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A strong convergence theorem for an iteration of nonexpansive mappings

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1 Introduction

Let C be a nonempty closed convex subset of a real Banach space E. Then, a mapping T of C into itself is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \qquad \textit{for all } x, y \in C.$$

We deal with the following iterative process, first considered by Halpern[2]:

$$A_0^\alpha x = x, \quad A_{n+1}^\alpha x = \alpha_{n+1}x + (1 - \alpha_{n+1})TA_n^\alpha x \; (n = 0, 1, 2, \cdots), \tag{1}$$

where $\alpha_n \in [0, 1]$. Recently, Wittmann[5] proved a strong convergence theorem of iterates $\{A_n^\alpha x\}$ defined by (1) in the case when E is a Hilbert space and $\{\alpha_n\}$ satisfies $0 \leq \alpha_n \leq 1$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = +\infty$ and $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < +\infty$; see[3].

In this paper, we extend Wittmann's result to a uniformly convex and uniformly smooth Banach space with a weakly sequentially continuous duality mapping.

2 Preliminaries

Let E be a real Banach space, and let $S_1[0] = \{x \in E : \|x\| = 1\}$ be its unit sphere. The norm of E is said to be Gâteaux diffrentiable (and E is said to be smooth), if

$$\lim_{t\to 0}\frac{\|x + ty\| - \|x\|}{t} \tag{2}$$

exists for each $x, y \in S_1[0]$. It is said to be Fréchet differentiable, if for each x in $S_1[0]$, this limit is attained uniformly for $y \in S_1[0]$. The norm of E is said to be uniformly Gâteaux differentiable, if for each y in $S_1[0]$, the limit (2) is approached uniformly as x varies over $S_1[0]$. Finally, it is said to be uniformly Fréchet differentiable (and E is said to be uniformly smooth), if the limit (2) is attained uniformly for x, y in $S_1[0] \times S_1[0]$. For a Banach space E, we denote by J the normalized duality mapping on E to 2^{E^*} given by

$$J(x) = \left\{ f \in E^* : \|f\|^2 = \|x\|^2 = \langle x, f \rangle \right\},$$

where E^* denotes the continuous dual space of E and $\langle x, f \rangle = f(x)$. It is well known that if E^* is strictly convex, then J is single valued, and if E^* is uniformly convex, then J is uniformly continuous on bounded sets; see[6]. Suppose that J is single valued. Then J is said to be weakly sequentially continuous, if for any $\{x_n\} \subseteq E$ with $x_n \rightharpoonup x$, $\{J(x_n)\}$ converges to $J(x)$ in weak-star topology, where $\rightharpoonup$ will denote weak convergence. We define, for any positive t,

$$\beta(t) = \sup \left\{ \frac{\|x + ty\|^2 - \|x\|^2}{t} - 2\langle y, J(x) \rangle : \|x\| \leq 1, \|y\| \leq 1 \right\}.$$

Clearly, $\beta : (0, +\infty) \to [0, +\infty)$ is nondecreasing continuous and $\beta(ct) \leq c\beta(t)$ for all $c \geq 1$. We also know Reich's result[4]:

Lemma 1 (Reich) *Let E be a uniformly smooth Banach space and let $\beta(t)$ be defined as above, then* $\lim_{t \to +0} \beta(t) = 0$ *and*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x) \rangle + \max \{\|x\|, 1\} \|y\| \beta(\|y\|)$$

for all $x, y \in E$.

3 Main result

We denote by N the set of positive integers. The following theorem is a generalization of Wittmann's result[5] which was proved in a Hilbert space.

Theorem 1 *Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $(0, 1)$ such that*
(i) $\lim_{n \to \infty} \alpha_n = 0$ *;*

(ii) $\sum_{n=1}^{\infty}\alpha_n = +\infty$;
(iii) $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < +\infty$.
Let E be a uniformly convex and uniformly smooth Banach space with a weakly sequentially continuous duality mapping $J : E \to E^*$, let C be a nonempty closed convex subset of E and let T be a nonexpansive mapping of C into itself such that $F(T) = \{x \in C : Tx = x\}$ is nonempty. Then for any $x \in C$, the sequence $\{A_n^\alpha x\}_{n=0}^{\infty}$ converges strongly to $p = Px$, where P is a sunny nonexpansive retraction of C onto $F(T)$.

Proof. We may assume that C is bounded and $0 \in F(T)$ without loss of generality. Then for any $x \in C$ and $n \in N$, we have $\|Tx\| \leq \|x\|$ and $\|A_n^\alpha x\| \leq \|x\|$.
Since

$$\begin{aligned} \|A_{n+1}^\alpha x - A_n^\alpha x\| &\leq |\alpha_{n+1} - \alpha_n|\|x\| + |\alpha_{n+1} - \alpha_n|\|TA_{n-1}^\alpha x\| \\ &+ (1 - \alpha_{n+1})\|TA_n^\alpha x - TA_{n-1}^\alpha x\| \\ &\leq 2|\alpha_{n+1} - \alpha_n|\|x\| + (1 - \alpha_{n+1})\|A_n^\alpha x - A_{n-1}^\alpha x\|, \end{aligned}$$

we get, for any integers m and n with $m < n$,

$$\|A_{n+1}^\alpha x - A_n^\alpha x\| \leq 2\sum_{i=m}^{n}|\alpha_{i+1} - \alpha_i|\|x\| + 2\|x\|\prod_{i=m}^{n}(1 - \alpha_{i+1}).$$

This implies

$$\limsup_{n\to\infty}\|A_{n+1}^\alpha x - A_n^\alpha x\| \leq 2\sum_{i=m}^{\infty}|\alpha_{i+1} - \alpha_i|\|x\|,$$

because $\lim_{n\to\infty}\prod_{i=m}^{n}(1 - \alpha_{i+1}) = 0$. Letting now m tend to infinity, by (iii), we have

$$\limsup_{n\to\infty}\|A_{n+1}^\alpha x - A_n^\alpha x\| = 0.$$

Combining this with

$$\begin{aligned} \|A_n^\alpha x - TA_n^\alpha x\| &\leq \|A_n^\alpha x - (1 - \alpha_n)TA_{n-1}^\alpha x\| \\ &+ (1 - \alpha_n)\|TA_{n-1}^\alpha x - TA_n^\alpha x\| + \alpha_n\|TA_n^\alpha x\| \\ &\leq 2\alpha_n\|x\| + \|A_{n-1}^\alpha x - A_n^\alpha x\|, \end{aligned}$$

by (i), we obtain

$$\lim_{n\to\infty}\|A_n^\alpha x - TA_n^\alpha x\| = 0. \tag{3}$$

On the other hand, by Reich[3] or Takahashi[6, p.128], there is a sunny nonexpansive retraction P of C onto $F(T)$. Let $p = Px$. Since P is a sunny nonexpansive retraction, we know from Takahashi[6, p179] that

$$\langle x - p, J(z - p)\rangle \leq 0 \qquad \textit{for any } z \in F(T). \tag{4}$$

We show next that

$$\limsup_{n \to \infty} \langle x - p, J(A_n^\alpha x - p)\rangle \leq 0.$$

If not, there exists a positive real number r such that

$$0 < r < \limsup_{n \to \infty} \langle x - p, J(A_n^\alpha x - p)\rangle.$$

So, there is a subsequence $\left\{A_{n_i}^\alpha x\right\}_{i=1}^\infty$ of $\{A_n^\alpha x\}$ such that

$$r < \langle x - p, J(A_{n_i}^\alpha x - p)\rangle.$$

By possibly replacing $\{n_i\}$ by another subsequence, we may also assume that $\left\{A_{n_i}^\alpha\right\}$ is weakly convergent to some $z_0 \in C$. Since E is uniformly convex, by Browder[1] and (3), we obtain $z_0 \in F(T)$. Since J is weakly sequentially continuous, we have

$$r \leq \langle x - p, J(z_0 - p)\rangle.$$

This contradicts (2). Hence we have

$$\limsup_{n \to \infty} \langle x - p, J(A_n^\alpha x - p)\rangle \leq 0.$$

So, for an arbitrary positive number ϵ, from (3), we can choose a nonnegative integer n_ϵ such that for any $n \geq n_\epsilon$,

$$\langle x - p, J(TA_n^\alpha x - p)\rangle < \epsilon$$

and

$$\max\{(1 - \alpha_{n+1})\|TA_n^\alpha x - p\|, 1\}\, \alpha_{n+1}\|x - p\|\beta(\alpha_{n+1}\|x - p\|) < \epsilon.$$

Then, by Lemma1, for any $n \geq n_\epsilon$, we have

$$\begin{aligned}
\|A_{n+1}^\alpha x - p\|^2 &\leq (1 - \alpha_{n+1})^2\|TA_n^\alpha x - p\|^2 \\
&+ 2\alpha_{n+1}(1 - \alpha_{n+1})\langle x - p, J(TA_n^\alpha x - p)\rangle \\
&+ \max\{(1 - \alpha_{n+1})\|TA_n^\alpha x - p\|, 1\}\, \alpha_{n+1}\|x - p\|\beta(\alpha_{n+1}\|x - p\|) \\
&\leq (1 - \alpha_{n+1})^2\|TA_n^\alpha x - p\|^2 + 3\alpha_{n+1}\epsilon.
\end{aligned}$$

Hence, we have, for $n > n_\epsilon$,

$$\|A_n^\alpha x - p\|^2 \leq 3\epsilon + \|A_{n_\epsilon}^\alpha x - p\|^2 \prod_{i=n_\epsilon+1}^{n}(1 - \alpha_i).$$

So, from $\lim_{n\to\infty}\prod_{i=n_\epsilon+1}^{n}(1 - \alpha_i) = 0$, we have

$$\limsup_{n\to\infty}\|A_n^\alpha x - p\|^2 \leq 3\epsilon.$$

Because $\epsilon > 0$ is arbitrary, we have

$$\lim_{n\to\infty} A_n^\alpha x = p.$$

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