, \beta, \gamma) : (\alpha, \beta, \gamma) \in [a, 1] \times [a, b] \times [0, 1]\} < h.$$

As in the proof of (i), choose a positive number r such that

$$max\{g(\alpha, \beta, \gamma) : (\alpha, \beta, \gamma) \in [a, 1] \times [a, b] \times [0, 1]\} < h - r.$$

Then, we obtain

$$h \leq \|x_{m+1} - w\| < h.$$

This is a contradiction. So, we obtain $Tz = z$. This completes the proof of (ii). To prove (iii), let $\{(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k})\}$ be a subsequence of $\{(\alpha_n, \beta_n, \gamma_n)\}$ such that $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \to (\alpha_0, \beta_0, \gamma_0)$. Without loss of generality, we assume that $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k})$

$\in [a,b] \times [0,b] \times [0,b]$ with some $a,b \in (0,1)$ for all $k \geq 1$. By the compactness of D, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}} \to z$ for some $z \in D$. Assume $Sz \neq z$. Then, as in the proof of (i), we have that $\|z - w\| = h > 0$ and

$$\|Tz - w\| \leq \|z - w\|.$$

So, we have that for any $\alpha \in [a,b], \beta = 0$ and $\gamma \in [0,b]$,

$$\begin{aligned}
& \|\alpha S[\beta Tz + (1-\beta)z] + (1-\alpha)[\gamma Sz + (1-\gamma)z] - w\| \\
= & \|\alpha Sz + (1-\alpha)[\gamma Sz + (1-\gamma)z] - w\| \\
= & \|\gamma[\alpha Sz + (1-\alpha)Sz] + (1-\gamma)[\alpha Sz + (1-\alpha)z] - w\| \\
\leq & \gamma\|z - w\| + (1-\gamma)\|\alpha Sz + (1-\alpha)z - w\| \\
< & h \qquad (5)
\end{aligned}$$

using strict convexity of E. Also we have that if $Tz \neq z$, then for any $\alpha \in [a,b]$,$\beta \in (0,1)$ and $\gamma \in [0,b]$,

$$\|\alpha S[\beta Tz + (1-\beta)z] + (1-\alpha)[\gamma Sz + (1-\gamma)z] - w\| < h$$

using strict convexity of E and (4). Further we have that if $Tz = z$, by (5), we also have, for any $\alpha \in [a,b], \beta \in (0,1)$ and $\gamma \in [0,b]$,

$$\begin{aligned}
& \|\alpha S[\beta Tz + (1-\beta)z] + (1-\alpha)[\gamma Sz + (1-\gamma)z] - w\| \\
= & \|\alpha Sz + (1-\alpha)[\gamma Sz + (1-\gamma)z] - w\| \\
< & h
\end{aligned}$$

using strict convexity of E. As in the proof of (i), we have

$$max\{g(\alpha,\beta,\gamma) : (\alpha,\beta,\gamma) \in [a,b] \times [0,b] \times [0,b]\} < h.$$

As in the proof of (i), choose a positive number r such that

$$max\{g(\alpha,\beta,\gamma) : (\alpha,\beta,\gamma) \in [a,b] \times [0,b] \times [0,b]\} < h - r.$$

Then, we obtain

$$h \leq \|x_{m+1} - w\| < h.$$

This is a contradiction. So, we obtain $Sz = z$. This completes the proof of (iii). To prove (iv), let $\{(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k})\}$ be a subsequence of $\{(\alpha_n, \beta_n, \gamma_n)\}$ such that $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \to (\alpha_0, \beta_0, \gamma_0)$. Without loss of generality, we assume that $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \in [a,b] \times [a,b] \times [0,b]$ with some $a,b \in (0,1)$ for all $k \geq 1$. By the compactness of D, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}} \to z$ for some $z \in D$. As in the proof of (ii) and (iii), we have $z \in F(S) \cap F(T)$. This completes the proof of (iv). To prove (v), let $\{(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k})\}$ be a subsequence of $\{(\alpha_n, \beta_n, \gamma_n)\}$ such that $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \to (\alpha_0, \beta_0, \gamma_0)$. Without loss of generality, we assume that $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \in [a,b] \times [a,1] \times [0,b]$ with some $a,b \in (0,1)$ for all $k \geq 1$. By the compactness of D, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}} \to z$ for some $z \in D$. Assume $STz \neq z$. Then, as in the proof of (i), we have $\|z - w\| = h > 0$ and $\|STz - w\| \leq \|z - w\|$. Then, we have that for any $\alpha \in [a, b], \beta = 1$ and $\gamma = 0$,

$$\begin{aligned} & \|\alpha S[\beta Tz + (1-\beta)z] + (1-\alpha)[\gamma Sz + (1-\gamma)z] - w\| \\ = & \|\alpha STz + (1-\alpha)z - w\| \\ < & h \end{aligned}$$

using strict convexity of E. Also, we have that if $Sz = z$, then for any $\alpha \in [a, b], \beta = 1$ and $\gamma \in (0, b]$,

$$\begin{aligned} & \|\alpha S[\beta Tz + (1-\beta)z] + (1-\alpha)[\gamma Sz + (1-\gamma)z] - w\| \\ = & \|\alpha STz + (1-\alpha)z - w\| \\ < & h \end{aligned}$$

using strict convexity of E. On the other hand, we have that if $Sz \neq z$, then for any $\alpha \in [a, b], \beta = 1$ and $\gamma \in (0, b]$,

$$\begin{aligned} & \|\alpha S[\beta Tz + (1-\beta)z] + (1-\alpha)[\gamma Sz + (1-\gamma)z] - w\| \\ \leq & \alpha \|STz - w\| + (1-\alpha) \|\gamma Sz + (1-\gamma)z - w\| \\ < & h \end{aligned}$$

using strict convexity of E. Also we have that for any $\alpha \in [a, b]$,$\beta \in (0, 1)$ and $\gamma \in [0, b]$,

$$\|\alpha S[\beta Tz + (1-\beta)z] + (1-\alpha)[\gamma Sz + (1-\gamma)z] - w\| < h$$

using strict convexity of E, $Sz \neq z$ and (4). As in the proof of (i), we have

$$max\{g(\alpha, \beta, \gamma) : (\alpha, \beta, \gamma) \in [a, b] \times [a, 1] \times [0, b]\} < h.$$

As in the proof of (i), choose a positive number r such that

$$max\{g(\alpha, \beta, \gamma) : (\alpha, \beta, \gamma) \in [a, b] \times [a, 1] \times [0, b]\} < h - r.$$

Then, we obtain

$$h \leq \|x_{m+1} - w\| < h.$$

This is a contradiction. So, we obtain $STz = z$. This completes the proof of (v). We shall prove (vi). Let $\{(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k})\}$ be a subsequence of $\{(\alpha_n, \beta_n, \gamma_n)\}$ such that $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \to (\alpha_0, \beta_0, \gamma_0)$. Without loss of generality, we assume that $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \in [a, b] \times [a, 1] \times [a, b]$ with some $a, b \in (0, 1)$ for all $k \geq 1$. By the compactness of D, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}} \to z$ for some $z \in D$. As in the proof of (i) and (v), we have $z \in F(S) \cap F(ST)$. Since $ST = TS$, we have that $Tz = TSz = STz = z$. This implies $z \in F(S) \cap F(T)$. This completes the proof of (vi). □

The following is a strong convergence theorem of iterates defined by (2) in a strictly convex Banach space.

Theorem 4.2 *Let C be a nonempty closed convex subset of a strictly convex Banach space E and let S, T be nonexpansive mappings of C into itself such that $S(C) \cup T(C)$ is contained in a compact subset of C and $F(S) \cap F(T)$ is nonempty. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and $x_{n+1} = \alpha_n S[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)[\gamma_n S x_n + (1-\gamma_n)x_n]$ for all $n \geq 1$, where $\alpha_n, \beta_n, \gamma_n \in [0,1]$. If there exists a cluster point $(\alpha_0, \beta_0, \gamma_0)$ of $\{(\alpha_n, \beta_n, \gamma_n)\}$ in $[a,b] \times [a,b] \times [0,b]$ for some $a, b \in (0,1)$, then $\{x_n\}$ converges strongly to a common fixed point of S and T. Further, assume S and T commute. If there exists a cluster point $(\alpha_0, \beta_0, \gamma_0)$ of $(\alpha_n, \beta_n, \gamma_n)$ in $[a,b] \times [a,1] \times [a,b]$ for some $a, b \in (0,1)$, then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof As in the proof of Theorem 4.1, $\lim_{n\to\infty} \|x_n - w\|$ exists for each $w \in F(S) \cap F(T)$. By Theorem 4.1, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to z_0 \in F(S) \cap F(T)$. Then, we get

$$\lim_{n\to\infty} \|x_n - z_0\| = \lim_{i\to\infty} \|x_{n_i} - z_0\| = 0.$$

Therefore, $\{x_n\}$ converges strongly to $z_0 \in F(S) \cap F(T)$. This completes the proof. □

Finally, we discuss the strong convergence of iterates defined by (3).

Theorem 4.3 *Let C be a nonempty closed convex subset of a strictly convex Banach space E and let S, T be nonexpansive mappings of C into itself such that $S(C) \cup T(C)$ is contained in a compact subset of C and $F(S) \cap F(T)$ is nonempty. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and $x_{n+1} = \alpha_n S[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)[\gamma_n T x_n + (1-\gamma_n)x_n]$ for all $n \geq 1$, where $\alpha_n, \beta_n, \gamma_n \in [0,1]$. Then:*

(i) If there exists a cluster point $(\alpha_0, \beta_0, \gamma_0)$ of $\{(\alpha_n, \beta_n, \gamma_n)\}$ such that $(\alpha_0, \beta_0, \gamma_0) \in [0,b] \times [0,1] \times [a,b]$ for some $a, b \in (0,1)$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to z \in F(T)$;

(ii) if there exists a cluster point $(\alpha_0, \beta_0, \gamma_0)$ of $\{(\alpha_n, \beta_n, \gamma_n)\}$ such that $(\alpha_0, \beta_0, \gamma_0) \in [a,1] \times [a,b] \times [0,1]$ for some $a, b \in (0,1)$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to z \in F(T)$;

(iii) if there exists a cluster point $(\alpha_0, \beta_0, \gamma_0)$ of $\{(\alpha_n, \beta_n, \gamma_n)\}$ such that $(\alpha_0, \beta_0, \gamma_0) \in [a,b] \times [0,b] \times [0,b]$ for some $a, b \in (0,1)$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to z \in F(S)$;

(iv) if there exists a cluster point $(\alpha_0, \beta_0, \gamma_0)$ of $\{(\alpha_n, \beta_n, \gamma_n)\}$ such that $(\alpha_0, \beta_0, \gamma_0) \in [a,b] \times [a,b] \times [0,b]$ for some $a, b \in (0,1)$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to z \in F(S) \cap F(T)$;

(v) if there exists a cluster point $(\alpha_0, \beta_0, \gamma_0)$ of $\{(\alpha_n, \beta_n, \gamma_n)\}$ such that $(\alpha_0, \beta_0, \gamma_0) \in [a,b] \times [a,1] \times [0,b]$ for some $a, b \in (0,1)$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to z \in F(ST)$;

(vi) *if there exists a cluster point* $(\alpha_0, \beta_0, \gamma_0)$ *of* $\{(\alpha_n, \beta_n, \gamma_n)\}$ *such that* $(\alpha_0, \beta_0, \gamma_0) \in [a, b] \times [0, 1] \times [a, b]$ *for some* $a, b \in (0, 1)$, *then there exists a subsequence* $\{x_{n_i}\}$ *of* $\{x_n\}$ *such that* $x_{n_i} \to z \in F(S) \cap F(T)$.

Proof By Mazur's theorem, $D = \overline{co}\{S(C) \cup T(C) \cup \{x_1\}\}$ is a compact subset of C containing $\{x_n\}$. Let w be a common fixed point of S and T. Also we have

$$\begin{aligned} & \|\alpha S[\beta Tz + (1-\beta)z] + (1-\alpha)[\gamma Tz + (1-\gamma)z] - w\| \\ \leq\ & \alpha \|S[\beta Tz + (1-\beta)z] - w\| + (1-\alpha)\|\gamma Tz + (1-\gamma)z - w\| \\ \leq\ & \alpha \|\beta Tz + (1-\beta)z - w\| + (1-\alpha)\|\gamma Tz + (1-\gamma)z - w\| \\ \leq\ & \|z - w\| \end{aligned} \tag{6}$$

for any $\alpha, \beta, \gamma \in [0, 1]$ and $z \in C$. Since $\{\|x_n - w\|\}$ is nonincreasing, we get that $\lim_{n\to\infty} \|x_n - w\|$ exists. Let $h = \lim \|x_n - w\|$. To prove (*i*), let $\{(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k})\}$ be a subsequence of $\{(\alpha_n, \beta_n, \gamma_n)\}$ such that $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \to (\alpha_0, \beta_0, \gamma_0)$. Without loss of generality, we assume that $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \in [0, b] \times [0, 1] \times [a, b]$ with some $a, b \in (0, 1)$ for all $k \geq 1$. By the compactness of D, there exists a subsequence $\{x_{n_{k_l}}\}$ such that $x_{n_{k_l}} \to z$ for some $z \in D$. Assume $Tz \neq z$. Since $x_{n_i} \to z$, we have $\|z - w\| = h$. From $Tz \neq z$, we have $h > 0$. We also know that $\|Tz - w\| \leq \|z - w\| = h$. For any $\alpha \in [0, 1), \beta \in [0, 1]$ and $\gamma \in (0, 1)$, we have

$$\|\alpha S[\beta Tz + (1-\beta)z] + (1-\alpha)[\gamma Tz + (1-\gamma)z] - w\| < h$$

using strictly convexity of E and (6). Further, consider a three variable real valued function g on $[0, 1] \times [0, 1] \times [0, 1]$ given by

$$g(\alpha, \beta, \gamma) = \|\alpha S[\beta Tz + (1-\beta)z] + (1-\alpha)[\gamma Tz + (1-\gamma)z] - w\|$$

for $(\alpha, \beta, \gamma) \in [0, 1] \times [0, 1] \times [0, 1]$. Then g is continuous. From compactness of $[0, b] \times [0, 1] \times [a, b]$, we have

$$max\{g(\alpha, \beta, \gamma) : (\alpha, \beta, \gamma) \in [0, b] \times [0, 1] \times [a, b]\} < h.$$

Choose a positive number r such that

$$max\{g(\alpha, \beta, \gamma) : (\alpha, \beta, \gamma) \in [0, b] \times [0, 1] \times [a, b]\} < h - r.$$

Then from $x_{n_{k_l}} \to z$, we obtain an integer $m \geq 1$ such that $\|x_m - z\| < r$. Hence we have

$$\begin{aligned} h \leq\ & \|x_{m+1} - w\| \\ \leq\ & \|x_{m+1} - \alpha_m S[\beta_m Tz + (1-\beta_m)z] - (1-\alpha_m)[\gamma_m Tz + (1-\gamma_m)z]\| \\ & + \|\alpha_m S[\beta_m Tz + (1-\beta_m)z] + (1-\alpha_m)[\gamma_m Tz + (1-\gamma_m)z] - w\| \\ \leq\ & \alpha_m \|S[\beta_m Tx_m + (1-\beta_m)x_m] - S[\beta_m Tz + (1-\beta_m)z]\| \\ & + (1-\alpha_m)\|\gamma_m(Tx_m - Tz) + (1-\gamma_m)(x_m - z)\| \\ & + \|\alpha_m S[\beta_m Tz + (1-\beta_m)z] + (1-\alpha_m)[\gamma_m Tz + (1-\gamma_m)z] - w\| \end{aligned}$$

$$
\begin{aligned}
&\le \alpha_m \|\beta_m(Tx_m - Tz) + (1-\beta_m)(x_m - z)\| \\
&\quad +(1-\alpha_m)\|\gamma_m(Tx_m - Tz) + (1-\gamma_m)(x_m - z)\| + h - r \\
&\le \alpha_m(\beta_m\|x_m - z\| + (1-\beta_m)\|x_m - z\|) \\
&\quad +(1-\alpha_m)(\gamma_m\|x_m - z\| + (1-\gamma_m)\|x_m - z\|) + h - r \\
&= \|x_m - z\| + h - r \\
&< h.
\end{aligned}
$$

This is a contradiction. So, we obtain $z = Tz$. This completes the proof of (i). To prove (ii), let $\{(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k})\}$ be a subsequence of $\{(\alpha_n, \beta_n, \gamma_n)\}$ such that $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \to (\alpha_0, \beta_0, \gamma_0)$. Without loss of generality, we assume that $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \in [a,1] \times [a,b] \times [0,1]$ with some $a, b \in (0,1)$ for all $k \ge 1$. By the compactness of D, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}} \to z$ for some $z \in D$. Assume $Tz \ne z$. Then, as in the proof of (i), we have $\|z - w\| = h > 0$ and $\|Tz - w\| \le \|z - w\|$. Further we have that for any $\alpha \in [a,1]$ and $\beta \in (0,1)$,

$$\|\alpha S[\beta Tz + (1-\beta)z] + (1-\alpha)[\gamma Tz + (1-\gamma)z] - w\| < h$$

using strict convexity of E and (6). As in the proof of (i), we also have

$$max\{g(\alpha,\beta,\gamma) : (\alpha,\beta,\gamma) \in [a,1] \times [a,b] \times [0,1]\} < h.$$

As in the proof of (i), choose a positive number r such that

$$max\{g(\alpha,\beta,\gamma) : (\alpha,\beta,\gamma) \in [a,1] \times [a,b] \times [0,1]\} < h - r.$$

Then we obtain $h \le \|x_{m+1} - w\| < h$. This is a contradiction. So, we obtain $z = Tz$. This completes the proof of (ii). To prove (iii), let $\{(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k})\}$ be a subsequence of $\{(\alpha_n, \beta_n, \gamma_n)\}$ such that $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \to (\alpha_0, \beta_0, \gamma_0)$. Without loss of generality, we assume that $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \in [a,b] \times [0,b] \times [0,b]$ with some $a, b \in (0,1)$ for all $k \ge 1$. By the compactness of D, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}} \to z$ for some $z \in D$. Assume $Sz \ne z$. Then, as in the proof of (i), we have $\|z - w\| = h > 0$. We have that for any $\alpha \in [a,b], \beta = 0$ and $\gamma \in [0,b]$,

$$
\begin{aligned}
&\|\alpha S[\beta Tz + (1-\beta)z] + (1-\alpha)[\gamma Tz + (1-\gamma)z] - w\| \\
&= \|\alpha Sz + (1-\alpha)[\gamma Tz + (1-\gamma)z] - w\| \\
&\le \gamma\,\|\alpha Sz + (1-\alpha)Tz - w\| + (1-\gamma)\,\|\alpha Sz + (1-\alpha)z - w\| \\
&\le \gamma\,\|z - w\| + (1-\gamma)\,\|\alpha Sz + (1-\alpha)z - w\| \\
&< h
\end{aligned}
$$

and using strict convexity of E. Also, we have that for any $\alpha \in [a,b], \beta \in (0,1)$ and $\gamma \in [0,b]$,

$$\|\alpha S[\beta Tz + (1-\beta)z] + (1-\alpha)[\gamma Tz + (1-\gamma)z] - w\| < h$$

using strict convexity of E and (6). As in the proof of (i), we also have

$$max\{g(\alpha,\beta,\gamma) : (\alpha,\beta,\gamma) \in [a,b] \times [0,b] \times [0,b]\} < h.$$

As in the proof of (i), choose a positive number r such that

$$max\{g(\alpha,\beta,\gamma) : (\alpha,\beta,\gamma) \in [a,b] \times [0,b] \times [0,b]\} < h - r.$$

Then we obtain $h \leq \|x_{m+1} - w\| < h$. This is a contradiction. So, we obtain $z = Sz$. This completes the proof of (iii). To prove (iv), let $\{(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k})\}$ be a subsequence of $\{(\alpha_n, \beta_n, \gamma_n)\}$ such that $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \to (\alpha_0, \beta_0, \gamma_0)$. Without loss of generality, we assume that $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \in [a,b] \times [a,b] \times [0,b]$ with some $a, b \in (0,1)$ for all $k \geq 1$. By the compactness of D, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}} \to z$ for some $z \in D$. As in the proof of (ii) and (iii), we have $z \in F(S) \cap F(T)$. This completes the proof of (iv). To prove (v), let $\{(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k})\}$ be a subsequence of $\{(\alpha_n, \beta_n, \gamma_n)\}$ such that $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \to (\alpha_0, \beta_0, \gamma_0)$. Without loss of generality, we assume that $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \in [a,b] \times [a,1] \times [0,b]$ with some $a, b \in (0,1)$ for all $k \geq 1$. By the compactness of D, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}} \to z$ for some $z \in D$. Assume $STz \neq z$. Then, as in the proof of (i), we have $\|z - w\| = h > 0, \|STz - w\| \leq \|z - w\|$ and $\|Tz - w\| \leq \|z - w\|$. Then we have that for any $\alpha \in [a,b], \beta = 1$ and $\gamma \in [0,1]$,

$$\begin{aligned}
& \|\alpha S[\beta Tz + (1-\beta)z] + (1-\alpha)[\gamma Tz + (1-\gamma)z] - w\| \\
= \; & \|\alpha STz + (1-\alpha)[\gamma Tz + (1-\gamma)z] - w\| \\
\leq \; & \gamma \|\alpha STz + (1-\alpha)Tz - w\| + (1-\gamma) \|\alpha STz + (1-\alpha)z - w\| \\
\leq \; & \gamma \|z - w\| + (1-\gamma) \|\alpha STz + (1-\alpha)z - w\| \\
< \; & h
\end{aligned}$$

using strict convexity of E. Also we have that if $Tz \neq z$, then for any $\alpha \in [a,b], \beta \in (0,1)$ and $\gamma \in [0,b]$,

$$\|\alpha S[\beta Tz + (1-\beta)z] + (1-\alpha)[\gamma Tz + (1-\gamma)z] - w\| < h$$

using strict convexity of E and (6). If $Tz = z$, we get

$$\begin{aligned}
& \|\alpha S[\beta Tz + (1-\beta)z] + (1-\alpha)[\gamma Tz + (1-\gamma)z] - w\| \\
= \; & \|\alpha STz + (1-\alpha)[\gamma Tz + (1-\gamma)z] - w\| \\
= \; & \|\alpha STz + (1-\alpha)z - w\|. \\
< \; & h
\end{aligned}$$

using strict convexity of E. As in the proof of (i), we also have

$$max\{g(\alpha,\beta,\gamma) : (\alpha,\beta,\gamma) \in [a,b] \times [a,1] \times [0,b]\} < h.$$

As in the proof of (i), choose a positive number r such that

$$max\{g(\alpha,\beta,\gamma) : (\alpha,\beta,\gamma) \in [a,b] \times [a,1] \times [0,b]\} < h - r.$$

Then we obtain $h \leq \|x_{m+1} - w\| < h$. This is a contradiction. So, we obtain $z = STz$. This completes the proof of (v). We shall prove (vi). let $\{(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k})\}$ be a subsequence of $\{(\alpha_n, \beta_n, \gamma_n)\}$ such that $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \to (\alpha_0, \beta_0, \gamma_0)$. Without loss of generality, we assume that $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \in [a,b] \times [a,1] \times [a,b] \cup [a,b] \times [0,b] \times [a,b]$ with some $a, b \in (0,1)$ for all $k \geq 1$. If $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \in [a,b] \times [0,b] \times [a,b]$, as in the proof of (i) and (iii), we have that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_{k_l}} \to z \in F(S) \cap F(T)$. Also we assume $(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \in [a,b] \times [a,1] \times [a,b]$. By the compactness of D, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}} \to z$ for some $z \in D$. As in the proof of (i) and (v), we have $z \in F(T) \cap F(ST)$. Then we have that $Sz = STz = z$. This implies $z \in F(S) \cap F(T)$. This completes the proof of (vi). □

The following is a strong convergence theorem of iterates defined by (3) in a strictly convex Banach space.

Theorem 4.4 *Let C be a nonempty closed convex subset of a strictly convex Banach space E and let S, T be nonexpansive mappings of C into itself such that $S(C) \cup T(C)$ is contained in a compact subset of C and $F(S) \cap F(T)$ is nonempty. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and $x_{n+1} = \alpha_n S[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)[\gamma_n T x_n + (1-\gamma_n)x_n]$ for all $n \geq 1$, where $\alpha_n, \beta_n, \gamma_n \in [0,1]$. If there exists a cluster point $(\alpha_0, \beta_0, \gamma_0)$ of $\{(\alpha_n, \beta_n, \gamma_n)\}$ in $[a,b] \times [a,b] \times [0,b]$ for some $a, b \in (0,1)$ or $[a,b] \times [0,1] \times [a,b]$ for some $a, b \in (0,1)$, then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof As in the proof of Theorem 4.3, $\lim_{n\to\infty} \|x_n - w\|$ exists for each $w \in F(S) \cap F(T)$. By Theorem 4.3, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to z_0 \in F(S) \cap F(T)$. Then, we get

$$\lim_{n\to\infty} \|x_n - z_0\| = \lim_{i\to\infty} \|x_{n_i} - z_0\| = 0.$$

Therefore, $\{x_n\}$ converges strongly to $z_0 \in F(S) \cap F(T)$. This completes the proof. □

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