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COVARIANCE IN BERNSTEIN'S INEQUALITY FOR OPERATORS

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ABSTRACT. Very recently, we discussed covariance in noncommutative probability based on Umegaki's idea, in which we pointed out the importance of the covariance-variance inequality. In this note, we examine Bernstein's inequality in the light of the covariance-variance inequality; we give improvements and generalizations of it.

1.Introduction. In [6], Furuta showed the following theorem which is an improvement of Bernstein's one in [1].

Theorem A. *If e is a unit eigenvector corresponding to an eigenvalue λ in a dominant operator A on a Hilbert space H , then*

$$|(g,e)|^2 \leq \frac{\|g\|^2\|Ag\|^2 - |(g,Ag)|^2}{\|(A-\lambda)g\|^2} \tag{1}$$

for all g in H for which $Ag \neq \lambda g$.

Here an operator A is called dominant if for each λ there is a real number $M_\lambda \geq 1$ such that $\|(A-\lambda)^*x\| \leq M_\lambda\|(A-\lambda)x\|$ for all x in H . We have to remark that $(A-\lambda)^*e = o$ under the dominance of A, that is, λ is a normal eigenvalue of A, i.e., there is a nonzero vector x in H such that $(A-\lambda)x = 0$ and $(A-\lambda)^*x = 0$. Under this consideration, we weakened the assumption of Theorem A to normality of the eigenvalue in [5]. More precisely,

Theorem B. *If e is a unit eigenvector corresponding to a normal eigenvalue λ of A on a Hilbert space H, then (1) holds for all g in H for which $Ag \neq \lambda g$.*

We also gave another generalization of Theorem A to normal approximate eigenvalues [2], i.e., a complex number λ is called a normal approximate eigenvalue of A if there exists a sequence $\{x_n\}$ of unit vectors such that $\|(A-\lambda)x_n\| \to 0$ and $\|(A-\lambda)^*x_n\| \to 0$.

Theorem C. *If $\{e_n\}$ is a sequence of unit vectors corresponding to a normal approximate eigenvalue λ of A, then*

$$\overline{\lim}\,|(g, e_n)|^2 \leq \frac{\|g\|^2\|Ag\|^2 - |(g, Ag)|^2}{\|(A-\lambda)g\|^2}$$

for all g in H for which $Ag \neq \lambda g$.

On the other hand, in [4] we recently discuss the variance and covariance of operators in the light of Umegaki's noncommutative probability [8]. Following J.I.Fujii's seminor talk, they are defined as follows: For a unit vector x and operators A, B

$$\mathrm{Cov}_x(A, B) = (A^*Bx, x) - (A^*x, x)(Bx, x) \tag{2}$$

and

$$\mathrm{Var}_x(A) = \|Ax\|^2 - |(Ax, x)|^2. \tag{3}$$

Since $\mathrm{Cov}(A, B)$ is a semi-inner product in the space of all operators on a Hilbert space, the Schwarz inequality implies the following covariance-variance inequality;

$$|\mathrm{Cov}(A, B)|^2 \leq \mathrm{Var}(A)\mathrm{Var}(B). \tag{4}$$

The covariance-variance inequality has many applications, e.g. the Kantorovich inequality, the Heinz-Kato-Furuta inequality [7] and the uncertainty principle [8].

In this note, we try to approach to Bernstein's inequality from the covariance-variance inequality; we give it improvements based on the covariance-variance inequality and discuss it in some general setting. For the latter, we introduce the sine of the covariance and the variance. As a matter of fact, we show that Pythagorean type theorem holds for the sine of the covariance, which includes Bernstein's inequality.

2. Results. We begin with the following improvement of Theorem B by the covariance variance inequality.

Theorem 1. *If e is a unit eigenvector corresponding to an eigenvalue $\overline{\lambda}$ of A^* on a Hilbert space H, then (1) holds for all g in H for which $Ag \neq \lambda g$.*

Proof. First of all, we note that the covariance is translation-invariant, i.e.,

$$\mathrm{Cov}(A - a, B - b) = \mathrm{Cov}(A, B)$$

for $a, b \in \mathbb{C}$, and so is the variance. We put $B = A - \lambda$ and may assume that $\|g\| = 1$. Now (1) can be rephrased as

$$(5) \qquad |(g,e)|^2 \|Bg\|^2 \leq \mathrm{Var}_g(B).$$

To prove (5), it suffices to take the projection E corresponding to the eigenvector e, i.e., $Ex = (x, e)e$ for $x \in H$. That is, we apply the covariance-variance inequality to E and B. Then we have

$$(6) \qquad |\mathrm{Cov}_g(E, B)|^2 \leq \mathrm{Var}_g(E)\mathrm{Var}_g(B).$$

Noting that $B^*e = 0$ by the assumption on λ, (6) is rewritten by

$$|(g,e)|^2 |(Bg, g)|^2 \leq \mathrm{Var}_g(B)(1 - |(g,e)|^2),$$

so that

$$|(g,e)|^2 \|Bg\|^2 = |(g,e)|^2 (|(Bg,g)|^2 + \mathrm{Var}_g(B)) \leq \mathrm{Var}_g(B),$$

as desired.

Remark. As seen in the proof above, we don't require that $Ae = \lambda e$, that is, λ is a normal eigenvalue of A with an eigenvector e. In addition, we cannot replace the assumption $A^*e = \overline{\lambda} e$ to the condition $Ae = \lambda e$. Actually we take, as a conterexample,

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \lambda = 2; g = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

It is easily checked that $(A-2)g \neq 0$ and $(g, e) \neq 0$, but

$$\|g\|^2 \|Ag\|^2 - |(Ag, g)|^2 = 0.$$

Next we generalize Bernstein's inequality (1). To do this, we introduce the sine of the covariance and the variance. For a unit vector x with $(A^*Bx, x) \neq 0$,

$$\mathrm{sCov}_x(A, B) = \frac{\mathrm{Cov}_x(A, B)}{(A^*Bx, x)}$$

and, for a unit vector x with $Ax \neq 0$,

$$\mathrm{sVar}_x(A) = \frac{\mathrm{Var}_x(A)}{\|Ax\|^2}.$$

Incidentally these definitions are available for arbitrary vectors x with suitable conditions $(A^*Bx, x) \neq 0$ or $Ax \neq 0$; we prepare the following definitions for these cases:

$$\mathrm{sCov}_x(A,B) = \frac{\|x\|^2(A^*Bx,x) - (A^*x,x)(Bx,x)}{(A^*Bx,x)}$$

and

$$\mathrm{sVar}_x(A) = \frac{\|x\|^2\|Ax\|^2 - |(Ax,x)|^2}{\|Ax\|^2}.$$

Since $|(Ax,x)|/\|Ax\|$ is regarded as the cosine between x and Ax, $\mathrm{sVar}_x(A)$ is the square of the sine between x and Ax. On the other hand, since $\mathrm{Cov}_x(A,B)$ is a semi-inner product, it may have Pythagorean properties. The following theorem can be understood from this viewpoint.

Theorem 2. *Let E be a projection such that $AE = EA = 0$ and $BE = EB = 0$. Then, for each $x \in H$*

$$\mathrm{sCov}_x(A,B) = \|Ex\|^2 + \mathrm{sCov}_{E^\perp x}(A,B). \tag{7}$$

In particular, if E is a projection such that $BE = EB = 0$, then

$$\mathrm{sVar}_x(B) = \|Ex\|^2 + \mathrm{sVar}_{E^\perp x}(B). \tag{8}$$

Proof. We put $y = E^\perp x$. Then we have

$$\begin{aligned}\mathrm{sCov}_x(A,B) &= \frac{\|x\|^2(A^*Bx,x) - (A^*x,x)(Bx,x)}{(A^*Bx,x)}\\ &= \frac{(\|Ex\|^2 + \|y\|^2)(A^*By,y) - (A^*y,y)(By,y)}{(A^*By,y)}\\ &= \|Ex\|^2 + \mathrm{sCov}_y(A,B).\end{aligned}$$

Remark. The above (8) also implies Theorem B. We keep the notations as in the proof of Theorem 1. Then we have $BE = EB = 0$ by the asumption. Since $\mathrm{sVar}_y(B)$ is nonnegative for all y, we obtain Theorem B.

Following our previous note [5], we finally give an improvement of Theorem C:

Theorem 3. *If $\{e_n\}$ is a sequence of unit vectors corresponding to an approximate eigenvalue $\overline{\lambda}$ of A^*, then*

$$\overline{\lim}\, |(g,e_n)|^2 \leq \frac{\|g\|^2\|Ag\|^2 - |(g,Ag)|^2}{\|(A-\lambda)g\|^2} \tag{9}$$

for all g in H for which $Ag \neq \lambda g$.

Proof. By a similar way to [5; Theorem 3], the proof is reduced to Theorem 1 via the Berberian representation. For the sake of convenience, we sketch it below.

For the sequence $\{|(g, e_n)|\}$, there is a generalized limit Lim such that

$$\mathrm{Lim}|(g, e_n)|^2 = \overline{\lim}|(g, e_n)|^2.$$

The Berberian representation $A \longrightarrow A^\circ$ is induced by Lim as follows, see [5]: The vector space V of all bounded sequences in H has a semi-inner product $< x^\circ, y^\circ >=$ $\mathrm{Lim}(x_n, y_n)$, so that a Hilbert space H° is given by the completion of V/N, where $N = \{x \in V; < x, y >= 0 \text{ for all } y \in V\}$. For an operator A on H, A° is defined by

$$A^\circ(\{x_n\} + N) = \{Ax_n\} + N.$$

Then it is known that it is an isometric *-isomorphism and converts the approximate eigenvalues of A to the eigenvalues of A°.

By the Berberian representation, we now obtain that

$$A^{*\circ}e^\circ = \overline{\lambda}e^\circ \quad \text{and} \quad | < g^\circ, e^\circ > |^2 = \overline{\lim}|(g, e_n)|^2,$$

where g° is the canonical embedding of g into H° and $e^\circ = \{e_n\} + N$. Hence it follows from Theorem 1 that

$$\begin{aligned}\overline{\lim}|(g, e_n)|^2 = | < g^\circ, e^\circ > |^2 &\leq \frac{\|g^\circ\|^2\|A^\circ g^\circ\|^2 - | < A^\circ g^\circ, g^\circ > |^2}{\|(A^\circ - \lambda)g^\circ\|^2} \\ &= \frac{\|g\|^2\|Ag\|^2 - |(Ag, g)|^2}{\|(A - \lambda)g\|^2}.\end{aligned}$$

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