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THE EXISTENCE OF GEODESIC LOOPS ON ALEXANDROV SURFACES

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§1. INTRODUCTION

In this paper our discussion is directed mainly to Alexandrov surfaces with curvature bounded below by a constant k and having no boundary. An Alexandrov space is a finite Hausdorff dimensional locally compact, complete length space satisfying Alexandrov convexity. The idea of the existence of geodesic loops was first proposed by Cohn-Vossen in analyzing the behavior of geodesics with variations of the total curvature. It had been understood that the geodesics of a Riemannian plane are all simple and not closed if there is no simple closed polygon bounding a compact domain of total curvature greater than π. Then the problem arose to find conditions under which many geodesic loops exit. In 1936, Cohn-Vossen showed the proper direction where there exist surprisingly many geodesic loops on a Riemannian plane under the hypothesis that the total curvature is strictly greater than π (Ref. [3], p. 144). Busemann extended this idea to Busemann G-planes admitting Busemann-total excess strictly greater than π in the case that the angular measure is uniform at π (Ref. [1], Theorem 44.9). By using the idea of Busemann-total excess, Machigashira (Ref. [10]) defined the total excess and Gaussian curvature of an Alexandrov surface. The purpose of this paper is to give a proof for the **Main Theorem,** extending Cohn-Vossen's idea to the more general case of Alexandrov surfaces.

Main Theorem. *Let X be a finitely connected Alexandrov surface with one end. If the total excess $C(X)$ satisfies the relation $C(X) > (2\chi(X) - 1)\pi$, then for any compact set C in X, there exists a bounded set N of X such that*

(1) $C \subset N$.

(2) *For any point $p \in X \backslash N$, there exists a geodesic loop γ_p whose base point is at p and C is contained in the disk domain bounded by γ_p.*

Here an Alexandrov surface is by definition finitely connected if it is homeomorphic to a closed surface without boundary from which finitely many points(ends) are removed. The definition of the total excess $C(X)$ will be given in §2. We note that $C(X)$ is equal to the total curvature of X if X is a smooth complete Riemannian manifold with dimension 2. The Euler-characteristic of X is denoted by $\chi(X)$. Throughout this paper we refer the basic tools of Alexandrov spaces to [4], [5] and [8].

§2. Preliminaries.

Throughout this section let X be an Alexandrov surface with curvature bounded below by k possessing no boundary. . An angle between two geodesics emanating from a point in X is naturally defined. For any point p in X, Σ_p denotes the space of directions at p which is a compact Alexandrov space with curvature bounded below by 1 with the Hausdorff dimension $\dim_H \Sigma_p$ is equal to 1. The length of Σ_p is less than or equal to 2π. A point p in X is called a singular point if Σ_p is not isometric to the unit circle $S^1(1)$. The set Sing(X) of all singular points in X is a countable set in X (Ref. [8], Theorem A). Now we will give the notion for the excess $\varepsilon(D)$ of a bounded domain D of X.

Since an Alexandrov surface is a topological manifold, for a point p in X and a sufficiently small geodesic triangle $\Delta = \Delta(abc)$ in a neighborhood U of p with the corners a, b, c enclosing a disk domain, the excess $\varepsilon_\circ(\Delta)$ of Δ is defined to be

$$\varepsilon_\circ(\Delta) := A + B + C - \pi,$$

where A, B and C are the inner angles of Δ at the corresponding corners.

If p is an interior point of Δ, by dividing Δ into three triangles, $\Delta_1 = \Delta(apb)$, $\Delta_2 = \Delta(bpc)$ and $\Delta_3 = \Delta(cpa)$, we have

$$\varepsilon_\circ(\Delta) = \sum_{i=1}^{3} \varepsilon_\circ(\Delta_i) + 2\pi - \mathrm{L}(\Sigma_p).$$

Theorem [(Ref. [1],Theorem 43.3) and (Ref. [10],Theorem 1.8)](The fundamental relation between the excess and the Euler-characteristic). *Let D be a domain whose boundary ∂D consists of a finite union of simple closed geodesic polygons. Let $\omega_1, \cdots, \omega_l$ be inner angles of all the corners of ∂D. If $\Delta \equiv \{\Delta_i\}_{i=1}^n$ is a finite simplical decomposition of D into small geodesic triangles, and $x_1 \cdots, x_k \in D$ are all the vertices of the Δ_i s lying in the interior of D, then*

$$\text{(2-1)} \qquad \varepsilon_\circ(D) + \sum_{i=1}^{k} (2\pi - L(\Sigma_{x_i})) = 2\pi\chi(D) - \sum_{j=1}^{l} (\pi - \omega_j),$$

where $\chi(D)$ denotes the Euler-characteristic of D, and $\varepsilon_\circ(D) := \sum_{i=1}^{n} \varepsilon_\circ(\Delta_i)$.

We have the important result proved by Machigashira(Ref. [10],Theorem 2.0) that $\varepsilon_\circ(D) \geq k\mathcal{H}^2(D)$, where $\mathcal{H}^2(D)$ denotes the two dimensional Hausdorff measure of D. Considering equation (2-1), we see that the right hand side is finite and independent of the choice of Δ, and thus so too is the left hand side. This fact and Machigashira's result together help us define the excess $\varepsilon(D)$ of D in the following way:

$$\varepsilon(D) := \liminf_{\delta \to 0 \ \Phi_\delta(D) \ni \Delta} \varepsilon_\circ(D),$$

where $\Phi_\delta(D)$ denotes the set of all finite simplical decompositions of D such that $|\Delta| < \delta$. Here $|\Delta|$ denotes the maximum of the circumferences of all geodesic triangles of Δ. Then the total excess $C(D)$ of D is defined as

$$\text{(2-2)} \qquad C(D) := \varepsilon(D) + \lim_{\delta \to 0,\ \Phi_\delta(D) \ni \Delta, i=1} \sum^{k} (2\pi - \mathrm{L}(\Sigma_{x_i}))$$

Let $\{D_i\}_{i=1,2\cdots}$ be a monotone increasing sequence of relatively compact domains in X such that $X = \bigcup_{i=1}^{\infty} D_i$ and ∂D_i consists of a finite union of simple closed geodesic polygons for each i.

Definition 2.1. We say that X admits a *total excess* $C(X)$ if and only if $C(X) := \lim_i C(D_i)$ exists, is bounded above, and its limit is independent of the choice of $\{D_i\}_{i=1,2,\cdots}$.

Definition 2.2. A subset U of X is called a *tube* if U is homeomorphic to $S^1 \times [0, \infty)$.

§3.Proof of the Main Theorem

In this section we prove the Main Theorem. The following Assertion and proposition are needed for the proof of our Main Theorem. For an arbitrary compact set $C \subset X$ we choose $C_\circ \supset C$ in X be a domain such that $X \backslash C_\circ$ is a tube whose boundary $\partial C_\circ$ is a simple closed geodesic polygon. Let $M := X \backslash C_\circ$. For any point $p \in X \backslash C_\circ$ we define

$\mathcal{A}_p := \{c | c:[0,1] \longrightarrow M$ is a simple closed curve which is freely homotopic to $\partial C_\circ$ in M with the base point at $p\}$

Then there exists a curve $\gamma_p \in \mathcal{A}_p$ such that $L(\gamma_p) = \inf_{c \in \mathcal{A}_p} L(c)$. Also the function $X \backslash C_\circ \ni p \longmapsto L(\gamma_p)$ is lipschitz continuous with lipschitz constant 2.

Let $\{p_j\}_{j=1,2,\ldots}$ be a divergent sequence of points such that $\lim_{j \to \infty} d(p_j, C_\circ) = \infty$, where d is the distance function defined on X and $\gamma_j (:= \gamma_{p_j}) \in \mathcal{A}_{p_j}$ satisfies $L(\gamma_j) = \inf_{c \in \mathcal{A}_{p_j}} L(c)$ for each j. Suppose that $\gamma_j \cap \partial C_\circ \neq \emptyset$. If ω_k, for $k = 1, 2, \ldots, b_j$, are all the inner angles at the vertices of D_j lying on $\partial C_\circ$ then clearly $\omega_k \leq \pi$, where D_j is the domain bounded by γ_j and containing $C_\circ$. Let $\gamma := \lim_{i \to \infty} \gamma_{j(i)}$ be the limit polygon. This $\gamma : \Re \longrightarrow M$ is parameterized by arc length such that $\gamma(0) \in \partial C_\circ$ and $\gamma(s) \notin \partial C_\circ$ for all $s > 0$. With these notations we have

Assertion (Ref. [7],Lemma (B)). *Let $\{\epsilon_j\}$ be a decreasing sequence of positive numbers tending to 0. For each j there exist large numbers $l_j, m_j, l'_j, -m'_j$ such that if*

$$\lambda_j : [0, l_j] \longrightarrow M \quad \text{and} \quad \mu_j : [0, m_j] \longrightarrow M$$

are minimizing geodesics with $\lambda_j(0) = \mu_j(0) = p_j$, $\lambda_j(l_j) = \gamma(l'_j) =: q_j$ and $\mu_j(m_j) = \gamma(m'_j) =: r_j$ then inner angles at q_j and r_j of the domain E_j bounded by γ, λ_j and μ_j are less than $\epsilon_j/2$.

Proof of the Assertion. We need only to find for a fixed j a point q_j on γ . Let $g(t) := d(p_j, \gamma(t))$ for all $t \geq 0$. Then by the triangle inequality we have

$$|t - g(t)| \leq d(p_j, \gamma(0)) < +\infty \quad \text{for all } t \geq 0. \tag{3-1}$$

Then g is lipschitz continuous function with lipschitz constant 1. Hence g is differentiable almost everywhere. Then the first variation formula(Ref. [8], Theorem

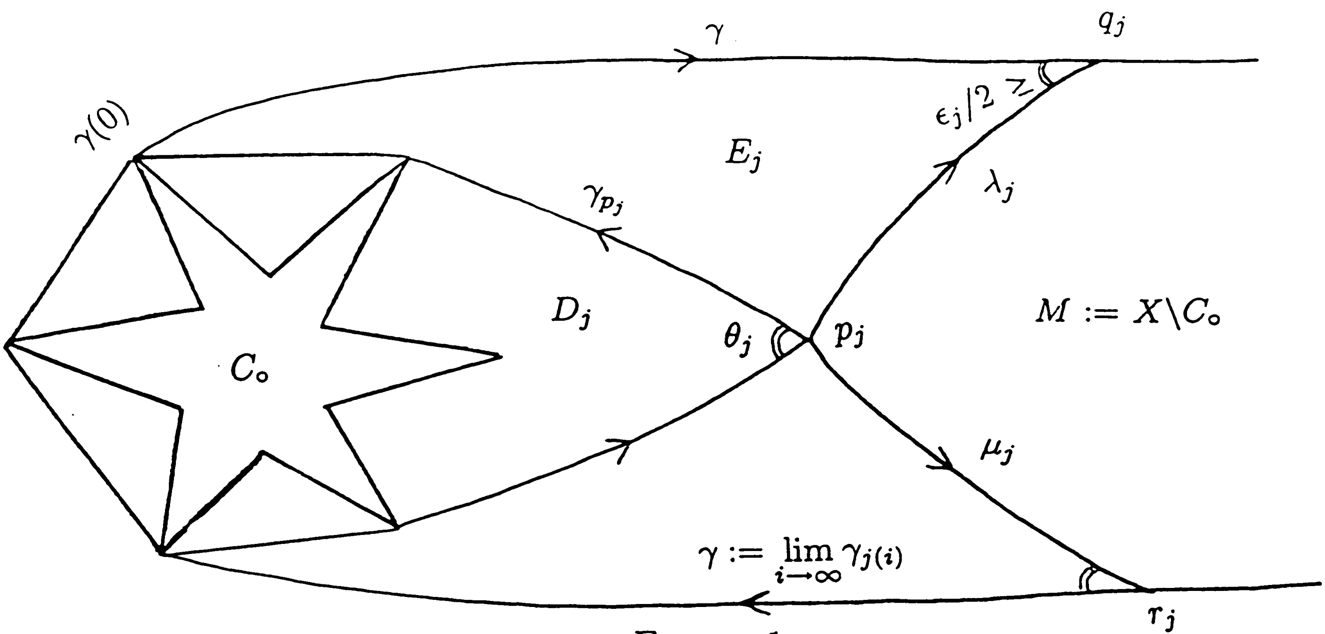

FIGURE 1

3.5) implies that $\frac{d}{dt}(g(t)) = \cos(\alpha_t)$ a.e., where α_t is the angle at $\gamma(t)$ of the domain E_j. For a large T,

$$T - g(T) = \int_{t_j}^{T} (1 - \cos(\alpha_t))\, dt \quad + \quad (t_j - g(t_j))$$

By fixing j and using (3-1), we obtain

$$0 \leq \int_{t_j}^{T} (1 - \cos(\alpha_t))\, dt \leq d(p_j, \gamma(0)) - (t_j - g(t_j)) < +\infty. \tag{3-2}$$

If for a given $\epsilon > 0$, there exists t_ϵ such that $1 - \cos(\alpha_t) > \epsilon$ for all $t > t_\epsilon$, then by (3-2) we have $+\infty > \int_{t_j}^{T}(1 - \cos(\alpha_t))\, dt \geq \epsilon(T - t_j)$. This is a contradiction for a large T. Therefore, for $\epsilon > 0$, there exists a monotone divergent sequence on which the integrand in (3-2) is less than ϵ. Thus the Assertion is proved.

Proposition (Ref. [6],Theorem C). *If $\{p_j\}$ is a divergent sequence of points in $X \backslash C_\circ$ with $\lim_{j\to\infty} d(p_j, \partial C_\circ) = \infty$ and for each j, $\gamma_{p_j} \in \mathcal{A}_{p_j}$ with $\gamma_{p_j} \cap \partial C_\circ \neq \emptyset$ and if θ_j is the inner angle at p_j of the domain D_j bounded by γ_{p_j} and containing $C_\circ$ then $\lim_{j\to\infty} \theta_j = 0$.*

Proof. Let ω_k, for $k = 1, 2, \ldots, b_j$, be all the inner angles at the vertices of D_j lying on $\partial C_\circ$. We apply the equation (2-1) to the domains D_j and E_j respectively:

$$\varepsilon_\circ(D_j) + \sum_{i=1}^{a_j} (2\pi - \mathrm{L}(\Sigma_{x_i})) = 2\pi\chi(D_j) - \sum_{k=1}^{b_j} (\pi - \omega_k) - (\pi - \theta_j),$$

where a_j has the same meaning as in the equation (2-1). By taking the $|\Delta_j| \to 0$, we have

$$C(D_j) = 2\pi\chi(X) - \pi - \sum_{k=1}^{b_j} (\pi - \omega_k) + \theta_j.$$

By taking the limit we have

$$\lim_{j\to\infty} C(D_j) = (2\chi(X) - 1)\pi - \lim_{j\to\infty} \sum_{k=1}^{b_j} (\pi - \omega_k) + \lim_{j\to\infty} \theta_j.$$

If U is the domain bounded by γ and containing $C_\circ$ then $\lim_{j\to\infty} C(D_j) = C(U)$ and

$$(3\text{-}3) \qquad C(U) = (2\chi(X)-1)\pi - \lim_{j\to\infty}\sum_{k=1}^{b_j}(\pi-\omega_k) + \lim_{j\to\infty}\theta_j .$$

Similarly,

$$C(E_j) \leq 2\pi\chi(E_j) - \sum_{k=1}^{b_j}(\pi-\omega_k) - 2\pi + \epsilon_j - \pi + \beta_j,$$

where β_j is the angle at p_j between λ_j and μ_j and clearly $\beta_j \leq 2\pi$. By taking the limit we have,

$$(3\text{-}4) \qquad C(U) \leq (2\chi(X)-1)\pi - \lim_{j\to\infty}\sum_{k=1}^{b_j}(\pi-\omega_k) + \lim_{j\to\infty}\epsilon_j .$$

From (3-3) and (3-4) we have $\lim_{j\to\infty}\theta_j \leq 0$, and hence $\lim_{j\to\infty}\theta_j = 0$.

Proof of the Main Theorem. Suppose the contrapositive of the Main Theorem. That is, there exists a compact set $C_\circ$ in X with the property that for any bounded set N containing $C_\circ$ in its interior, there exists a point p in $X\backslash N$ such that there is no geodesic loop γ_p whose base point is at p, such that the domain enclosed by γ_p contains $C_\circ$.

Let $M := X\backslash C_\circ$ be any tube. There is a point $p \in X\backslash N$ such that γ_p is not a geodesic loop and hence $\gamma_p \cap \partial C_\circ \neq \emptyset$.

Let $\{p_j\}$; $p_j \in X\backslash N$ be a divergent sequence of points with $\lim_{j\to\infty} d(p_j, \partial C_\circ) = \infty$ such that $\gamma_{p_j} \cap \partial C_\circ \neq \emptyset$ for each j. Let $\{C_j\}$ be a monotone increasing sequence of compact sets such that $C_\circ \subset C_1 \subset C_2 \subset \ldots$, $\bigcup_j C_j = X$, where ∂C_j is a geodesic polygon. Then for a given $\epsilon > 0$, there exists $j(\epsilon)$ such that p_j for each $j > j(\epsilon)$ has the following properties:

(1) $\mathcal{A}_{p_j} = \{c \mid c : [0,1] \longrightarrow X\backslash C_j$ is a simple closed curve which is freely homotopic to ∂C_j in $X\backslash C_j$ with the base point at $p_j\}$

Then there exists a curve

$$\mathbb{P}_j \in \mathcal{A}_{p_j} \quad \text{such that} \quad L(\mathbb{P}_j) = \inf_{c\in\mathcal{A}_{p_j}} L(c).$$

$\mathbb{P}_j$ is a geodesic polygon such that all of its vertices with the exception of p_j are on ∂C_j.

(2) The inner angle θ_j at p_j of $\mathbb{P}_j$ is less than or equal to ϵ_j.

Let D_j be a domain bounded by $\mathbb{P}_j$ containing C_j. Choose a monotone increasing subsequence $\{D_k\}$ of $\{D_j\}$ such that $C_k \subset D_k \subset C_{k+1} \subset D_{k+1}$. Then $\bigcup_k D_k = X$. By applying (2-1) to D_k, we have

$$\varepsilon_\circ(D_k) + \sum_{i=1}^{a_k}(2\pi - \mathrm{L}(\Sigma_{x_i})) = 2\pi\chi(D_k) - \sum_{j=1}^{b_k}(\pi-\omega_j) - (\pi-\theta_k),$$

$$\varepsilon_\circ(D_k) + \sum_{i=1}^{a_k}(2\pi - \mathrm{L}(\Sigma_{x_i})) \leq (2\chi(X)-1)\pi + \theta_k.$$

Then by taking the $|\Delta_k| \to 0$, we have $C(D_k) \leq (2\chi(X) - 1)\pi + \epsilon_k$. Then $\lim_{k\to 0} C(D_k) = C(X)$ leads to a contradiction.

Remark. According to Cohn-Vossen(Ref. [2]) tubes are classified into two groups as expanding and contracting. Our Main Theorem always holds for the contracting case without any restriction on the excess.

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