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A note on Polysurface groups

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Introduction.

We shall define a group Γ to be a polysurface group of length n if there is a filtration $\{\Gamma_i\}_{0\leqq i\leqq n}$ of Γ such that

(1) Γ is torsion free,
(2) $1=\Gamma_0\subset\Gamma_1\subset\cdots\subset\Gamma_{n-1}\subset\Gamma_n=\Gamma$,
and
(3) for each i, $\Gamma_i\lhd\Gamma_{i+1}$ and Γ_{i+1}/Γ_i is the fundamental group of an orientable surface.

We call a group Γ a polysurface group without abelian factor if for each i, Γ_{i+1}/Γ_i is the fundamental group of an orientable surface with genus $\geqq 2$ and a polysurface group of length 1 a surface group.

In a series of his works ([J1],[J2]), F.E.A.Johnson has studied the smooth realization of polysurface group without abelian factor. Here a group Γ is called to be smoothly realizable when there exists a smooth manifold X_Γ whose fundamental group is Γ.

Being motivated by his works, we shall consider the following

Problem. *Does every polysurface group Γ embed as a discrete cocompact subgroup of a non-compact connected Lie group G without compact factor ?*

If a polysurface group Γ embeds as a discrete cocompact subgroup of a non-compact Lie group G, then Γ is realized as the fundamental group of smooth closed aspherical manifold $\Gamma\backslash G/K$, where K is a maximal compact subgroup of G.

In this note, we shall use the following notations.

1. A Lie group is assumed to be connected,non-compact,unless the contrary stated explicitly.
2. For a Lie group G, G° denotes the identity component.
3. $\tilde{X}$ denotes the universal covering space of X.
4. H^4, H^3 or H^2 denotes 4-dimensional, 3-dimensional or 2-dimensional hyperbolic space,respectively.
5. For a group G, $Z(G)$ denotes the center of G.
6. For a subgroup H of G, $N_G(H)$ or $C_G(H)$ denotes the normalizer or centralizer of H in G, respectively.
7. $\chi(X)$ denotes the Euler characteristic of X.

1. Preliminaries

We shall prove the following

Theorem 1. *Let Γ be a polysurface group without abelian factor and G a non-compact Lie group such that G/R has no compact factor, where R is the radical of G. If Γ is a discrete cocompact subgroup of G, then R is trivial, i.e. G is semisimple.*

Proof. Assume R is not trivial. Since $R/R \cap \Gamma$ is compact (see [R]) and G/R is non-compact, $\Gamma \cap R$ is a non-trivial solvable normal subgroup of Γ. It is easy to see that a polysurface group without abelian factor has no non-trivial solvable normal subgroup. Hence $\Gamma \cap R$ is trivial. $\square$

2. Polysurface groups without abelian factor

In this section, we shall consider only polysurface groups without abelian factor and hence we call such groups polysurface group simply.

We have the following

Lemma 1. *Let Γ be a polysurface group of length n ($n \geqq 2$). If Γ is embeddable in the product group $PSL_2(R)^n$ as a discrete cocompact subgroup, then Γ is reducible.*

For the definition and more informations about reducibility, see [R] (Chap.5 in [R]).

Proof. If Γ is irreducible, then it follows from a result in [BW] (Corollary 4.6 p.220 in [BW]) that the first Betti number $b_1(\Gamma) = 0$. Note that Γ is given by an exact sequence;

$$1 \longrightarrow S_1 \longrightarrow \Gamma \longrightarrow S_2 \longrightarrow 1,$$

where S_2 is a surface group. Then it is easy to see that there exists a surjection $\Gamma/[\Gamma,\Gamma] \to S_2/[S_2,S_2]$. Since rank $S_2/[S_2,S_2] \neq 0$, rank $\Gamma/[\Gamma,\Gamma] \neq 0$. This completes the proof. $\square$

Theorem 2. *Let Γ be a polysurface group without abelian factor of length n, i.e. Γ is defined by the following exact sequence;*

$$1 \longrightarrow S_1 \longrightarrow \Gamma \xrightarrow{p} S_2 \longrightarrow 1,$$

where S_1 is a polysurface group without abelian factor of length $n-1$ and S_2 is a surface group. Assume S_1 is embedded in the product $PSL_2(R) \times \cdots \times PSL_2(R)$ ($n-1$ times) as a discrete cocompact subgroup. Then Γ is embedded in the product $PSL_2(R) \times \cdots \times PSL_2(R)$ (n times) as a discrete cocompact subgroup if and only if the operator homomorphism $\theta : S_2 \to Out(S_1)$ has finite image.

Proof. First we assume that the operator homomorphism $\theta : S_2 \to Out(S_1)$ has finite image. Let S_2' be the kernel of θ and Γ' the preimage of S_2' by p. We have the following commutative diagram;

$$\begin{array}{ccccccccc} 1 & \longrightarrow & S_1 & \longrightarrow & \Gamma & \xrightarrow{p} & S_2 & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & S_1 & \longrightarrow & \Gamma' & \longrightarrow & S_2' & \longrightarrow & 1 \end{array}$$

where the horizontal sequences are exact.

By assumption, S_2' is of finite index in S_2, hence Γ' is of finite index in Γ. Because of $H^2(S_2'; Z(S_1)) = 0$ and the operator homomorphism $\theta : S_2' \to Out(S_1)$ is trivial, we have $\Gamma' \cong S_1 \times S_2'$. Since S_1 is a reducible discrete cocompact subgroup of $PSL_2(R) \times \cdots \times PSL_2(R)$ ($n-1$ times) (by Lemma 1),S_1 contains a normal subgroup $S_{11} \times \cdots \times S_{1n-1}$ of finite index, where S_{1i} is a surface group for $i = 1, \cdots, n-1$. Hence Γ' contains the subgroup $S_{11} \times \cdots \times S_{1n-1} \times S_2'$ of finite index. Then we have the following exact sequence;

$$1 \longrightarrow S_1' \longrightarrow \Gamma' \longrightarrow F \longrightarrow 1,$$

where S_1' is a product of n surface groups and F is a finite group. By the result in [J], Γ is embeddable in the product Lie group $PSL_2(R) \times \cdots \times PSL_2(R)$ (n times) as a discrete cocompact subgroup. Thus we have completed the proof of sufficiency.

Conversely, we assume that Γ is embedded in $PSL_2(R) \times \cdots \times PSL_2(R)$ (n times) as a discrete cocompact subgroup. Note that Γ is reducible. Then Γ contains $\Gamma_1 \times \cdots \times \Gamma_n$ as a normal subgroup of finite index, where Γ_i is a fundamental group of an orientable surface of genus $\geqq 2$.

We have the following four cases;

(1) $p(\Gamma_1 \times \cdots \times \Gamma_{n-1}) = 1$ and $p(\Gamma_n) = 1$
(2) $p(\Gamma_1 \times \cdots \times \Gamma_{n-1}) = 1$ and $p(\Gamma_n) \neq 1$
(3) $p(\Gamma_1 \times \cdots \times \Gamma_{n-1}) \neq 1$ and $p(\Gamma_n) = 1$
(4) $p(\Gamma_1 \times \cdots \times \Gamma_{n-1}) \neq 1$ and $p(\Gamma_n) \neq 1$.

It is clear that the case (1) does not occur by cohomological dimension argument.

Consider the case (4). Since $p(\Gamma_n)$ is a finitely generated normal subgroup of S_2, it is a surface group and of finite index in S_2. Choose a non-trivial element $x \in p(\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_{n-1})$. Then we have $x^m \in p(\Gamma_n)$ for some power of x. Since $x \in C_{S_2}(p(\Gamma_n))$, we have $x^m \in C_{S_2}(p(\Gamma_n)) \cap p(\Gamma_n) = Z(p(\Gamma_n)) = 1$ and hence we have $x^m = 1$. This contradicts the torsion freeness of S_2. Thus case (4) does not occur.

Consider the case (2). First we show that $p|\Gamma_n : \Gamma_n \to S_2$ is injective. It is sufficient to show that $\Gamma_n \cap S_1 = 1$. Note that $S_1 \cap \Gamma \cong \Gamma_1 \times \cdots \times \Gamma_{n-1} \times (S_1 \cap \Gamma_n)/\Gamma_1 \times \cdots \times \Gamma_{n-1}$ is a subgroup of $S_1/(\Gamma_1 \times \cdots \times \Gamma_{n-1})$. Since $cd(S_1) = cd(\Gamma_1 \times \cdots \times \Gamma_{n-1})$, $S_1/(\Gamma_1 \times \cdots \times \Gamma_{n-1})$ is finite, $S_1 \cap \Gamma_n$ is so, which implies our assertion.

Thus we have the following commutative diagram;

$$\begin{array}{ccccccccc}
1 & \longrightarrow & S_1 & \longrightarrow & \Gamma & \xrightarrow{p} & S_2 & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Gamma_1 \times \cdots \times \Gamma_{n-1} & \longrightarrow & \Gamma_1 \times \cdots \times \Gamma_n & \longrightarrow & p(\Gamma_n) & \longrightarrow & 1
\end{array}$$

where the horizontal sequences are exact.

We shall define a homomorphism $\lambda : p(\Gamma_n) \to \Gamma_n$ by the inverse of $p|\Gamma_n$. Then we have $\lambda(p(\Gamma_n)) \subset C_\Gamma(\Gamma_1 \times \cdots \times \Gamma_{n-1})$.

Put $S_2' = p(\Gamma_n)$ and $S_1' = \Gamma_1 \times \cdots \times \Gamma_{n-1}$. We shall identify elements $s' \in S_2'$ and $\lambda(s')$.

We define a homomorphism $\phi : S_2' \to Aut(S_1)$ by the formula $\phi(s') = c_{s'}$, where $c_{s'}$ denotes the conjugation by s'. Since $c_{s'}|S_1' = id$, we have a homomorphism $\bar{\phi} : S_2' \to Aut(S_1/S_1')$. This define a homomorphism $\psi : \phi \mapsto \bar{\phi}$ naturally. We shall show that ψ is injective. Assume $\psi(\phi(s')) = 1$. Then, for every element $s_1 \in S_1$, we have $s_1^{-1} s' s_1 s'^{-1} = s_1' \in S_1'$. Since S_1, S_1' are normal in Γ and $S_1 \cap S_1' = 1$, we have $s_1^{-1} s' s_1 s'^{-1} = 1$ and $s_1' = 1$. Thus we have $s' s_1 s'^{-1} = s_1$. This implies ψ is injective. Thus ϕ has finite image, because $\bar{\phi}$ has finite image. It follows that the restriction of the operator homomorphism θ to S_2' has finite image and hence there exists a subgroup S_2'' of S_2 of finite index such that $\theta|S_2''$ is trivial. Since S_2'' is a subgroup of S_2 with finite index, the image of θ is finite. This completes the proof of our assertion.

Finally consider the case (3). This case is easily reduced to the case (2) or (4). Thus we have completed the proof of Theorem. $\square$

We shall consider the case of length 2 in more detail. We are given an exact sequence ;

$$(*) \qquad 1 \longrightarrow S_1 \longrightarrow \Gamma \xrightarrow{p} S_2 \longrightarrow 1,$$

where S_1 and S_2 are surface groups. Let $\theta : S_2 \to Out(S_1)$ be the operator homomorphism.

Recall that θ is defined as follows. Let γ_2 be an element of S_2. Choose an element $\gamma \in \Gamma$ such that $p(\gamma) = \gamma_2$. Consider the conjugation $c_\gamma : S_1 \to S_1$,which is the automorphism of S_1, uniquely determined up to $Inn(S_1)$. In other words, we have a homomorphism $\theta : S_2 \to Out(S_1)$.

We have the following

Proposition 1. *Assume the operator homomorphism θ is not injective. Then if Γ is embeddable in a non-compact Lie group G without compact factor as a discrete cocompact subgroup, then G is not simple.*

Proof. Since θ is not injective, there exists an non trivial element $\gamma_2 \in S_2$ such that $\theta(\gamma_2) = 1$. This implies that there exists an element $\gamma \in \Gamma$ such that $\gamma\, \gamma_1\, \gamma^{-1} = \gamma'_1\, \gamma_1\, {\gamma'}_1^{-1}$ for every element $\gamma_1 \in S_1$ for some element ${\gamma'}_1^{-1} \in S_1$. We have $({\gamma'}_1^{-1}\gamma)\gamma_1({\gamma'}_1^{-1}\gamma)^{-1} = \gamma_1$. Since γ is not in S_1, $\gamma\gamma_1{}^{-1} \neq 1$. Thus we have $C_\Gamma(S_1) \neq 1$. Since S_1 is centerless, $C_\Gamma(S_1) \cap S_1 = 1$.

Choose a non trivial element γ from $C_\Gamma(S_1)$. Since S_1 is a normal subgroup of Γ, we have $\Gamma \subset N_G(S_1)$. Then we have $\Gamma \subset N_G(A(S_1))$, where $A(*)$ denotes the algebraic closure of $*$, the smallest algebraic subgroup of G which contains $*$. In fact, for any element $\gamma \in \Gamma$, S_1 is stable under the conjugation $c_\gamma : G \to G$ and hence, for any algebraic subgroup K which contains S_1, $c_\gamma(K)$ is also algebraic subgroup containing S_1. This implies $c_\gamma(A(S_1)) = A(S_1)$, in other words, $\Gamma \subset N_G(A(S_1))$. Since $N_G(A(S_1))$ is an algebraic subgroup of G,

$A(\Gamma) \subset N_G(A(S_1))$. By the same argument, we have $G \subset N_G(A(\Gamma))$. Assume G is simple. Then it follows that we have $A(\Gamma) = G$ and $G = N_G(A(S_1))$. If $\gamma \in C_G(S_1)$, then γ is also in $C_G(A(S_1)) = C_G(G)$, in particular $Z(\Gamma) \neq 1$, which is a contradiction. □

By results in [J], there are three types of polysurface group of length 2;

Type 1. The operator homomorphism θ has finite image.
Type 2. θ is not injective and its image is not finite.
Type 3. θ is injective.

We have the following

Theorem 3. *(1) If Γ is of type 1, then Γ is embedded in $PSL_2(R) \times PSL_2(R)$ as a discrete cocompact subgroup.*

(2) If Γ is of type 2, then Γ is not embedded in any non-compact Lie group as a discrete cocompact subgroup.

Proof. (1) follows from Theorem 1.

Let Γ be of type 2. Assume $\Gamma \subset G$. It follows from Proposition 1 that G is not simple and hence $G = PSL_2(R) \times PSL_2(R)$. Again it follows from Theorem 1 that θ has finite image. □

Remark: We don't know that the group of type 3 can be embedded in $O(4,1)^{\circ}$. But the following Proposition follows immediately from Proposition 1.

Proposition 2. *Let Γ be a polysurface group of length 2. If Γ is embedded in $O(4,1)^{\circ}$ as a discrete cocompact subgroup, then θ is injective.*

Remark: Let Γ be a polysurface group of length 2. It is proved that Γ is not embedded in $SU(2,1)$ as a cocompact discrete subgroup.

In fact, we have the following Proposition

Proposition 3. *Let Γ be a polysurface group of length 2. Then Γ is not embedded in $SU(2,1)$ as a cocompact discrete subgroup.*

Proof. Assume the contrary. It follows from a result in [W] that the 4-manifold X_Γ with the fundamental group $\pi_1(X_\Gamma) = \Gamma$ has non-zero signature $\sigma(X_\Gamma)$. On the other hand, X_Γ is a fiber bundle over a surface with surface as a fiber. Then it is easy to see that the signature of X_Γ is zero. □

3. Polysurface group of length 2

In this section, we shall consider polysurface groups of length 2, i.e. the group Γ given by the following exact sequences;

$$(1) \qquad 1 \longrightarrow Z^2 \longrightarrow \Gamma \longrightarrow S \longrightarrow 1,$$

$$(2) \qquad 1 \longrightarrow S \longrightarrow \Gamma \longrightarrow Z^2 \longrightarrow 1,$$

and

$$(3) \qquad 1 \longrightarrow Z^2 \longrightarrow \Gamma \longrightarrow Z^2 \longrightarrow 1.$$

We shall omit the case (3) since the group Γ is a poly Z group and such a group is already studied by many people.

3.1 Polysurface group of type (1)

In this subsection, we shall consider the poly surface group of length 2 of type (1). It follows from results in [W] that if Γ is embeddable in a non compact Lie group G without compact factor, then G is one of $Iso(R^2 \times H^2)^\circ$ or $Iso(R \times \widetilde{PSL_2(R)})^\circ$. In fact, it is clear that G is not solvable. It follows from results in [W] that G is one of

$$Iso(H^4)^\circ, Iso(H^2 \times H^2)^\circ, Iso(R^2 \times H^2)^\circ, Iso(R \times \widetilde{PSL_2(R)})^\circ, Iso(R \times H^3)^\circ$$

We show that G is not $Iso(H^4)^\circ$, $Iso(H^2 \times H^2)^\circ$.

First we shall show that Γ is not embedded in $Iso(H^4)^\circ$.

Assume that $\Gamma \subset Iso(H^4)^\circ = O(4,1)^\circ$. Note that Γ is realized as the fundamental group of a 4-dimensional hyperbolic manifolds M. It follows that $S_1 \neq Z^2$.

Assume $\Gamma \subset Iso(H^2 \times H^2)^\circ$. Since Γ is reducible (see Lemma 1), we have the following commutative diagram;

$$\begin{tikzcd}
 & 1 & 1 & 1 & \\
1 \arrow[r] & p_2(S_1) \arrow[r] \arrow[u] & p_2(\Gamma) \arrow[r] \arrow[u] & F \arrow[r] \arrow[u] & 1 \\
1 \arrow[r] & Z^2 \arrow[r] \arrow[u] & \Gamma \arrow[r, "p"] \arrow[u, "p_2"] & S_2 \arrow[r] \arrow[u] & 1, \\
1 \arrow[r] & Z^2 \cap PSL_2(R) \arrow[r] \arrow[u] & \Gamma_1 \arrow[r] \arrow[u] & p(\Gamma_1) \arrow[r] \arrow[u] & 1 \\
 & 1 \arrow[u] & 1 \arrow[u] & 1 \arrow[u] &
\end{tikzcd}$$

where $\Gamma_1 = PSL_2(R) \cap \Gamma$ and the lower vertical maps are inclusions and S_2 are surface groups, Note that $PSL_2(R) \cap \Gamma$ and $p_2(\Gamma)$ are discrete and cocompact subgroups of $PSL_2(R)$.

If $S_1 = Z^2$, then $S_1 \cap PSL_2(R) = 1$ and $S_1 \lhd p_2(\Gamma)$, which is absurd.

It follows from a result in [W] (Prop. 10.4 in [W]) that Γ is not embeddable in $Iso(R \times H^3)^\circ$, since Γ contains Z^2 as a normal subgroup.

We note that a torsion free group Γ embeds as a discrete cocompact subgroup of $G = Iso(R^2 \times H^2)^\circ$ or $Iso(R \times \widetilde{PSL_2}(R))^\circ$, then the manifold $M = \Gamma \backslash G/K$, where K is a maximal compact subgroup of G, is a Seifert manifold admitting a geometry structure. It follows from results in [U] (Theorem B, section 5 in [U]) that we have the following

Proposition 3. *The group Γ has the following presentation;*
Generators*:*$\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, t_1, t_2$
Relations*:*
$\alpha_i(t_1, t_2)\alpha_i^{-1} = (t_1, t_2)A_i,$
$\beta_i(t_1, t_2)\beta_i^{-1} = (t_1, t_2)B_i,$
$[t_1, t_2] = 1, \Pi[\alpha_i, \beta_i] = t_1^a t_2^b$
where $A_i, B_i \in SL_2(Z)$.
Moreover,
The case $\Gamma \subset Iso(R^2 \times H^2)^\circ$.
(1) A_i, B_i are powers of a common periodic matrix in $SL_2(Z)$.
(2) If A_i, B_i are all trivial, then the rational Euler class is zero.

The case $\Gamma \subset Iso(R \times \widetilde{PSL_2}(R))^\circ$.
(1) $A_i = B_i = 1$
(2) The rational Euler class is non zero.

Corollary. *(A) If $\Gamma \subset Iso(R^2 \times H^2)^\circ$, then Γ is an extension;*

$$1 \longrightarrow Z^2 \longrightarrow \Gamma \longrightarrow S \longrightarrow 1$$

such that
(1) the operator homomorphism $\theta : S \to Aut(Z^2)$ has finite image;
(2) $[\Gamma] \in H^2(S, Z^2)$ has finite order.

(B) If $\Gamma \subset Iso(R \times \widetilde{PSL_2}(R))^\circ$, then Γ is an extension;

$$1 \longrightarrow Z^2 \longrightarrow \Gamma \longrightarrow S \longrightarrow 1$$

such that
(1) the operator homomorphism is trivial;
(2) $[\Gamma] \in H^2(S, Z^2)$ has infinite order.

Now we shall consider the converse of Corollary. In other words, let Γ be a torsion free extension;

$$1 \longrightarrow Z^2 \longrightarrow \Gamma \longrightarrow S \longrightarrow 1$$

where S is a surface group of genus$\geqq 2$.

We have the following

Theorem 4. *Let Γ be such a group as above. Assume that*
(1) the operator homomorphism has finite image;
(2) $[\Gamma] \in H^2(S, Z^2)$ has finite order.
Then Γ embeds as a discrete subgroup of $Iso(R^2 \times H^2)^\circ$.

This follows from results in [J] (see Theorem 5.1 in [J]).

Theorem 5. *Let Γ be such a group as above. Assume that*
(1) the operator homomorphism is trivial;
(2) $[\Gamma] \in H^2(S, Z^2)$ has infinite order.
Then Γ embeds as a discrete subgroup of $Iso(R \times \widetilde{PSL_2(R)})^\circ$.

Proof. Note that Γ is an extension;

$$1 \longrightarrow Z^2 \longrightarrow \Gamma \longrightarrow S \longrightarrow 1$$

where Z^2 is the center of Γ. Let $i : Z^2 \to R^2$ be the inclusion. Since $[\Gamma]$ has infinite order, $i_*([\Gamma]) \in H^2(S, R^2)$ is not zero. We have the following exact sequences;

$$1 \longrightarrow R \longrightarrow Iso(\widetilde{PSL_2(R)})^\circ \xrightarrow{k} Iso(H^2)^\circ \longrightarrow 1$$

and

$$1 \longrightarrow R \times R \longrightarrow R \times Iso(\widetilde{PSL_2(R)})^\circ \longrightarrow Iso(H^2)^\circ \longrightarrow 1$$

where $R \times R$ is the center of $Iso(R \times \widetilde{PSL_2(R)})^\circ$.

Put $S' = k^{-1}(S)$ and $\bar{S} = R \times S'$. Then we have the following commutative diagram;

$$\begin{array}{ccccccccc}
1 & \longrightarrow & R^2 & \longrightarrow & \bar{S} & \longrightarrow & S & \longrightarrow & 1 \\
 & & {\scriptstyle pr}\big\downarrow & & {\scriptstyle pr}\big\downarrow & & {\scriptstyle =}\big\downarrow & & \\
1 & \longrightarrow & R & \longrightarrow & S' & \longrightarrow & S & \longrightarrow & 1
\end{array}$$

Clearly $pr_* : H^2(S, R^2) \to H^2(S, R)$ sends $[\bar{S}]$ to $[S']$. Since $H^2(S, R^2) \cong R^2$, there exists a linear isomorphism $\epsilon : R^2 \to R^2$ such that $\epsilon_* i_*[\Gamma] = [\bar{S}]$. Then we have the following commutative diagram;

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & Z^2 & \longrightarrow & \Gamma & \longrightarrow & S & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \epsilon\cdot i} & & \Big\downarrow & & \Big\downarrow & & \\
1 & \longrightarrow & R^2 & \longrightarrow & \bar{S} & \longrightarrow & S & \longrightarrow & 1 \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\
1 & \longrightarrow & R^2 & \longrightarrow & R \times Iso(\widetilde{PSL_2(R)})^\circ & \longrightarrow & Iso(H^2)^\circ & \longrightarrow & 1
\end{array}
$$

It follows from a result in [LR] that Γ is a discrete cocompact subgroup of $Iso(R \times \widetilde{PSL_2(R)}))^\circ$. $\square$

3.2 Polysurface group of type (2)

In this section, we shall consider polysurface group of type (2). As in subsection 3.2, the Lie group in which Γ is embeddable as a cocompact discrete subgroup is one of the followings;

$$Iso(H^4)^\circ, Iso(H^2 \times H^2)^\circ, Iso(R^2 \times H^2)^\circ, Iso(R \times \widetilde{PSL_2(R)})^\circ, Iso(R \times H^3)^\circ$$

We have the following

Lemma 2. *Let Γ be an extension;*

$$1 \longrightarrow S_1 \longrightarrow \Gamma \longrightarrow Z^2 \longrightarrow 1 \quad .$$

Then the aspherical manifold M realizing Γ has zero Euler characteristic.

Proof. Since $S_2 = Z^2$, then Γ is realized as the fundamental group of an aspherical manifold M. Clearly M is a fiber bundle over 2-dimensional torus T^2 with a surface as a fiber. Note that the bundle along the fiber is orientable. It follows from a result in [M] that the spectral sequence with rational coefficient collapses and hence the Euler characteristic $\chi(M)$ is zero.

First we shall show that Γ is not embedded in $Iso(H^4)^\circ$. Assume that $\Gamma \subset Iso(H^4)^\circ = O(4,1)^\circ$.

Note Γ is realized as the fundamental group of a 4-dimensional hyperbolic manifolds M. In this case, it follows from Lemma 2 that we have $\chi(M) = 0$, which contradicts a result in [K].

Assume $\Gamma \subset Iso(H^2 \times H^2)^\circ$. Since Γ is reducible (see Lemma 1), we have the following commutative diagram;

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & p_2(S_1) & \longrightarrow & p_2(\Gamma) & \longrightarrow & F & \longrightarrow & 1 \\
 & & \uparrow & & {\scriptstyle p_2}\uparrow & & \uparrow & & \\
1 & \longrightarrow & S_1 & \longrightarrow & \Gamma & \xrightarrow{p} & Z^2 & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & S_1 \cap PSL_2(R) & \longrightarrow & \Gamma_1 & \longrightarrow & p(\Gamma_1) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 1 & & 1 & & & &
\end{array}$$

where $\Gamma_1 = PSL_2(R) \cap \Gamma$ and the lower vertical maps are inclusion and Note that $PSL_2(R) \cap \Gamma$ and $p_2(\Gamma)$ are discrete and cocompact subgroups of $PSL_2(R)$.

On the other hand, M is finitely covered by a product $\Sigma_1 \times \Sigma_2$ of orientable surfaces of genus $\geqq 2$ and hence $\chi(M) \neq 0$, which contradicts Lemma 2.

We have the following

Theorem 6. *Assume Γ is embeddable in a non-compact Lie group G without compact factor as a cocompact discrete subgroup and Γ is type (2). Then we have*

(1) *G is one of the following*

$$Iso(R^2 \times H^2)^\circ, Iso(R \times H^3)^\circ$$

(2) *G is $Iso(R^2 \times H^2)^\circ$ if and only if the operator homomorphism $\theta : Z^2 \to Out(S)$ has finite image.*

(3) *G is $Iso(R \times H^3)^\circ$ if and only if the operator homomorphism $\theta : Z^2 \to Out(S)$ is not injective and infinite image.*

Proof. (1) Assume $G = Iso(R \times \widetilde{PSL_2(R)})^\circ$. We have the following commutative diagram;

$$\begin{array}{ccccccccc}
1 & \longrightarrow & R \times R & \longrightarrow & G & \xrightarrow{p} & Iso(H^2)^\circ & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & (R \times R) \cap \Gamma & \longrightarrow & \Gamma & \longrightarrow & p(\Gamma) & \longrightarrow & 1.
\end{array}$$

It is well known that $p(\Gamma)$ is a discrete cocompact subgroup of $Iso(H^2)^\circ$ and $(R \times R) \cap \Gamma \cong Z^2$ is central in Γ. We have the following commutative diagram;

$$\begin{tikzcd}
 & & 1 & 1 & \\
1 \arrow[r] & S \arrow[r] & p(\Gamma) \arrow[r] \arrow[u] & F \arrow[r] \arrow[u] & 1 \\
1 \arrow[r] & S \arrow[r] \arrow[u, "="] & \Gamma \arrow[r] \arrow[u] & Z^2 \arrow[r] \arrow[u] & 1, \\
 & 1 \arrow[r] \arrow[u] & Z^2 \arrow[r, "="] \arrow[u] & {Z^2}_{(1)} \arrow[r] \arrow[u] & 1 \\
 & & 1 \arrow[u] & 1 \arrow[u] &
\end{tikzcd}$$

where F is a finite group. It follows that the operator homomorphism $\theta : Z^2 \to Out(S)$ restricts trivial homomorphism ${Z^2}_{(1)} \to Out(S)$, i.e. θ has finite image. Hence Γ has a normal subgroup $Z^2 \times S$ of finite index. This implies $b_1(\Gamma)$ is even. This contradicts a result in [W], which states that the geometry $R \times Iso(\widetilde{PSL_2(R)})$ has odd first Betti number.

(2) If $\Gamma \subset Iso(R^2 \times H^2)^\circ$, then it is clear that the operator homomorphism has finite image. Assume that the operator homomorphism θ has finite image. Then, up to finite index, Γ contains Z^2 as a normal subgroup, which means that G is $Iso(R^2 \times H^2)^\circ$ by the Wall's classification in [W].

(3) Assume that $\Gamma \subset Iso(R \times H^3)^\circ$. It follows from a result in [W] (Prop. 10.4 in [W]) that Γ contains a normal free abelian group of rank 1 whose quotient is a lattice of $Iso(H^3)$. We have the following commutative diagram;

$$\begin{tikzcd}
 & 1 & 1 & 1 & \\
1 \arrow[r] & S \arrow[r] \arrow[u] & \Gamma_1 \arrow[r, "p"] \arrow[u] & Z \oplus F \arrow[r] \arrow[u] & 1 \\
1 \arrow[r] & S \arrow[r] \arrow[u, "="] & \Gamma \arrow[r] \arrow[u] & Z^2 \arrow[r] \arrow[u] & 1. \\
 & 1 \arrow[r] \arrow[u] & Z \arrow[r, "="] \arrow[u] & Z_{(1)} \arrow[r] \arrow[u] & 1 \\
 & & 1 \arrow[u] & 1 \arrow[u] &
\end{tikzcd}$$

This implies that the restriction of the operator homomorphism to the subgroup $Z_{(1)}$ is trivial. Since Γ contains no Z^2 as a normal subgroup, the operator homomorphism has infinite image. It is not difficult to show that if the operator homomorphism is not injective and has infinite image, then G is $Iso(R \times H^3)^\circ$.

This completes the proof of Theorem.

Remark:

(1) Let Γ be an abstract group defined by the following exact sequence;

$$1 \longrightarrow S \longrightarrow \Gamma \longrightarrow Z^2 \longrightarrow 1$$

where S is a surface group and the operator homomorphism of the sequence has finite image. Then it follows from a result in [J] (Theorem 6.3 in [J]) that Γ is embeddable in $Iso(R^2 \times H^2)^\circ$ as a cocompact discrete subgroup.

(2) We have the following Problem:

Problem Let Γ be a torsion free extension;

$$1 \longrightarrow S \longrightarrow \Gamma \longrightarrow Z^2 \longrightarrow 1,$$

where S is a surface group of genus $\geqq 2$, the operator homomorphism $\theta : Z^2 \to Out(S)$ is not injective and infinite image. Then is Γ embeddable in $Iso(R \times H^3)^\circ$?

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