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CONTROLLABILITY OF NONLINEAR FUNCTIONAL INTEGRO-DIFFERENTIAL SYSTEMS IN BANACH SPACE

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1. Introduction.

Controllability of linear and nonlinear systems represented by ordinary differential equations in finite-dimension space has been extensively studied. Several authors have extended the concept to infinite-dimension systems represented by evolution equations with bounded operators in Banach spaces(Ref.[4]) for Volterra integrodifferential systems, I and Kwun(Ref.[3]) studied the approximate controllability for delay Volterra systems with bounded linear operators in Banach space. The purpose of this paper is to study the controllability of abstract functional integrodifferential systems in Banach space by using the Schauder fixed point theorem and we give an example. The abstract functional integrodifferential equations are arised many physical phenomena.

2. Preliminaries.

Let X be a Banach space with norm $\|\cdot\|$ and let $C = C([-r,0],X)$ be the Banach space of continuous functions defined on $[-r,0], r > 0$ with supermum norm $\|\cdot\|_C$. If x is continuous function from $[-r,T], T > 0$ to X and $t \in [0,T] = J$, then x_t denotes the element of C given by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r,0]$. We consider the following abstract functional integrodifferential equation

$$\tag{1} \begin{aligned} \frac{d}{dt}x(t) + Ax(t) =& (Bu)(t) + \int_0^t [a(t,s)g(s,x_s) + h(t,s,x_s)]ds \\ &+ f(t,x_t), \quad t \in [0,T] = J \\ x(t) =& \phi(t), \quad -r \le t \le 0. \end{aligned}$$

where the state $x(\cdot)$ takes values in the Banach space X and the control function $u(\cdot)$ is given in $L^2(J;U)$, a Banach space of admissable control functions, with U a Banach space.
Here $-A$ is an infinitesimal generator of a strongly continuous semigroup $S(t), t \ge 0$ on X, and B is a bounded linear operator from U into X. The nonlinear functions $g : J \times C \to X$, $h : J \times J \times C \to X$, $f : J \times C \to X$ and the kernel $a : J \times J \to R$ (R denotes the set of real numbers) are continuous.

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We list the following hypotheses;

(i) A is the infinitesimal generator of a strongly continuous semigroup $S(t), t \geq 0$ satisfying $\|S(t)\| \leq M_1$.

(ii) The linear operator W from U into X, defined by

$$Wu = \int_0^T S(T-s)Bu(s)ds,$$

has an invertible operator W^{-1} defined on $L^2(J;U)/\ker W$, and there exists positive constants M_2, M_3 such that $\|B\| \leq M_2, \|W^{-1}\| \leq M_3$.

(iii) Let $b_1, b_3 : J \to R^+, b_2 : J \times J \to R^+$ be continuous functions such that

$$\begin{aligned}
&\|g(t,\phi) - g(t,\psi)\| \leq b_1(t)\|\phi - \psi\|_C, \\
&\|h(t,s,\phi) - h(t,s,\psi)\| \leq b_2(t,s)\|\phi - \psi\|_C, \\
&\|f(t,\phi) - f(t,\psi)\| \leq b_3(t)\|\phi - \psi\|_C,
\end{aligned}$$

where $\|g(t,0)\| = \|h(t,s,0)\| = \|f(t,0)\| = 0 \quad \text{for} \quad t,s \in J, \quad \phi,\psi \in C.$

(iv) The function $a(t,s)$ is Hölder continuous with exponent α, i.e., there exists a positive constant a_0 such that

$$|a(t_1,s_1) - a(t_2,s_2)| \leq a_0(|t_1 - t_2|^\alpha + |s_1 - s_2|^\alpha)$$

for $t_1, t_2, s_1, s_2 \in J, 0 < \alpha \leq 1$.

Then, for the system (1), there exists a mild solution of the following form(Ref.[2]):

$$\begin{aligned}
x(t) =& S(t)\phi(0) + \int_0^t S(t-s)Bu(s)ds \\
&+ \int_0^t S(t-s)\left\{\int_0^s [a(s,\tau)g(\tau,x_\tau) + h(s,\tau,x_\tau)]d\tau \right. \\
&\left. + f(s,x_s)\right\}ds, \quad t \in [0,T] = J \qquad (2) \\
x(t) =& \phi(t), \quad -r \leq t \leq 0.
\end{aligned}$$

Definition 2.1. The system (1) is said to be controllable on the interval J if, for every continuous initial function $\phi \in C$ and $x_1 \in X$, there exists a control $u \in L^2(J;U)$ such that the solution $x(t)$ of (1) satisfies $x(T) = x_1$.

3. Main Result.

Theorem 3.1. *If the hypotheses (i)~(iv) are satisfied, then the system (1) is controllable on J.*

Proof. Using the hypothsis (ii), define the control

$$\begin{aligned}
u(t) =& W^{-1}[x_1 - S(T)\phi(0) \\
&- \int_0^T S(T-s)\left\{\int_0^s [a(s,\tau)g(\tau,x_\tau) + h(s,\tau,x_\tau)]d\tau + f(s,x_s)\right\}ds](t).
\end{aligned}$$

Now, it is shown that, when using this control, the operator defined by

$$
\begin{aligned}
(\Phi x)(t) =& S(t)\phi(0) + \int_0^t S(t-\eta)BW^{-1}[x_1 - S(T)\phi(0) \\
& - \int_0^T S(T-s)\left\{\int_0^s [a(s,\tau)g(\tau,x_\tau) + h(s,\tau,x_\tau)]d\tau \ + f(s,x_s)\right\}ds](\eta)d\eta \\
& + \int_0^t S(t-s)\left\{\int_0^s [a(s,\tau)g(\tau,x_\tau) + h(s,\tau,x_\tau)]d\tau \ + f(s,x_s)\right\}ds \\
(\Phi x)(t) =& \phi(t), \quad t \in [-r,0]
\end{aligned} \tag{3}
$$

has a fixed point. This fixed point is then a solution of Equation (2). Clearly $(\Phi x)(T) = x_1$, which means that control u steers the abstract functional integrodifferential system from the initial function ϕ to x_1 in time T, provided we can obtain a fixed point of nonlinear operator Φ. Define the function $\phi' \in C([-r,T];X)$ by

$$
\begin{aligned}
&\phi'_0 = \phi \\
&\phi'(t) = S(t)\phi(0), \quad t \in J
\end{aligned}
$$

and

$$X_0 = \{x \in C([-r,T];X) : x_0 = 0, \|x\|_C \le d, 0 \le t \le T\},$$

where the positive constant d is given by
$d = M_1M_2M_3[\|x_1\| + M_1NK_1T^2 + M_1K_2T + K_3]T + M_1[NK_1T + K_2T + K_3]T$.
Then, X_0 is bounded, closed convex subset of $C([-r,T];X)$. Consider the transformation

$$Y : X_0 \to C([-r,T];X)$$

defined by

$$
\begin{aligned}
(Yx)_0 =& 0, \\
(Yx)(t) =& \int_0^t S(t-\eta)BW^{-1}[x_1 - \int_0^T S(T-s)\left\{\int_0^s [a(s,\tau)g(\tau,\phi'_\tau + x_\tau)\right. \\
& \left. + h(s,\tau,\phi'_\tau + x_\tau)]d\tau + f(s,\phi'_s + x_s)\right\}ds](\eta)d\eta \\
& + \int_0^t S(t-s)\left\{\int_0^s [a(s,\tau)g(\tau,\phi'_\tau + x_\tau) + h(s,\tau,\phi'_\tau + x_\tau)]d\tau \right. \\
& \left. + f(s,\phi'_s + x_s)\right\}ds, \quad t \in J.
\end{aligned} \tag{4}
$$

Finding a fixed point of Y, and thus proving the theorem, is equivalent to finding a fixed point of Φ, and hence the solution (2) for the system (1). For that, if x is a fixed point of Y, then we can define

$$v_t = \phi'_t + x_t, \quad t \in J.$$

Then

$$
\begin{aligned}
v_0 =&\phi, \\
v(t) =&S(t)\phi(0) + \int_0^t S(t-\eta)BW^{-1}[x_1 - S(T)\phi(0) \\
&- \int_0^T S(T-s)\Big\{\int_0^s [a(s,\tau)g(\tau,v_\tau) + h(s,\tau,v_\tau)]d\tau + f(s,v_s)\Big\}ds](\eta)d\eta \\
&+ \int_0^t S(t-s)\Big\{\int_0^s [a(s,\tau)g(\tau,v_\tau) + h(s,\tau,v_\tau)]d\tau + f(s,v_s)\Big\}ds, \quad t \in J.
\end{aligned}
$$

We claim that $Y : X_0 \to X_0$. It is easy to observe from hypothesis (iv) that there exists a constant N such that $|a(t,s)| \le N$, $t, s \in J$. By virtue of hypothesis (iii) and the continuity of function g, h, f, there exist constants $K_i \ge 0, i = 1,2,3$ such that $\|g(\tau, \phi'_\tau + x_\tau)\| \le K_1$, $\|h(s,\tau,\phi'_\tau + x_\tau)\| \le K_2$ and $\|f(s,\phi'_s + x_s)\| \le K_3$, for $0 \le \tau \le s \le T$, $t \in J$ and $x \in X_0$. Thus, we have

$$
\begin{aligned}
\|(Yx)(t)\| \le& \Big\| \int_0^t S(t-\eta)BW^{-1}[x_1 - \int_0^T S(T-s)\Big\{\int_0^s [a(s,\tau)g(\tau,\phi'_\tau + x_\tau) \\
&+ h(s,\tau,\phi'_\tau + x_\tau)]d\tau + f(s,\phi'_s + x_s)\Big\}ds](\eta)d\eta \Big\| \\
&+ \Big\| \int_0^t S(t-s)\Big\{\int_0^s [a(s,\tau)g(\tau,\phi'_\tau + x_\tau) + h(s,\tau,\phi'_\tau + x_\tau)]d\tau \\
&+ f(s,\phi'_s + x_s)\Big\}ds \Big\| \\
&\le M_1M_2M_3[\|x_1\| + M_1NK_1T^2 + M_1K_2T^2 + M_1K_3T]T \\
&+ M_1[NK_1T + K_2T + K_3]T = d,
\end{aligned}
$$

which implies that

$$\|(Yx)_t\|_C \le d.$$

It follows that Y is also continuous and maps X_0 into itself. Moreover, Y maps X_0 into a precompact subset of X_0. To prove this, we first show that, for every fixed $t \in J$, the set

$$X_0(t) = \{(Yx)(t) : x \in X_0\}$$

is precompact in X. This is clear for $t = 0$, since $X_0(0) = \{0\}$. Let $t > 0$ be fixed and for $o < \varepsilon < t$ define

$$
\begin{aligned}
(Y_\varepsilon x)(t) =& \int_0^{t-\varepsilon} S(t-\eta)BW^{-1}[x_1 - \int_0^T S(T-s)\Big\{\int_0^s [a(s,\tau)g(\tau,\phi'_\tau + x_\tau) \\
&+ h(s,\tau,\phi'_\tau + x_\tau)]d\tau + f(s,\phi'_s + x_s)\Big\}ds](\eta)d\eta \\
&+ \int_0^{t-\varepsilon} S(t-s)\Big\{\int_0^s [a(s,\tau)g(\tau,\phi'_\tau + x_\tau) + h(s,\tau,\phi'_\tau + x_\tau)]d\tau \\
&+ f(s,\phi'_s + x_s)\Big\}ds
\end{aligned}
$$

Since $S(t)$ is compact for every $t > 0$, the set

$$X_\varepsilon(t) = \{(Y_\varepsilon x)(t) : x \in X_0\}$$

is precompact in X for every $\varepsilon, 0 < \varepsilon < t$. Furthermore, for $x \in X_0$, we have

$$\begin{aligned}
&\|(Yx)(t) - (Y_\varepsilon x)(t)\| \\
&\leq \| \int_{t-\varepsilon}^{t} S(t-\eta) B W^{-1}[x_1 - \int_0^T S(T-s) \Big\{ \int_0^s [a(s,\tau) g(\tau, \phi'_\tau + x_\tau) \\
&\quad + h(s,\tau,\phi'_\tau + x_\tau)] d\tau + f(s, \phi'_s + x_s) \Big\} ds](\eta) d\eta \| \\
&\quad + \| \int_{t-\varepsilon}^{t} S(t-s) \Big\{ \int_0^s [a(s,\tau) g(\tau, \phi'_\tau + x_\tau) + h(s,\tau,\phi'_\tau + x_\tau)] d\tau \\
&\quad + f(s, \phi'_s + x_s) \Big\} ds \| \\
&\leq \varepsilon M_1 M_2 M_3 [\|x_1\| + M_1 N K_1 T^2 + M_1 K_2 T^2 + M_1 K_3 T] \\
&\quad + M_1 \varepsilon [N K_1 T + K_2 T + K_3]
\end{aligned}$$

which implies that $X_0(t)$ is totally bounded, that is, precompact in X. We want to show that

$$Y(X_0) = \{Yx : x \in X_0\}$$

is an equicontinuous family of functions. For that, let $t_2 > t_1 > 0$. Then, we have

$$\begin{aligned}
&\|(Yx)(t_1) - (Yx)(t_2)\| \\
&\leq \| \int_0^{t_1} [S(t_1-\eta) - S(t_2-\eta)] B W^{-1}[x_1 - \int_0^T S(T-s) \\
&\quad \cdot \Big\{ \int_0^s [a(s,\tau) g(\tau, \phi'_\tau + x_\tau) + h(s,\tau,\phi'_\tau + x_\tau)] d\tau + f(s, \phi'_s + x_s) \Big\} ds](\eta) d\eta \\
&\quad - \int_{t_1}^{t_2} S(t_2-\eta) B W^{-1}[x_1 - \int_0^T S(T-s) \Big\{ \int_0^s [a(s,\tau) g(\tau, \phi'_\tau + x_\tau) \\
&\quad + h(s,\tau,\phi'_\tau + x_\tau)] d\tau + f(s, \phi'_s + x_s) \Big\} ds](\eta) d\eta \| \\
&\quad + \| \int_0^{t_1} [S(t_1-s) - S(t_2-s)] \qquad (5) \\
&\quad \cdot \Big\{ \int_0^s [a(s,\tau) g(\tau, \phi'_\tau + x_\tau) + h(s,\tau,\phi'_\tau + x_\tau)] d\tau + f(s, \phi'_s + x_s) \Big\} ds \\
&\quad - \int_{t_1}^{t_2} S(t_2-s) \Big\{ \int_0^s [a(s,\tau) g(\tau, \phi'_\tau + x_\tau) + h(s,\tau,\phi'_\tau + x_\tau)] d\tau \\
&\quad + f(s, \phi'_s + x_s) \Big\} ds \|
\end{aligned}$$

$$\leq \int_0^{t_1} \|S(t_1-s)-S(t_2-s)\| M_2M_3[\|x_1\| + M_1NK_1T^2 + M_1K_2T^2 + M_1K_3T]ds$$
$$+ \int_{t_1}^{t_2} \|S(t_2-s)\| M_2M_3[\|x_1\| + M_1NK_1T^2 + M_1K_2T^2 + M_1K_3T]ds$$
$$+ \int_0^{t_1} \|S(t_1-s)-S(t_2-s)\|[NK_1T + K_2T + K_3]ds$$
$$+ \int_{t_1}^{t_2} \|S(t_2-s)\|[NK_1T + K_2T + K_3]ds$$

The compactness of $S(t), t > 0$ implies that $S(t)$ is continuous in the uniform operator topology for $t > 0$. Thus, the right hand side of (5), which is indepentent of $x \in X_0$, tends to zero as $t_1 - t_2 \to 0$. So, $Y(X_0)$ is an equicontinuous family of functions. Also, $Y(X_0)$ is bounded in X_0, and so by the Arzela-Ascoli Theorem, $Y(X_0)$ is precompact. Hence, from the Schauder Fixed point theorem, Y has a fixed point in X_0. So that, any fixed point of Φ is a mild solution of (1) on J satisfying

$$(\Phi x)(t) = x(t) \in X.$$

Thus, the system (1) is controllable on J.

4. Example.

Consider the abstract functional integrodifferential equation of the form

$$\begin{aligned}
& y_t(t,x) - (k(x)y_x(t,x))_x \\
& \quad = (Bu)(t) + \int_0^t [a(t,s)g(s,y(s-r,x)) + h(t,s,y(s-r,x))]ds \\
(6) \qquad & \quad + f(t,y(t-r,x)), \quad x \in [0,1] = I, t \in J, \\
& y(t,0) = y(t,1) = 0, \quad t \in J \\
& y(t,x) = \phi(t,x), \quad -r \leq t \leq 0, x \in I,
\end{aligned}$$

where $B : U \to X$, with $U \subset J$ and $X = L^2[I,R]$, is a linear operator such that there exists an invertible operator W^{-1} on $L^2(J,U)/\ker W$, where W is defined by

$$Wu = \int_0^T S(T-s)Bu(s)ds,$$

$S(t)$ is a compact semigroup. The functions a, g, h and f in (6) satisfy the following conditions;

(i) $a : J \times J \to R$ is Hölder continuous with exponent α,

(ii) The functions $g, f : J \times R \to R, h : J \times J \times R \to R$ are continuous and such that

$$\begin{aligned}
&|g(t,x) - g(t,\overline{x})| \leq L_1|x - \overline{x}|, \\
&|h(t,s,x) - h(t,s,\overline{x})| \leq L_2|x - \overline{x}|, \\
&|f(t,x) - f(t,\overline{x})| \leq L_3|x - \overline{x}|, \\
&|g(t,0)| = |h(t,s,0)| = |f(t,0)| = 0,
\end{aligned}$$

for $0 \leq s \leq t \leq T$ and $x, \overline{x} \in R$, where L_1, L_2, L_3 are nonnegative constants. Let $X = L^2(I, R)$. We define an operator

$$A : X \to X \quad \text{by} \quad (Ay)(t)(x) = -(k(x)y_x(t,x))_x$$

with

$$D(A) = \{x \in X : (k(\cdot)y_x(\cdot,\cdot))_x \in X, y(t,0) = y(t,1) = 0\}.$$

Define the mapping $G, F : J \times C \to X$ and $H : J \times J \times C \to X$ by $G(t,\phi)(x) = g(t, \phi(-r)x)$, $H(t,s,\phi)(x) = h(t,s,\phi(-r)x)$ and $F(t,\phi)(x) = f(t,\phi(-r)x)$. Equation (6) can be formulated abstractly as

$$\begin{aligned} y'(t) + Ay(t) =& (Bu)(t) + \int_0^t [a(t,s)G(s,y_s) + H(t,s,y_s)]ds \\ & + F(t,y_t), \quad t \in [0,T] = J \\ y(t) =& \phi(t), \quad -r \leq t \leq 0. \end{aligned} \tag{7}$$

Further, all the conditions satisfied in the above theorem are satisfied. Hence, the system (6) is controllable on J.

References

1. K.Balachandran, P.Balasuramaniam and J.P.Dauer, *Locally Null Controllability of Nonlinear Functional Differential systems in Banach space*, J.Opti.Theory Appl. **88 no.1** (1996), 61-75.
2. M.B.Dhakne and B.G.Pachpatte, *On some Abstract Functional Integrodifferential Equations*, Indian J.Pure Appl.Math. **22 no.2** (1991), 109-134.
3. Y.C.Kwun, J.Y.Park and J.W.Ryu, *Approximate Controllability and Controllability for Delay Volterra systems*, Bull.Korea Math.Soc. **28** (1991), 131-145.
4. K.Naito, *On Controllabiltiy for a Nonlinear Volterra Equation*, Nonl.Anal.T.M.A. **18** (1992), 99-108.
5. K.Naito and J.Y.Park, *Approximate Controllability for Trajectories of a Delay Volterra Control system*, J.Opti.Theory Appl. **61** (1989), 271-279.
6. S.Nakagiri and M.Yamamoto, *Controllability and Observability of Linear retarded systems in Banach space*, Intern.J. Control **49** (1989), 1489-1504.

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