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OPTIMAL CONTROL FOR RETARDED CONTROL SYSTEM

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1. Introduction

In this paper we deal with the control problem for retarded functional differential equation:

$$\frac{d}{dt}x(t) = A_0x(t) + A_1x(t-h) + \int_{-h}^{0} a(s)A_2x(t+s)ds + B_0u(t), \tag{1.1}$$

$$x(0) = g^0, \quad x(s) = g^1(s), \quad s \in [-h, 0) \tag{1.2}$$

in the Hilbert space H. After we consider the regularity of solution of the retarded system, we proceed to necessary optimality condition of the optimal solution for given cost function J in set of a admissible controls that is a closed and convex.

As for the regularity of solution we reduce the results of G. Blasio, K. Kunlsch and A. Sinestrari [2] regarding term by term. There exists a many literatures which studies optimal control problems of control systems in Banach spaces. However, most studies have been devoted to the systems without delay and the papers treating the retarded system with unbounded operators are not so many([cf. see [3.8] in case where with bounded operators).

In section 2, we consider some basic results on existence, uniqueness, and a representation formular functional differential equations in Hilbert spaces. We establish a form of a mild solution which is

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described by the integral equation in terms of fundamental solution using structural operator. In section 3,4, we shall give two forms of quadratic cost functions, which are called the quadratic cost criteria in linear dynamic system and the feedback control law for regulator problem. First we consider results on the existence and uniqueness of optimal control in the closed convex admissable set. So we present the necessary conditions of optimality which are described by the adjoint state and integral inequality. Maximal principle and bang-bang principle for technologically important costs are also given.

2. Functional differential equation with time delay

Let V and H be two Hilbert spaces The norm on V(resp. H) will be denoted by $||\cdot||$ (resp. $|\cdot|$) and the corresponding scalar products will be denoted by $((\cdot,\cdot))$(resp. $(\cdot,\cdot)$). Assume $V \subset H$, the injection of V into H is continuous and V is dense in H. H will be identified with its dual space. If V^* denotes the dual space, H may be identified with a subspace of V^* and may write $V \subset H \subset V^*$. Since V is dense in H and H is dense in V^* and the corresponding injections are continuous. If an operator A_0 is bounded linear operator from V to V^* and generates an analytic semigroup, then it is easily seen that

$$H = \{x \in V^* : \int_0^T ||A_0 e^{tA_0} x||_*^2 dt < \infty\}, \tag{2.1}$$

for the time $T > 0$ where $||\cdot||_*$ is the norm of the element of V^*. The realization of A_0 in H which is the restriction of A_0 to

$$D(A_0) = \{u \in V : A_0 u \in H\}$$

is also denoted by A_0. Therefore, in terms of the intermediate theory we can see that

$$(V, V^*)_{\frac{1}{2},2} = H \tag{2.2}$$

and hence we can also replace the intermediate space F in the paper [2] with the space H. Hence, from now on we derive the same results of G. Blasio, K. Kunisch and A. Sinestrari [2]. Let $a(u,v)$ be a bounded sesquilinear form defined in $V \times V$ satisfying Gårding's inequality

$$\text{Re } a(u,v) \geq c_0||u||^2 - c_1|c|^2, \quad c_0 > 0, \quad c_1 \geq 0.$$

Let A_0 be the operator associated with a sesquilinear form

$$(A_0 u, v) = -a(u, v), \quad u,\ v \in V.$$

Then A_0 generates an analytic semigroup in both H and V^* and so the equation (1.1) and (2.2) may be considered as an equation in both H and V^*:

Let the operators A_1 and A_2 be a bounded linear operators from V to V^*. The function $a(\cdot)$ is assume to be a real valued Hölder continous in $[-h, 0]$ and the controller operator B_0 is a bounded linear operator from some Banach space U to H. Under these conditions, from (2.2) Theorem 3.3 of [2] we can obtain following result.

PROPOSITION 2.1. *Let $g = (g^0, g^1) \in Z = H \times L^2(-h, 0; V)$ and $u \in L^2(0, T; U)$. Then for each $T > 0$, a solution x of the equation (1.1) and (1.2) belongs to*

$$L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H).$$

According to S. Nakagiri [7], we define the fundamental solution $W(t)$ for (1.1) and (1.2) by

$$W(t)g^0 = \begin{cases} x(t; 0, (g^0, 0)), & t \geq 0 \\ 0 & t < 0 \end{cases}$$

for $g^0 \in H$. Since we assume that $a(\cdot)$ is Hölder continuous the fundamental solution exists as seen in [11]. It is known that $W(t)$ is strongly continuous and $A_0 W(t)$ and $dW(t)/dt$ are strongly continuous except at $t = nr$, $n = 0,\ 1,\ 2,\ \ldots$.

For each $t > 0$, we introduce the structual operator $F(\cdot)$ from $H \times L^2(0, T; V)$ to $H \times L^2(0, T; V^*)$ defined by

$$\begin{aligned} &Fg = ([Fg]^0, [Fg]^1), \\ &[Fg]^0 = g^0, \\ &[Fg]^1 = A_1 g^1(-h - s) + \int_{-h}^{s} a(\tau) A_2 g^1(\tau - s) d\tau \end{aligned}$$

for $g = (g^0, g^1) \in H \times L^2(0,T;V)$. The solution $x(t) = x(t; g, u)$ of (1.1) and (1.2) is represented by

$$x(t) = W(t)g^0 + \int_{-h}^{0} W(t+s)[Fg]^1(s)ds + \int_{0}^{t} W(t-s)B_0u(s)ds$$

for $t \geq 0$.

Let $I = [0,T]$, $T > 0$ be a finite interval. We introduce the transposed system which is exactly same as in S. Nakagiri[8]. Let $q_0^* \in X^*$, $q_1^* \in L^1(I;H)$. The retarded transposed system in H is defined by

(2.3)

$$\frac{dy(t)}{dt} + A_0^*y(t) + A_1^*y(t+h) + \int_{-h}^{0} a(s)A_2y(t-s)ds$$
$$+ q_1^*(t) = 0 \qquad \text{a.e.} \quad t \in I,$$

(2.4)

$$y(T) = q_0^*, \quad y(s) = 0 \quad \text{a.e. } s \in (T, T+h].$$

Let $W^*(t)$ denote the adjoint of $W(t)$. Then as proved in S. Nakagiri [8], the mild solution of (2.3) and (2.4) is defined as follows:

$$y(t) = W^*(T-t)(q_0^*) + \int_{t}^{T} W^*(\xi - t)q_1^*(\xi)d\xi,$$

for $t \in I$ in the weak sence. The tranposed system is used to present a concrete form of the optimality conditions for control optimization problems.

3. Optimality contion for quadratic cost function

With every control $u \in L^2(0,T;U)$ we associate the following cost function:

$$J(u) = \int_{0}^{T} ||Cx_u(t) - z_d(t)||_X^2 dt + \int_{0}^{T} (Nu(t), u(t))dt$$

where the operator C is a bounded from H to another Hilbert space X and $z_d \in L^2(I;X)$. Finally we are given N is a self adjoint and positive definite:

$$N \in B(X), \quad \text{and} \quad (Nu, u) \geq c||u||, \quad c > 0,$$

where $B(X)$ denotes the space of bounded operators on X. Let $x_u(t)$ stands for solution of (1.1) and (1.2) associated with the control $u \in L^2(0,T;U)$. Let U_{ad} be a closed convex subset of $L^2(0,T;U)$.

THEOREM 3.1. *Let the operators C and N satisfy the conditions mentioned above. then there exists a unique element $u \in U_{ad}$ such that*

$$J(u) = \inf_{v \in U_{ad}} J(v). \tag{3.1}$$

Furthermore, it is holds the following inequality:

$$\int_0^T (-\Lambda_U^{-1} B_0^* y(s) + Nu(s), v(s) - u(s))ds \geq 0$$

where $y(t)$ is a solution of (2.3) and (2.4) for initial condition $y(s) = 0$ for $s \in [T, T+h]$ substituting q_1^* by $-C^*\Lambda_X(Cx_u(t) - z_d)$. That is, $y(t)$ satisfies the following transposed system:

(3.2)

$$\frac{dy(t)}{dt} + A_0^* y(t) + A_1^* y(t+h) + \int_{-h}^0 a(s) A_2 y(t-s)ds$$
$$- C^*\Lambda_X(Cx_u(t) - z_d) = 0 \quad \text{a.e.} \quad t \in I,$$

(3.3)

$$y(T) = 0, \quad y(s) = 0 \quad \text{a.e. } s \in (T, T+h]$$

in the weak sense. Here, the operator Λ_U(resp. Λ_X) is the canonical isomorphism of U(resp. X) onto U^*(resp. X^*).

Proof. Let $x(t) = x(t; g, 0)$. Then it holds that

$$\begin{aligned}
J(v) &= \int_0^T ||Cx_v(t) - z_d(t)||^2 dt + \int_0^T (Nv(t), v(t))dt \\
&= \int_0^T ||C(x_v(t) - x(t)) + Cx(t) - z_d(t)||^2 dt + \int_0^T (Nv(t), v(t))dt \\
&= \pi(v, v) - 2L(v) + \int_0^T ||z_d(t) - Cx(t)||^2 dt
\end{aligned}$$

where

$$\begin{aligned}
\pi(u, v) = &\int_0^T (C(x_u(t) - x(t)), C(x_v(t) - x(t)))dt \\
&+ \int_0^T (Nu(t), v(t))dt
\end{aligned}$$

$$L(v) = \int_0^T (z_d(t) - Cx(t), C(x_v(t) - x(t)))dt.$$

The form $\pi(u,v)$ is a continuous bilinear form in $L^2(0,T;U)$ and from assumption of the positive definite of the operator N we have

$$\pi(v,v) \geq c||v||^2 \quad v \in L^2(0,T;U).$$

Therefore in virtue of Theorem 1.1 of Chapter 1 in [6] there exists a unique $u \in L^2(0,T;U)$ such that (3.1). If u is an optimal control (cf. Theorem 1.3. Chapter 1 in [6]), then

$$J'(u)(v-u) \geq 0 \quad u \in U_{ad}, \tag{3.4}$$

where $J'(u)v$ means thr Fréchet derivative of J at u, applied to v. It is easily seen that

$$\begin{aligned} x'_u(t)(v-u) &= (v-u, x'_u(t)) \\ &= x_v(t) - x_u(t). \end{aligned}$$

Therefore, (3.4) is equivalent to

$$\begin{aligned} &\int_0^T (Cx_u(t) - z_d(t), C(x_v(t) - x_u(t)))dt + \int_0^T (Nu, v-u)dt = \\ &\int_0^T (C^*\Lambda_X(Cx_u(t) - z_d(t), x_v(t) - x_u(t))dt + \int_0^T (Nu, v-u)dt \\ &\geq 0 \end{aligned}$$

Note that $C^* \in B(X^*, H^*)$ and for ϕ and ψ in H we have $(C^*\Lambda_X C\psi, \phi)$ $= (C\psi, C\phi)$ where duallity pairing is also denoted by $(\cdot,\cdot)$. From Fubini's theorem and

$$x_u(t) - x_v(t) = \int_0^t W(t-s)B_0(v(s) - u(s))ds$$

we have

$$\int_0^T \int_0^t ((\Lambda_U^{-1} B_0^* W^*(t-s) C^* \Lambda_X (Cx_u(t) - z_d(t)) + Nu(s),$$

$$v(s) - u(s)))ds\,dt$$

$$= \int_0^T (\int_s^T ((\Lambda_U^{-1} B_0^* W^*(t-s) C^* \Lambda_X (Cx_u(t) - z_d(t)))dt + Nu(s),$$

$$v(s) - u(s)))ds$$

$$= \int_0^T (-\Lambda_U{}^{-1} B_0^* y(s) + Nu(s), v(s) - u(s)))ds$$

$$\geq 0$$

where $y(s)$ is given by (3.2) and (3.3), that is, $y(s)$ is following form:

$$y(s) = -\int_s^T W^*(t-s) C^* \Lambda_X (Cx_u(t) - z_d(t))dt.$$

REMARK. Identifying the antidual U with U (and also in case X) we need not use the canonical isomorphism Λ_U. But in case where $U \subset V^*$ this leads to difficulties since H has already been identified with its dual.

COROLLARY 3.1 (MAXIMAL PRINCIPLE). *Let U_{ad} be bounded and $N = 0$. If u be an optimal solution for J then*

$$\max_{v \in U_{ad}} (v, \Lambda_U^{-1} B_0^* y(s)) = (u, \Lambda_U^{-1} B_0^*(s) y(s))$$

where $y(s)$ is given by in Theorem 3.1.

Proof. We note that if U_{ad} is bounded then the set of elements $u \in U_{ad}$ such that (3.1) is a nonempty, closed and convex set in U_{ad}. Let t be a Lebesque point of u, $v \in U_{ad}$ and $t < t + \epsilon < T$. Further, put

$$v_\epsilon(s) = \begin{cases} v, & \text{if } \ t < s < t+\epsilon \\ u(s), & \text{otherwise.} \end{cases}$$

Then Substituting v_ϵ for v in (3.4) and deviding thr resulting inequality by ϵ, we obtain

$$\frac{1}{\epsilon}\int_t^{t+\epsilon}(-\Lambda_U^{-1}B_0^*y(s), v(s)-u(s))ds \geq 0.$$

Thus by letting $\epsilon \to 0$, the proof is complete.

THEOREM 3.2 (BANG-BANG PRINCIPLE). *Let U_{ad} be bounded and $N=0$. Let B_0^* and C be one to one mappings. If there is not the control u such that $Cx_u(t) = z_d(t)$ a.e, then the optimal control $u(t)$ is a bang-bang control, i.e, $u(t)$ satisfies $u(t) \in \partial U_{ad}$ for almost all t where ∂U_{ad} denotes the boundary of U_{ad}.*

Proof. On account of Corollary 3.1 it is enough to show that $\Lambda_U^{-1}B_0^*(t)y(t) \neq 0$ for almost all t. If $B_0^*(t)y(t) = 0$, then since

$$y(s) = -\int_s^T W^*(t-s)C^*\Lambda_X(Cx_u(t)-z_d(t))dt,$$

it follows that

$$Cx_u(t)-z_d(t) = 0 \quad a.e..$$

It is a contraction.

4. Optimality condition for regular cost function

In this section, the optimal control problem is to find a control u which minimizes the cost function

$$J(u) = (Gx(T), x(T))_H + \int_0^T((D(t)x(t), x(t))_H + (Q(t)u(t), U(t))_U)dt$$

where $x(\cdot)$ is a solution of (1.1) and (1.2), $G \in B(H)$ is self adjoint and nonnegative, and $D \in \mathcal{B}_\infty(0,T;H,H)$ which is a set of all essentially bounded operators on $(0,T)$ and $Q \in \mathcal{B}_\infty(0,T;U,U)$ are self adjoint and ninnegative, with $Q(t) \geq m$ for some $m > 0$, for almost all t.

THEOREM 4.1. *Let U_{ad} be closed convex in $L^2(0,T;U)$. Then there exists a unique element $u \in U_{ad}$ such that*

$$J(u) = \inf_{v \in U_{ad}} J(v). \tag{4.1}$$

Moreover, it is holds the following inequality:

$$\int_0^T (B_0^* y(s) + Qu(s), v(s) - u(s))ds \geq 0$$

where $y(t)$ is a solution of (2.3) and (2.4) for initial condition that $y(T) = Gx_u(T)$ and $y(s) = 0$ for $s \in (T, T+h]$ substituting $q_1^*(t)$ by $D(t)x_u(t)$. That is, $y(t)$ satisfies the following transposed system:

(4.2)

$$\frac{dy(t)}{dt} + A_0^* y(t) + A_1^* y(t+h) + \int_{-h}^0 a(s) A_2 y(t-s)ds$$
$$+ D(t)(x_v(t) - x_u(t)) = 0 \quad \text{a.e.} \quad t \in I,$$

(4.3)

$$y(T) = Gx_u(T), \quad y(s) = 0 \quad \text{a.e. } s \in (T, T+h]$$

in the weak sense.

Proof. Under the hypotheses on G, D, and Q, there exists a unique u which minimizes J. Then $J'(u)(v-u) \geq 0$. Since

$$\begin{aligned} J'(u)(v-u) = {} & 2(Gx_u(T), x_v(T) - x_u(T)) \\ & + 2\int_0^T (D(t)x_u(t), x_v(t) - x_u(t)) \\ & + 2(Q(t)u(t), v(t) - u(t))dt, \end{aligned}$$

(4.1) is equivalent to that

$$\begin{aligned} & \int_0^T (B_0^* W^*(T-s)(Gx_u(T), v(s) - v(s))ds + \\ & \int_0^T (B_0^* \int_s^T W^*(t-s)D(t)x_u(t)dt + Qu(s), v(s) - u(s))ds \\ & \geq 0. \end{aligned}$$

Hence

$$y(s) = W^*(T-s)Gx_u(T) + \int_s^T W^*(t-s)D(t)x_u(t)dt$$

is solves (4.2) and (4.3).

From now on, we consider the case where $U_{ad} = L^2(0,T;U)$. Let $x_u(t) = x(t;g,0) + \int_0^t W(t-s)B_0u(s)ds$ be solution of (1.1) and (1.2). Define $T \in B(H, L^2(0,T;H))$ and $T_T \in B(L^2(0,T;H),H)$ by

$$(T\phi)(t) = \int_0^t W(t-s)\phi(s)ds,$$
$$T_T\phi = \int_0^T W(T-s)\phi(s)ds.$$

Then we can write the cost function as

(4.4)
$$\begin{aligned} J(u) =&(G(x(T;g,0)+T_TB_0u),(x(T;g,0)+T_TB_0u))_H \\ &+ (D(x(\cdot;g,0)+TB_0u), x(\cdot;g,0)+TB_0u)_{L^2(0,T;H)} \\ &+ (Qu,u)_{L^2(0,T;U)}. \end{aligned}$$

The adjoint oprators T^* and T_T^* are given by

$$(T^*\phi)(t) = \int_t^T W^*(s-t)\phi(s)ds,$$
$$(T_T^*\phi)(t) = W^*(T-t)\phi.$$

THEOREM 4.2. *Let $U_{ad} = L^2(0,T;U)$. Then there exists a unique control u such that (4.1) and*

$$u(t) = -A^{-1}B_0^*y(t)$$

for almost all t, where $A = Q + B_0^*T^*DTB_0 + B_0^*T_T^*GT_T^*B_0$ and where $y(t)$ is a solution of (2.3) and (2.4) for initial condition that $y(T) = Gx(T)$ and $y(s) = 0$ for $s \in (T, T+h]$ substituting $q_1^*(t)$ by $Dx(t)$.

Proof. The optimal control for J is unique solution of

(4.5) $$J'(u)v = 0.$$

From (4.4) we have

$$\begin{aligned} J'(u)v =& 2(G(x(T;g,0)+T_T B_0 u), T_T B_0 v)) \\ &+2(D(x(\cdot;g,0)+TB_0u), TB_0v) \\ &+2(Qu,v) \\ =&2((Q+B_0^*T^*DTB_0^*+BT_T^*GT_TB_0)u,v) \\ &+2(B_0^*T^*Dx(\cdot;g,0)+B_0^*T_T^*Gx(T;g,0),v). \end{aligned}$$

Hence (4.5) is equivalent to that

$$((A+B_0^*T^*Dx(t;g,0)+B_0^*T_T^*Gx(T;g,0))u,v)=0$$

since $A^{-1} \in \mathcal{B}_\infty(0,T;H,U)$ (see Appendix of [3]). Hence from The definitions of T and T_T it follows that

$$y(t)=W^*(T-t)Gx(T)+\int_t^T W^*(s-t)Dx(t)ds.$$

Therefore, the proof is complete.

REMARK. For the cost function J in section 4 we can also obtain the pointwise maximal principle and bang -bang principle.

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