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A free group of rotations with rational entries on the 3-dimensional unit sphere

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ABSTRACT. Dekker showed the existence of a free group of rank 2 of rotations (special orthogonal matrices) acting on the 3-dimensional unit sphere whose non-trivial rotations does not have a fixed point. In this paper, we find a free group on the 3-dimensional unit sphere which satisfies the same condition and whose entries of matrices are rational.

Introduction.

Banach and Tarski proved a very curious paradox which enables us to partition a golf ball into finite number of pieces and reconstruct the earth from these pieces:

Paradox A [BT,W]. *If U and V are any bounded subsets of the 3-dimensional Euclidean space $\mathbb{R}^3$, each having non-empty interior, then U and V can be both partitioned into the same finite number of respectively (orientation-preserving) congruent pieces. Formally,*

$$U = \bigcup_{l=0}^{m-1} U_l, \quad V = \bigcup_{l=0}^{m-1} V_l,$$

$U_l \cap U_{l'} = \emptyset = V_l \cap V_{l'}$ if $0 \leqq l \lneqq l' \lneqq m$, and there are (orientation-preserving) isometries $\alpha_0, \ldots, \alpha_{m-1}$ such that, for each $0 \leqq l \lneqq m$, $\alpha_l(U_l) = V_l$.

Paradox A owes the following paradox essentially:

Paradox B [Si,W]. *The 2-dimensional unit sphere $\mathbb{S}^2 = \{\vec{r} \in \mathbb{R}^3 : |\vec{r}| = 1\}$ admits three decompositions into disjoint pieces*

$$\mathbb{S}^2 = A_0 \cup \cdots \cup A_{p-1} \cup B_0 \cup \cdots \cup B_{q-1},$$

$$\mathbb{S}^2 = A'_0 \cup \cdots \cup A'_{p-1}, \quad \text{and} \quad \mathbb{S}^2 = B'_0 \cup \cdots \cup B'_{q-1},$$

and $A_i \approx A'_i$ $(i = 0, 1, \ldots, p-1)$, $B_j \approx B'_j$ $(j = 0, 1, \ldots, q-1)$, where $C \approx C'$ means $\gamma(C) = C'$ for a suitable rotation γ.

Robinson [R,W] constructed decompositions for $p = q = 2$, which is the fewest piece's decompositions. For a given set E, it is a very important problem to find

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a free group of rank 2 acting on E whose element different from the identity has no fixed point. If such a free group exists, E has a Hausdorff decomposition (see [Sa0], etc.), i.e., E is partitioned into three subsets P, Q, and R such that

$$P \approx Q \approx R \approx P \cup Q \approx Q \cup R \approx R \cup P.$$

The existence of a Hausdorff decomposition implies the existence of 2 + 2 pieces decompositions, but the converse is not always true. For example, the 2-dimensional unit sphere $\mathbb{S}^2$ does not have a Hausdorff decomposition because every non-trivial rotation has an axis. So such a free group on $\mathbb{S}^2$ does not exist. In higher dimensional space, does there exist a free group of rank 2 of rotations (special orthogonal matrices) on the 3-dimensional unit sphere $\mathbb{S}^3 = \{\vec{r} \in \mathbb{R}^4 : |\vec{r}| = 1\}$ whose elements distinct from the identity have no fixed point? Dekker [D,W] proved that there exists a free group of rank 2 on the 3-dimensional unit sphere $\mathbb{S}^3$ whose non-trivial rotations have no fixed point. So $\mathbb{S}^3$ has a Hausdorff decomposition. This paper proves the existence of such a free group on $\mathbb{S}^3$ whose entries of matrices are all rational numbers. We show affirmative answers of Problem A and Problem B of [Sa1] in the case of $n \equiv 0, 3 \pmod 4$.

Free Groups on the unit spheres.

The following is the main theorem in this paper which gives a concrete example of a pair of matrices with rational entries which generates a free group of rotations whose non-trivial element has no fixed point:

Theorem. *Let σ, τ be rotations on $\mathbb{S}^3$ given, respectively, by*

$$\frac{1}{7}\begin{pmatrix} 2 & -6 & -3 & 0 \\ 6 & 2 & 0 & 3 \\ 3 & 0 & 2 & -6 \\ 0 & -3 & 6 & 2 \end{pmatrix}, \quad \frac{1}{7}\begin{pmatrix} 2 & 0 & -3 & -6 \\ 0 & 2 & -6 & 3 \\ 3 & 6 & 2 & 0 \\ 6 & -3 & 0 & 2 \end{pmatrix}.$$

Then σ and τ generate freely a group of rotations such that no element of this group other than unity has any fixed point on $\mathbb{S}^3$.

Proof. We denote the matrix $\begin{pmatrix} c & -x & -y & -z \\ x & c & -z & y \\ y & z & c & -x \\ z & -y & x & c \end{pmatrix}$ by $\begin{bmatrix} c \\ x \\ y \\ z \end{bmatrix}$ for $c, x, y, z \in \mathbb{R}$. Four rotations in $H = \{\begin{bmatrix} c \\ x \\ y \\ z \end{bmatrix} : c, x, y, z \in \mathbb{R}\}$ are defined by

$$s = \frac{1}{\sqrt{14}}\begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad t = \frac{1}{\sqrt{14}}\begin{bmatrix} 3 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \sigma = s^2, \text{ and } \tau = t^2.$$

It is enough to show that every non-empty reduced word w in $\{s^{-1}, s, t^{-1}, t\}$ has no fixed points on $\mathbb{S}^3$. For $w = \begin{bmatrix} c_w \\ x_w \\ y_w \\ z_w \end{bmatrix}$, we prove $\det(w-1) \neq 0$. We define the operation $*$ on $K = \{(c, \begin{pmatrix} x \\ y \\ z \end{pmatrix}) : c, x, y, z \in \mathbb{R}\}$ by the following:

$$(c', \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}) * (c, \begin{pmatrix} x \\ y \\ z \end{pmatrix}) = (c'c - x'x - y'y - z'z, \begin{pmatrix} x'c + c'x + y'z - z'y \\ y'c + c'y + z'x - x'z \\ z'c + c'z + x'y - y'x \end{pmatrix}).$$

Then $(H, \cdot)$ is isomorphic to $(K, *)$, so [Sa1] implies ${x_w}^2 + {y_w}^2 + {z_w}^2 \gneqq 0$. Hence we get

$$\det(w-1) = ((c_w - 1)^2 + {x_w}^2 + {y_w}^2 + {z_w}^2)^2 \gneqq 0. \quad \square$$

Theorem answers affirmatively Problem A in [Sa1] for $n = 4k$:

Corollary 1. *$< \phi_{4k}, \psi_{4k} >$, the group generated by ϕ_{4k} and ψ_{4k}, is a free group of rotations such that all the elements of $< \phi_{4k}, \psi_{4k} >$ different from the identity have no fixed point on $\mathbb{S}^{4k-1}$, where*

$$\phi_{4k} = \begin{pmatrix} \sigma & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \sigma \end{pmatrix} \quad \text{and} \quad \psi_{4k} = \begin{pmatrix} \tau & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \tau \end{pmatrix}.$$

Theorem and [Sa1] answer affirmatively Problem B of [Sa1] for $n = 4k-1$:

Corollary 2. *$< \phi_{4k-1}, \psi_{4k-1} >$ is a free group of rotations which has no non-trivial element fixing a point on $\mathbb{S}^{4k-2} \cap \mathbb{Q}^{4k-1}$ and is such that if two elements $f, g \in < \phi_{4k-1}, \psi_{4k-1} >$ have a common fixed point on $\mathbb{S}^{4k-2}$ then $fg = gf$, where*

$$\phi_{4k-1} = \begin{pmatrix} \mu & 0 & \dots & 0 \\ 0 & \sigma & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \sigma \end{pmatrix} \quad \text{and} \quad \psi_{4k-1} = \begin{pmatrix} \nu & 0 & \dots & 0 \\ 0 & \tau & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \tau \end{pmatrix}.$$

$$\mu = \frac{1}{7}\begin{pmatrix} 6 & 2 & 3 \\ 2 & 3 & -6 \\ -3 & 6 & 2 \end{pmatrix} \quad \text{and} \quad \nu = \frac{1}{7}\begin{pmatrix} 2 & -6 & 3 \\ 6 & 3 & 2 \\ -3 & 2 & 6 \end{pmatrix}.$$

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