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REAL HYPERSURFACES IN COMPLEX HYPERBOLIC SPACE WITH η-RECURRENT SECOND FUNDAMENTAL TENSOR

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ABSTRACT. Recently, Hamada [4] has proved that there do not exist any real hypersurfaces in complex projective space $P_n(C)$ with recurrent second fundamental tensor. From this point of view, he introduce the notion of η-recurrent second fundamental tensor for real hypersurfaces in $P_n(C)$. In this paper we also consider the notion of η-recurrent second fundamental tensor for real hypersurfaces in complex hyperbolic space $H_n(C)$ and classified such kind of real hypersurfaces under the condition that the structure vector field ξ is principal.

1. Introduction

A complex $n(\geq 2)$-dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$. A complete and simply connected complex space form is a complex projective space $P_n(C)$, a complex Euclidean space C^n or a complex hyperbolic space $H_n(C)$, according as $c > 0$, $c = 0$ or $c < 0$. The induced almost contact metric structure of a real hypersurface M of $M_n(c)$ is denoted by (ϕ, ξ, η, g).

There exist many studies about real hypersurfaces of $M_n(c)$. One of the first researches is the classification of homogeneous real hypersurfaces in a complex projective space $P_n(C)$ by Takagi [14], who showed that these hypersurfaces of $P_n(C)$ could be divided into six types which are said to be of type A_1, A_2, B, C, D, and E, and in [3] Cecil-Ryan and [7] Kimura proved that they are realized as the tubes of constant radius over Kaehlerian submanifolds if the structure vector field ξ is principal. Also Berndt [2] showed recently that all real hypersurfaces with constant principal curvatures of a complex hyperbolic space $H_n(C)$ are realized as the tubes of constant radius over certain submanifolds when the structure vector field ξ is principal. Nowadays in $H_n(C)$ they are said to be of type A_0, A_1, A_2, and B.

On the other hand, in [9] Kobayashi and Nomizu have introduced the notion of recurrent tensor field of type (r, s) on a manifold M with a linear connection. That is, a non-zero tensor field K of type (r, s) on M is said to be *recurrent* if there exists a 1-form α such that

$$\nabla K = K \otimes \alpha.$$

Moreover, they gave some geometric interpretation of a manifold M with recurrent curvature tensor in terms of holonomy group, see also [15].

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Now let us denote by A the second fundamental form of real hypersurfaces in $M_n(c)$, $c\neq 0$. Recently, Hamada [4] applied this notion of recurrent second fundamental form to real hypersurfaces M in a complex projective space $P_n(C)$, which is defined in such a way that

$$\nabla A = \alpha \otimes A$$

for a certain 1-form α defined on M, and proved the following

Theorem A. *$P_n(C)$ do not admit any real hypersurfaces with recurrent second fundamental tensor.*

Now let T_0 be a distribution defined by a subspace $T_0(x) = \{X \in T_x M : X \perp \xi_{(x)}\}$ in the tangent space $T_x M$. Then by virtue of Theorem A Hamada [5] considered the notion of *η-recurrent* second fundamental form defined by

$$g((\nabla_X A)Y, Z) = \alpha(X) g(AY, Z)$$

for a certain 1-form α defined on T_0 and any X, Y and Z in T_0 and classified such kind of real hypersurfaces in $P_n(C)$ by the following

Theorem B. *Let M be a real hypersurface in a complex projective space $P_n(C)$ with η-recurrent second fundamental form and ξ is principal. Then M is locally congruent to a tube of some radius r over one of the following Kaehler submanifolds:*

(A_1) *hyperplane $P_{n-1}(C)$, where $0 < r < \frac{\pi}{2}$,*
(A_2) *totally geodesic $P_k(C)$ $(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$,*
(B) *complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$.*

Now its geometrical meaning of *η-recurrency* can be interpreted that *the eigen space of the shape operator A are parallel along the curve γ orthogonal to ξ.* Here, the eigen spaces of the shape operator A are said to be *parallel* along γ if they are invariant with respect to any parallel translations along γ, see [13].

In this paper we also consider the notions of *recurrent* second fundamental form and *η-recurrent* second fundamental form for real hypersurfaces in a complex hyperbolic space $H_n(C)$ and proved the followings

Theorem 1. *$H_n(C)$ do not admit any real hypersurfaces with recurrent second fundamental tensor.*

Theorem 2. *Let M be a real hypersurface in $H_n(C)$ with η-recurrent second fundamental form and ξ is principal, then M is congruent to one of real hypersurfaces*

(A_0) *a horosphere in $H_n(C)$, i.e., a Montiel tube,*
(A_1) *a tube over a totally geodesic hyperplane $H_k(C)$ $(k = 0$ or $n-1)$,*
(A_2) *a tube over a totally geodesic $H_k(C)$ $(1 \leq k \leq n-2)$.*
(B) *a tube over a real hyperbolic space $H_n(R)$.*

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2. Preliminaries

First of all, we recall fundamental properties of real hypersurfaces of a complex space form. Let M be a real hypersurface of a complex n-dimensional complex space form $M_n(c)$ of constant holomorphic sectional curvature $c(\neq 0)$ and let C be a unit normal field on a neighborhood of a point x in M. We denote by J an almost complex structure of $M_n(c)$. For a local vector field X on a neighborhood of x in M, the transformation of X and C under J can be represented as

$$JX = \phi X + \eta(X)C, \qquad JC = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on a neighborhood of x in M, respectively. Moreover, it is seen that $g(\xi, X) = \eta(X)$,where g denotes the induced Riemannian metric on M. By properties of the almost complex structure J, the set (ϕ, ξ, η, g) of tensors satisfies

$$\phi^2 = -I + \eta\otimes\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1,$$

where I denotes the identity transformation. Accordingly, the set is so called an *almost contact metric structure*. Furthermore the covariant derivative of the structure tensors are given by

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{2.1}$$

where ∇ is the Riemannian connection of g and A denotes the shape operator with respect to the unit normal C on M.

Since the ambient space is of constant holomorphic sectional curvature c, the equation of Gauss and Codazzi are respectively given as follows

$$\begin{aligned} R(X,Y)Z =& \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & - 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.2}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}, \tag{2.3}$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X.

Now let us suppose that the structure vector ξ is a principal vector with principal curvature β, that is, $A\xi = \beta\xi$. Then, differentiating this, we have

$$(\nabla_X A)\xi = (X\beta)\xi + \beta\phi AX - A\phi AX, \tag{2.4}$$

where we have used (2.1). Then it follows

$$\begin{aligned} g((\nabla_X A)Y, \xi) =& (X\beta)\eta(Y) + \beta g(Y, \phi AX) \\ & - g(Y, A\phi AX) \end{aligned} \tag{2.5}$$

for any tangent vector fields X and Y on M. By the equation of Codazzi (2.3), we have

$$2A\phi AX - \frac{c}{2}\phi X = \beta(\phi A + A\phi)X. \tag{2.6}$$

3. Proof of Theorems 1 and 2

It is well known that the complex hyperbolic space $H_n(C)$ admits the Bergmann metric normalized so that the constant holomorphic sectional curvature c is -4.

Now let us prove Theorem 1 given in the introduction. From the assumption of recurrent second fundamental form we have

$$g((\nabla_X A)Y, \xi) = \alpha(X)g(AY, \xi). \tag{3.1}$$

From this let us put $A\xi = \beta\xi + \gamma U$, where U is orthogonal to ξ. Then (3.1) implies

$$g((\nabla_X A)\xi, Y) = \beta\alpha(X)\eta(Y) + \gamma\alpha(X)g(U, Y). \tag{3.2}$$

Now if we use the equation of Codazzi (2.3), we have

$$\begin{aligned} g((\nabla_X A)\xi, Y) &= g((\nabla_\xi A)X + \phi X, Y) \\ &= g((\nabla_\xi A)X, Y) + g(\phi X, Y) \\ &= \alpha(\xi)g(AX, Y) + g(\phi X, Y). \end{aligned} \tag{3.3}$$

for any X, Y in M. Thus (3.2) and (3.3) give the following

$$\alpha(\xi)g(AX, Y) = \beta\alpha(X)\eta(Y) + \gamma\alpha(X)g(U, Y) - g(\phi X, Y). \tag{3.4}$$

From this, putting $X = U, Y = \xi$, we have

$$\alpha(\xi)\gamma = \beta\alpha(U). \tag{3.5}$$

Similarly, putting $X = U, Y = \phi U$ and $X = \phi U, Y = U$ in (3.4) respectively, we have

$$\begin{aligned} \alpha(\xi)g(AU, \phi U) &= -g(\phi U, \phi U) = -1, \quad \text{and} \\ \alpha(\xi)g(A\phi U, U) &= \gamma\alpha(\phi U)g(U, U) - g(\phi^2 U, U) \\ &= \gamma\alpha(\phi U) + 1. \end{aligned} \tag{3.6}$$

So it follows

$$\gamma\alpha(\phi U) = -2. \tag{3.7}$$

Also putting $X = Y = \phi U$ in (3.4), we have

$$\alpha(\xi)g(A\phi U, \phi U) = 0.$$

From this, together with (3.5) and (3.6), it follows

$$\beta\alpha(U)g(A\phi U, \phi U) = \gamma\alpha(\xi)g(A\phi U, \phi U) = 0. \tag{3.8}$$

On the other hand, by the equation of Codazzi (2.3) for $c = -4$ we have

$$(\nabla_U A)\phi U - (\nabla_{\phi U} A)U = 2\xi.$$

From this and (3.8), together with the recurrency of M in $H_n(C)$ we have

$$\begin{aligned} 0 &= \beta\alpha(U)g(A\phi U, \phi U) \\ &= \beta g((\nabla_U A)\phi U, \phi U) \\ &= \beta g((\nabla_{\phi U} A)U + 2\xi, \phi U) \\ &= \beta\alpha(\phi U)g(AU, \phi U). \end{aligned}$$

From this and (3.6), (3.7) we know $\beta = 0$. Thus (3.5) and (3.6) gives $\gamma = 0$. This makes a contradiction to (3.7). So there do not exist any real hypersurfaces M in $H_n(C)$ with recurrent second fundamental tensor. This completes the proof of Theorem 1.

Now the formula (2.6) gives the following equation for real hypersurfaces in $H_n(C)$ when the structure vector field ξ is principal

$$2A\phi AX + 2\phi X = \beta(\phi A + A\phi)X \tag{3.9}$$

for any vector field X in M. It follows that if $AX = \lambda X$ for any X in T_0, which is a distribution defined by a subspace $T_0(x) = \{X \in T_x M : X \perp \xi_{(x)}\}$ in the tangent space $T_x M$, then

$$(2\lambda - \beta)A\phi X = (\beta\lambda - 2)\phi X. \tag{3.10}$$

Now we introduce a lemma proved by Ki and the second author [6]

Lemma 3.1. *Let M be a real hypersurface in a complex hyperbolic space $H_n(C)$. If ξ is a principal curvature vector with principal curvature β, then β is locally constant.*

Hereafter, we are going to prove Theorem 2 in the introduction. The second fundamental form of M in a complex hyperbolic space $H_n(C)$ is said to be η-*recurrent* if and only if there exists a 1-form α such that

$$g((\nabla_X A)Y, Z) = \alpha(X)g(AY, Z)$$

for any X, Y and Z in T_0. When the 1-form α defined on T_0 vanishes, the second fundamental form of M is said to be η-*parallel*.

Motivated by Theorem 1, we classify real hypersurfaces in $H_n(C)$ with η-recurrent second fundamental form and principal structure vector field ξ. In order to prove Theorem 2, let us introduce a theorem proved by the second author [12].

Theorem 3.2. *Let M be a real hypersurface in a complex hyperbolic space $H_n(C)$ with η-parallel second fundamental form and ξ is principal. Then M is locally congruent to one of real hypersurfaces of type A_0,A_1, A_2 or B.*

Remark 3.3. Kimura and Maeda [8] have proved that a real hypersurface of a complex projective space $P_n(C)$ with *η-parallel* second fundamental form and principal structure vector ξ is locally congruent to one of real hypersurfaces of type A_1, A_2 and B.

By Theorem 3.2 we know that the second fundamental form of real hypersurfaces of type A_0,A_1,A_2 or B are η-recurrent and its structure vector field ξ is principal.

Conversely, let us prove Theorem 2. Under the assumption of *η-recurrency* and ξ principal it suffices to show that all of principal curvatures of M in $H_n(C)$ are constant. Then by a theorem of Berndt [1], we know that M is congruent to one of real hypersurfaces of type A_0,A_1, A_2 and B.

Now let us show that every principal curvatures of M are constant. From the notion of η-recurrency and the equation of Codazzi (2.3) we have

$$\alpha(X)g(AY,Z) = \alpha(Y)g(AX,Z) = \alpha(Z)g(AX,Y)$$

for any X, Y and Z in T_0. This implies

$$\alpha(X)AY - \alpha(Y)AX = b\xi$$

for a certain smooth function b on M.

In order to show that every principal curvatures are constant we consider the following cases:

Case I. Let us consider the open set $\mathcal{U}$ consisting of points, at which there exist two distinct principal curvatures.

In this case $T_0(x) = \{X{\in}T_xM : X{\perp}\xi\} = T_\lambda$ for any point x in $\mathcal{U}$. So, by a theorem of Montiel [10] or Montiel and Romero [11] M is locally congruent to a horosphere (or said to be of a *Montiel* tube) or a geodesic hypersphere. Of course, every principal curvatures of these hypersurfaces are known to be constant.

Case II. Let us consider the open set $\mathcal{V} = Int(M - \mathcal{U})$ consisting of points, at which there exist more than 3 distinct principal curvatures.

Then among them let us take out any two distinct principal curvatures λ and μ different from β. Then on this $\mathcal{V}$ we can consider the following subcases:

Sub. II.1: Let $\mathcal{W} = \{p{\in}\mathcal{V}|\lambda(p){\neq}0, \mu(p){\neq}0\}$. Then λ and μ are non-vanishing at any point of $\mathcal{W}$.

In this case we can decompose the distribution T_0 into the direct sum of eigenspces such that

$$T_0 = T_\lambda{\oplus}T_\mu{\oplus}T_{\mu_1}{\oplus}\cdots{\oplus}T_{\mu_k},$$

where $\mu_1, \cdots, \mu_k$ denote principal curvatures different from λ and μ, and T_λ, T_μ and T_{μ_i} denote the eigenspaces of principal vectors in T_0 with corresponding principal curvatures λ, μ and μ_i.

Choose $X \in T_\lambda$, $Y \in T_\mu$ such that X and Y are orthogonal to ξ, then we have

$$\alpha(X)\mu Y - \alpha(Y)\lambda X = 0.$$

Then

$$\alpha(X)\mu = 0 \quad \text{and} \quad \alpha(Y)\lambda = 0 \tag{3.11}$$

for any $X \in T_\lambda$ and $Y \in T_\mu$. So it follows that

$$\begin{aligned} X\mu &= g((\nabla_X A)Y, Y) + g(A\nabla_X Y, Y) \\ &= \alpha(X) g(AY, Y) + \mu g(\nabla_X Y, Y) \\ &= \alpha(X)\mu \\ &= 0, \end{aligned} \tag{3.12}$$

where we have used the notion of *η-recurrency* in the second equality for any $X \in T_\lambda$ and $Y \in T_\mu$. Since λ and μ are non-zero, (3.11) implies $\alpha(X) = 0 = \alpha(Y)$. This means

$$Y\mu = 0 \tag{3.13}$$

for any $Y \in T_\mu$. Moreover, for any $Z \in T_{\mu_i}$ the η-recurrency implies

$$\mu\alpha(Z)Y - \mu_i \alpha(Y) Z = 0.$$

This means $\alpha(Z)\mu = 0$ and $\alpha(Y)\mu_i = 0$. So

$$\begin{aligned} Z\mu &= g((\nabla_Z A)Y, Y) + g(A\nabla_Z Y, Y) \\ &= \alpha(Z) g(AY, Y) + \mu g(\nabla_Z Y, Y) \\ &= \alpha(Z)\mu \\ &= 0. \end{aligned} \tag{3.14}$$

On the other hand, by (2.4), we get the following for any $Y \in T_\mu$

$$\begin{aligned} \xi\mu &= \xi g(AY, Y) \\ &= g((\nabla_\xi A)Y, Y) + g(A\nabla_\xi Y, Y) + g(AY, \nabla_\xi Y) \\ &= g((\nabla_Y A)\xi, Y) \\ &= g((Y\beta)\xi + \beta\phi AY - A\phi AY, Y) \\ &= \beta\mu g(\phi Y, Y) - \mu^2 g(\phi Y, Y) \\ &= 0 \end{aligned} \tag{3.15}$$

where in the third equality we have used the equation of Codazzi (2.3) and the fact $AY = \mu Y$. From these (3.12),(3.13),(3.14), and (3.15) we know $X\mu = 0$ for any $X \in T_0$. So μ is constant on $\mathcal{W}$. Similarly, we know that λ is also constant on $\mathcal{W}$.

Sub. II.2: Let us consider the open subset $Int(\mathcal{V}-\mathcal{W})$ of $\mathcal{V}$. Then on this open subset either λ or μ vanishes identically. Thus for convenience sake we consider such a situation

$$Int(\mathcal{V}-\mathcal{W}) = \{p\in\mathcal{V}|\lambda(p) = 0, \mu(p)\neq 0\}.$$

Now we want to show that μ is constant on $Int(\mathcal{V}-\mathcal{W})$.

Then in this case the distribution T_0 is decomposed into the direct sum of eigenspaces such that

$$T_0 = T_{\lambda=0}\oplus T_{\mu\neq 0}\oplus T_{\mu_1}\oplus\cdots\oplus T_{\mu_k}.$$

When $\mu_i = 0$ for all $i = 1, ..., k$, we consider such a situation that $T_0 = T_{\lambda=0}\oplus T_{\mu\neq 0}$. Then (3.10) gives

$$\beta A\phi X = 2\phi X.$$

So $A\phi X = \frac{2}{\beta}\phi X$. In this case, by Lemma 3.1, $\mu = \frac{2}{\beta}$ is constant.

Next for convenience sake we consider the case that

$$T_0 = T_{\lambda=0}\oplus T_{\mu\neq 0}\oplus T_{\nu\neq 0}.$$

From the formulas

$$b\xi = \alpha(X)AZ - \alpha(Z)AX = \alpha(X)\nu Z, \quad \text{and}$$

$$b\xi = \alpha(Y)AZ - \alpha(Z)AY = \alpha(Y)\nu Z - \alpha(Z)\mu Y$$

for any $X\in T_{\lambda=0}, Y\in T_{\mu\neq 0}$ and $Z\in T_{\nu\neq 0}$, we have $\alpha(X) = \alpha(Y) = \alpha(Z) = 0$. So $W\mu = 0$ for any $W\in T_0$. From this together with the fact $\xi\mu = 0$, we know μ is constant on $Int(\mathcal{V}-\mathcal{W})$. Thus accordingly, by the continuity of principal curvatures the set $\mathcal{W}$ is empty or $\mathcal{V}$ itself. From this the principal curvatures λ and μ are constant on $\mathcal{V}$.

Summing up the above Cases I and II, by the continuity of principal curvatures again $\mathcal{U}$ is empty or the whole set M. When $\mathcal{U}$ is empty, the open $\mathcal{V}$ should be the whole set M. From this we conclude that every principal curvatures of T_0 are constant on M. Together with Lemma 3.1 every principal curvatures of M are constant. Now we have completed the proof of Theorem 2.

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