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SEMINORMAL OPERATORS AND WEYL SPECTRA

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ABSTRACT. In this paper we show that the Weyl spectrum of a seminormal operator T satisfies the spectral mapping theorem for any analytic function f on a neighborhood of $\sigma(T)$ and Weyl's theorem holds for $f(T)$. Finally we give conditions for an operator to be of the form unitary + compact and answer an old question of Oberai.

0. Introduction. Throughout this paper let H denote an infinite dimensional Hilbert space and $B(H)$ the set of all bounded linear operators on H. If $T \in B(H)$, we write $\sigma(T)$ for the spectrum of T, $\pi_0(T)$ for the set of eigenvalues of T, $\pi_{0f}(T)$ for the set of eigenvalues of finite multiplicity, and $\pi_{00}(T)$ for the isolated points of $\sigma(T)$ that are eigenvalues of finite multiplicity. If E is a subset of $\mathbb{C}$, we write iso E for the set of isolated points of E. An operator $T \in B(H)$ is said to be *Fredholm* if its range ran T is closed and both the null spaces ker T and ker T^* are finite dimensional. The *index* of a Fredholm operator T, denoted by $i(T)$, is defined by

$$i(T) = \dim\ker T - \dim\ker T^*.$$

The *essential spectrum* of T, denoted by $\sigma_e(T)$, is defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Fredholm}\}.$$

A Fredholm operator of index zero is called a *Weyl* operator. The *Weyl spectrum* of T, denoted by $\omega(T)$, is defined by

$$\omega(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Weyl}\}.$$

It was shown ([2]) that for any operator T, $\sigma_e(T) \subset \omega(T) \subset \sigma(T)$, and $\omega(T)$ is a nonempty compact subset of $\mathbb{C}$.

Recall ([9], [12]) that an operator $T \in B(H)$ is said to be *seminormal* if either T or T^* is hyponormal. Every hyponormal operator is seminormal,

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but the converse is not true in general. Unilateral shifts are examples of seminormal operators.

An operator $T \in B(H)$ is said to be *dominant* ([4], [13]) if for every $z \in \mathbb{C}$ there exists $M_z > 0$ such that

$$(T-z)(T-z)^* \leq M_z(T-z)^*(T-z)$$

In this case, if $\sup_{z\in\mathbb{C}} M_z = M < \infty$, T is said to be $M-$hyponormal, and if $M = 1$, T is hyponormal. Evidently,

$$T \text{ is hyponormal} \Longrightarrow T \text{ is } M\text{−hyponormal} \Longrightarrow T \text{ is dominant}$$

If T is both Fredholm and seminormal, then either $i(T) \leq 0$ or $i(T) \geq 0$. It was known that the mapping $T \to \omega(T)$ is upper semi-continuous, but not continuous at T ([7]). However if $T_n \to T$ with $T_nT = TT_n$ for all $n \in \mathbb{N}$ then

$$\lim \omega(T_n) = \omega(T). \tag{0.1}$$

It was known that $\omega(T)$ satisfies the one-way spectral mapping theorem for analytic funcions: if f is analytic on a neighborhood of $\sigma(T)$ then

$$\omega(f(T)) \subset f(\omega(T)). \tag{0.2}$$

The inclusion (0.2) may be proper(see [2, Example 3.3]). If T is normal then $\sigma_e(T)$ and $\omega(T)$ coincide. Thus if T is normal and f is analytic on a neighborhood of $\sigma(T)$, it follows that $\omega(f(T)) = f(\omega(T))$ since $f(T)$ is also normal.

In this paper we show that the Weyl spectrum of a seminormal operator T satisfies the spectral mapping theorem for any analytic function f on a neighborhood of $\sigma(T)$ and Weyl's theorem holds for $f(T)$. Finally we give conditions for an operator to be of the form unitary + compact and answer an old question of Oberai.

1. Weyl spectral properties. It was shown ([2]) that for any operator T, $\omega(T^*) = \omega(T)^*$ and

$$\omega(T) = \cap_{K\in\mathcal{K}}\sigma(T+K).$$

The Weyl spectrum of an operator is the disjoint union of the essential spectrum and a particular open set ([2]): For any operator T in $B(H)$,

$$\omega(T) = \sigma_e(T) \cup \{\lambda : T-\lambda \text{ is Fredholm and } i(T-\lambda) \neq 0\}. \tag{1.1}$$

We have the concrete form of $\omega(T)$ provided T is seminormal:

THEOREM 1. *If T is a seminormal operator, then*

$$\omega(T) = \cap\{\sigma(T+K) : TK = KT \text{ and } K \text{ is normal compact}\}.$$

Proof. Let $E = \{\sigma(T+K) : TK = KT$ and K is normal compact$\}$. Then by [2, Theorem 2.5], $\omega(T) \subset E$. Since T is seminormal, by [10] there exists a normal compact operator K such that $KT = TK$ and $\sigma(T+K) = \omega(T)$. Thus $E \subset \omega(T)$. This completes the proof.

THEOREM 2. *If $\pi(T)$ is seminormal in $B(H)/\mathcal{K}$ and if $\omega(T) \subset \{\lambda : |\lambda| = 1\}$, then T is of form unitary + compact.*

Proof. By hypothesis, 0 is not in $\omega(T)$ and so $T = S + K$, where S is invertible and K is compact. Hence $\pi(T) = \pi(S)$. Since $\sigma(\pi(T)) \subset \omega(T) \subset \{\lambda : |\lambda| = 1\}$, by [9, p. 59 Corollary], $\pi(T)$ is unitary in $B(H)/\mathcal{K}$ and so $\pi(S^*S) = \pi(I)$. But square roots of a positive element of a C^*-algebra are unique, so $\pi((S^*S)^{1/2}) = \pi(I)$. Let the polar decomposition of S be given by $S = U(S^*S)^{1/2}$, where U is unitary. Then

$$\begin{aligned}\pi(T) = \pi(S) = \pi(U(S^*S)^{1/2}) &= \pi(U)\pi((S^*S)^{1/2}) \\ &= \pi(U)\pi(I) = \pi(U),\end{aligned}$$

so that $T - U$ is compact.

COROLLARY 3. *If $\pi(T)$ is normal in $B(H)/\mathcal{K}$ and if $\omega(T) \subset \{\lambda : |\lambda| = 1\}$, then T is of form unitary + compact.*

For an example, consider $T = U \oplus U^*$, where U is the unilateral shift. In this case, $\omega(T) = \{\lambda : |\lambda| = 1\} = \sigma_e(T)$. But T is not a normal operator. Since $I - UU^*$ and $UU^* - I$ are rank one operators, $\pi(T)$ is normal. By Corollary 3, $T = U \oplus U^*$ is of the form unitary + compact.

LEMMA 4([13]). *If S and T are commuting dominant operators, then*

$$S, T \text{ Weyl} \iff ST \text{ Weyl}. \tag{1.2}$$

If the "dominant" condition is dropped in the above lemma, then the backward implication may fail even though T_1 and T_2 commute: For example, if U is the unilateral shift on l_2, consider the following operators on $l_2 \oplus l_2$: $T_1 = U \oplus I$ and $T_2 = I \oplus U^*$.

THEOREM 5. *If T is seminormal and f is analytic on a neighborhood of $\sigma(T)$, then $\omega(f(T)) = f(\omega(T))$.*

Proof. If T is hyponormal, then it follows from [13, Theorem 2.2].
Suppose that T^* is hyponormal and $p(t)$ is any polynomial. Let

$$p(T) - \lambda I = a_0(T - \mu_1 I)\cdots(T - \mu_n I).$$

Since T^* is hyponormal, $T^* - \overline{\mu}_i I$ are commuting hyponormal operators for each $i = 1, 2, \cdots, n$. It thus follows from Lemma 4 and $\omega(T^*) = \omega(T)^*$ that

$$\begin{aligned}
\lambda \notin \omega(p(T)) &\Longleftrightarrow p(T) - \lambda I = \text{Weyl} \\
&\Longleftrightarrow a_0(T - \mu_1 I)\cdots(T - \mu_n I) = \text{Weyl} \\
&\Longleftrightarrow \overline{a}_0(T^* - \overline{\mu}_1 I)\cdots(T^* - \overline{\mu}_n I) = \text{Weyl} \\
&\Longleftrightarrow T^* - \overline{\mu}_i I = \text{Weyl for each } i = 1, 2, \cdots, n \\
&\Longleftrightarrow T - \mu_i I = \text{Weyl for each } i = 1, 2, \cdots, n \\
&\Longleftrightarrow \mu_i \notin \omega(T) \text{ for each } i = 1, 2, \cdots, n \\
&\Longleftrightarrow \lambda \notin p(\omega(T))
\end{aligned}$$

which says that

$$\omega(p(T)) = p(\omega(T)). \tag{1.3}$$

Next suppose r is any rational function with no poles in $\sigma(T)$. Write $r = p/q$, where p and q are polynomials and q has no zeros in $\sigma(T)$. Then

$$r(T) - \lambda I = (p - \lambda q)(T)(q(T))^{-1}.$$

By (1.3),

$$(p - \lambda q)(T) \text{ Weyl} \Longleftrightarrow p - \lambda q \text{ has no zeros in } \omega(T).$$

Thus we have

$$\begin{aligned}
\lambda \notin \omega(r(T)) &\Longleftrightarrow (p - \lambda q)(T) = \text{Weyl} \\
&\Longleftrightarrow p - \lambda q \text{ has no zeros in } \omega(T) \\
&\Longleftrightarrow ((p - \lambda q)(x))q(x)^{-1} \neq 0 \text{ for any } x \in \omega(T) \\
&\Longleftrightarrow \lambda \notin r(\omega(T))
\end{aligned}$$

which says that $\omega(r(T)) = r(\omega(T))$. If f is analytic on a neighborhood of $\sigma(T)$, then by Runge's theorem ([4]), there is a sequence $\{r_n(t)\}$ of rational functions with no poles in $\sigma(T)$ such that $\{r_n\}$ converges to f uniformly on a neighborhood of $\sigma(T)$. Since $\{r_n(T)\}$ converges to $f(T)$ and each $r_n(T)$ commutes with $f(T)$, by [7]

$$f(\omega(T)) = \lim r_n(\omega(T)) = \lim \omega(r_n(T)) = \omega(f(T)).$$

An operator T is said to be *polynomially compact* ([2]) if there exists a polynomial p such that $p(T)$ is compact. Thus we see that T is polynomially compact if and only if T^* is polynomially compact. From Theorem 5 and [2, Corollary 6.6], we can obtain the following result:

THEOREM 6. *If T is seminormal and satisfies condition (i), then T is normal $(i = 1, 2, 3)$.*

(1) *T is polynomially compact.*
(2) *There exists an analytic function f on $\sigma(T)$ such that $f(T)$ is compact and f has finitely many zeros on $\omega(T)$.*
(3) *$\omega(T)$ is finite.*

We say that *Weyl's theorem holds for T* if

$$\omega(T) = \sigma(T) - \pi_{00}(T).$$

There are several classes of operators including normal, hyponormal, and seminormal operators on a Hilbert space for which Weyl's theorem holds. Also it was shown in [8] that Weyl's theorem holds for any spectral operator of finite type on a Banach space. Oberai has raised the following question: Does there exist a hyponormal operator T such that Weyl's theorem does not hold for T^2? Note that T^2 may not be hyponormal even if T is hyponormal ([5, Problem 209]). We will show that Weyl's theorem holds for $p(T)$ when T is a seminormal operator and p is a polynomial. Thus we answer the old question of Oberai since every hyponormal operator is seminormal.

Recall ([8]) that $T \in B(H)$ is said to be *isoloid* if iso $\sigma(T) \subset \pi_0(T)$.

LEMMA 7([8]). *If $T \in B(H)$ is isoloid and f is analytic on a neighborhood of $\sigma(T)$, then $f(\sigma(T) - \pi_{00}(T)) = \sigma(f(T)) - \pi_{00}(f(T))$.*

THEOREM 8. *If $T \in B(H)$ is seminormal, then for any analytic function f on a neighborhood of $\sigma(T)$, Weyl's theorem holds for $f(T)$.*

Proof. By [7] and [12], every seminormal operator T is isoloid and Weyl's theorem holds for any seminormal operator T. Hence by Theorem 5 and Lemma 7,

$$\omega(f(T)) = f(\omega(T)) = f(\sigma(T) - \pi_{00}(T)) = \sigma(f(T)) - \pi_{00}(f(T)).$$

Therefore Weyl's theorem holds for $f(T)$.

Since every hyponormal operator is seminormal, we obtain the following result which answers to the old question of Oberai.

COROLLARY 9. *If $T \in B(H)$ is hyponormal, then for any polynomial $p(t)$ Weyl's theorem holds for $p(T)$.*

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