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The adjacency operators of the infinite directed graphs and the von Neumann algebras generated by partial isometries

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1. Introduction.

A directed graph $G = (V, E)$ is a pair of countable sets V and E. An element $v \in V$ is called a vertex and an element $(v, u) \in E$ is called an arc with an initial vertex v and a terminal vertex u. For each vertex $v \in V$, the outdegree $d^+(v)$, the indegree $d^-(v)$ and the valency $d(v)$ are defined as follows ;

$$d^+(v) = |\{(v,u)\ ;\ (v,u) \in E\}|,\ d^-(v) = |\{(u,v)\ ;\ (u,v) \in E\}|$$

and

$$d(v) = d^+(v) + d^-(v)$$

respectively where $|\{\cdot\}|$ means the cardinal number of a set $\{\cdot\}$. A graph has bounded valency if there is a constant $\alpha > 0$ such that $d(v) \leqq \alpha$ for every vertex $v \in V$. Throughout this paper, we assume that a graph is a directed graph without multiple arcs and has bounded valency.

Mohar defined an adjacency operator for infinite undirected graphs in [2], and Fujii, Sasaoka and Watatani defined one for infinite graphs in [1]. An adjacency operator is in general unbounded, but we treat only bounded adjacency operator under the our assumption that a graph has bounded valency.

Let H be the Hilbert space $\ell^2(V)$ with the canonical basis $\{e_v; v \in V\}$ defined by $e_v(u) = \delta_{v,u}$ for $u, v \in V$.

Let us define a closed operator $A = A(G)$ with the domain $D(A)$ by

$$D(A) = \left\{ x = \sum_{v \in V} x_v e_v \in H\ ;\ \sum_{u \in V} |\sum_{(v,u) \in E} x_v|^2 < \infty \right\}$$

$$Ax = \sum_{u \in V} \sum_{(v,u) \in E} x_v e_u \quad \text{for } x \in D(A).$$

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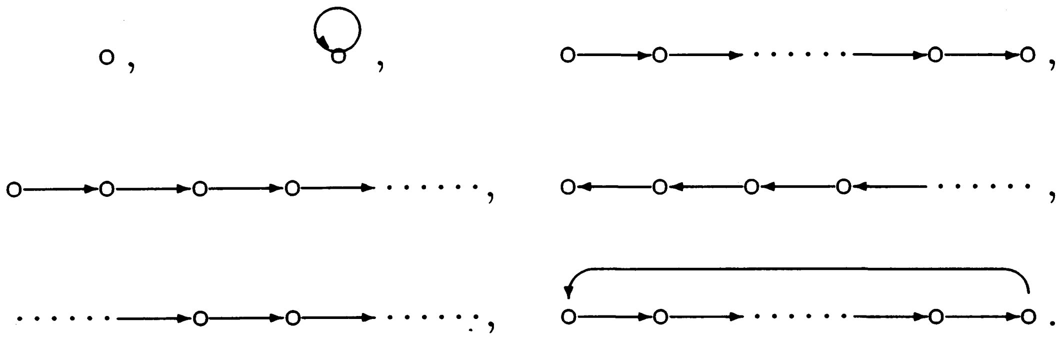
Now, we call $A = A(G)$ the adjacency operator of G. Then we have the following lemma [1 ; Theorem 2].

Lemma 1. *Let A be the adjacency operator of a graph G. Then, A is bounded if and only if G has bounded valency.*

We firstly show some properties of an adjacency operator which is a partial isometry.

An operator T on a Hilbert space K is a partial isometry if A^*A and AA^* are projections. In [1], a characterization that an adjacency operator is a partial isometry was given as follows :

Lemma 2. *Let A be the adjacency operator of a graph G. Then, A is a partial isometry if and only if the connected components of G are one of the following ;*



2. Results.

An operator T on K is called a power partial isometry if T^m is a partial isometry for $m = 1, 2, \cdots$.

If we treat the adjacency operator $A = A(G)$ of an infinite directed graph G and A is a partial isometry, then we can show by considering Lemma 2 that A is a power partial isometry.

Proposition 3. *Let A be the adjacency operator of a graph G. If A is a partial isometry, then A is a power partial isometry.*

Proof. By Lemma 2, the operator A is represented by the direct sum in the following ;

$$A = \sum_{n=1}^{\infty} \oplus U_n \quad \text{on} \quad H = \sum_{n=1}^{\infty} \oplus H_n$$

where U_n is an operator satisfying one of the following conditions ;

(1) $U_n = 0$ on H_n,

(2) $U_n =$ the identity on H_n,

(3) $\dim H_n = s < \infty$ and $\{e_{n(k)}\}_{k=1}^{s}$ is a basis for H_n, and then

$$U_n e_{n(k)} = e_{n(k+1)} \quad (1 \leqq k \leqq s-1) \quad \text{and} \quad U_n e_{n(s)} = 0,$$

(4) $\dim H_n = s < \infty$ and $\{e_{n(k)}\}_{k=1}^{s}$ is a basis for H_n, and then

$$U_n e_{n(k)} = e_{n(k+1)} \quad (1 \leqq k \leqq s-1) \quad \text{and} \quad U_n e_{n(s)} = e_{n(1)},$$

(5) $\dim H_n = \infty$ and $\{e_{n(k)}\}_{k=1}^{\infty}$ is a completely orthonormal basis for H_n, and then

$$U_n e_{n(k)} = e_{n(k+1)} \quad (k = 1,\ 2,\ \cdots),$$

(6) $\dim H_n = \infty$ and $\{e_{n(k)}\}_{k=1}^{\infty}$ is a completely orthonormal basis for H_n, and then

$$U_n e_{n(1)} = 0 \quad \text{and} \quad U_n e_{n(k)} = e_{n(k-1)} \quad (k = 2,\ 3,\ \cdots),$$

(7) $\dim H_n = \infty$ and $\{e_{n(k)}\}_{k=-\infty}^{\infty}$ is a completely orthonormal basis for H_n, and then

$$U_n e_{n(k)} = e_{n(k+1)} \quad (k \in Z).$$

Then, each subspace H_n reduces A.

In the case of (1) (resp. (2)), $U_n^m = 0$ (resp. $U_n^m =$ the identity) on H_n $(m = 1,\ 2,\ \cdots)$. And so, U_n is a power partial isometry on H_n.

In the case of (3), $U_n^m e_{n(k)} = e_{n(k+m)}$ $(1 \leqq k \leqq s-m)$ and $U_n^m e_{n(k)} = 0$ $(s-m \leqq k \leqq s)$. And so U_n is a power partial isometry on H_n.

In the case of (4) and (7), since U_n is a unitary operator on H_n, U_n is a power partial isometry on H_n.

Furthermore, in the cases of (5) and (6), since U_n is a unilateral shift on H_n, U_n is a power partial isometry on H_n.

Since each subspace H_n reduces A, A is a power partial isometry on H by the above mentioned arguments. This gives the proof of Proposition 3.

As a property for the power partial isometries, Saito [5] showed the following result :

Let T be a power partial isometry on a Hilbert space K which is quasi-nilpotent. Then the von Neumann algebra $M(T)$ generated by T is of type I where $M(T)$ means the von Neumann algebra generated by T and the identity I.

Furthermore, Saito denoted as a remark in [5] that the type of von Neumann algebra generated by a general power partial isometry may be unknown.

For this remark, we can give an answer for a power partial isometry which is not necessarily quasi-nilpotent.

In particular, the following theorem is an extension of result obtained by Saito [5 ; Theorem 3].

Let U be an operator on a Hilbert space K. Then U is called a truncated shift of index n ($n = 1, 2, \cdots$) if U is the operator such that K is the n-fold direct sum $K = K_0 \oplus K_0 \oplus \cdots \oplus K_0$, and $U = 0$ if $n = 1$ and

$$U\langle f_1, f_2, \cdots, f_n\rangle = \langle 0, f_1, f_2, \cdots, f_{n-1}\rangle$$

if $n > 1$. Then, Saito showed the following result [5 ; Theorem 3].

Let T be the operator represented by the finite direct sum of truncated shifts

$$T = \sum_{k=1}^{r} \oplus U_{n(k)} \quad (1 \leq n(1) < n(2) < \cdots < n(r))$$

where $U_{n(k)}$ is a truncated shift of index $n(k)$. Then the von Neumann algebra $M(T)$ generated by T is of type I.

Even if T is the operator represented by the infinite direct sum of truncated shifts, we show in the following theorem that the von Neumann algebra $M(T)$ is of type I.

Theorem 4. *Let T be the operator acting on a Hilbert space K represented by the infinite direct sum of truncated shifts*

$$T = \sum_{k=1}^{\infty} \oplus U_{n(k)} \quad (1 \leq n(1) < n(2) < \cdots < n(k) < \cdots)$$

where $U_{n(k)}$ is a truncated shift of index $n(k)$. Then the von Neumann algebra $M(T)$ generated by T is of type I.

Proof. Let T be the infinite direct sum of truncated shifts

$$T=\sum_{k=1}^{\infty}\oplus U_{n(k)} \quad \text{acting on} \quad K=\sum_{k=1}^{\infty}\oplus K_{n(k)}$$

where each $K_{n(k)}$ is the $n(k)$-fold direct sum $K_{n(k)}=K_0^{(k)}\oplus K_0^{(k)}\oplus\cdots\oplus K_0^{(k)}$. Then, every $K_{n(k)}$ reduce T $(k=1,\ 2,\ \cdots)$. Let $E^{(k)}$ be the projection of K onto $K_{n(k)}$, then $E^{(k)}$ is an element of the commutant $M(T)'$ of $M(T)$. Since

$$T^{*^{n(1)-1}}T^{n(1)-1}=(E^{(1)}-E_1^{(1)})\oplus(E_{12}^{(2)}\oplus O_{11}^{(2)})\oplus\cdots\oplus(E_{12}^{(k)}\oplus O_{11}^{(k)})\oplus\cdots,$$

where $E_1^{(1)}$ is the projection on $K_0^{(1)}\oplus O_0^{(1)}\oplus O_0^{(1)}\oplus\cdots\oplus O_0^{(1)}$ and each $O_{11}^{(k)}$ is the zero operator on the $(n(1)-1)$-fold direct sum

$$(O_0^{(k)}\oplus O_0^{(k)}\oplus\cdots\oplus O_0^{(k)}\oplus)\overbrace{K_0^{(k)}\oplus\cdots\oplus K_0^{(k)}}^{n(1)-1}$$

and $E_{12}^{(k)}$ is the projection on the $(n(k)-n(1)+1)$-fold direct sum

$$\overbrace{K_0^{(k)}\oplus\cdots\oplus K_0^{(k)}}^{n(k)-n(1)+1}(\oplus O_0^{(k)}\oplus O_0^{(k)}\oplus\cdots\oplus O_0^{(k)}) \quad (k=2,\ 3,\ \cdots),$$

$$I-T^{*^{n(1)-1}}T^{n(1)-1}=E_1^{(1)}\oplus(O_{12}^{(2)}\oplus E_{11}^{(2)})\oplus\cdots\oplus(O_{12}^{(k)}\oplus E_{11}^{(k)})\oplus\cdots$$

is an element of $M(T)$ where $O_{12}^{(k)}$ is the zero operator on the $(n(k)-n(1)+1)$-fold direct sum

$$\overbrace{K_0^{(k)}\oplus\cdots\oplus K_0^{(k)}}^{n(k)-n(1)+1}(\oplus O_0^{(k)}\oplus O_0^{(k)}\oplus\cdots\oplus O_0^{(k)})$$

and $E_{11}^{(k)}$ is the projection of K onto the $(n(1)-1)$-fold direct sum

$$(O_0^{(k)}\oplus O_0^{(k)}\oplus\cdots\oplus O_0^{(k)}\oplus)\overbrace{K_0^{(k)}\oplus\cdots\oplus K_0^{(k)}}^{n(1)-1}.$$

Furthermore, since

$$TT^* = (E^{(1)} - E_1^{(1)}) \oplus (E^{(2)} - E_1^{(2)}) \oplus \cdots \oplus (E^{(k)} - E_1^{(k)}) \oplus \cdots$$

where $E_1^{(k)}$ is the projection on

$$K_0^{(k)} \oplus \overbrace{O_0^{(k)} \oplus O_0^{(k)} \oplus \cdots \oplus O_0^{(k)}}^{n(k)-1} \quad (k = 1,\ 2,\ 3,\ \cdots),$$

$$I - TT^* = E_1^{(1)} \oplus E_1^{(2)} \oplus E_1^{(3)} \oplus \cdots \oplus E_1^{(k)} \oplus \cdots$$

is an element of $M(T)$. Thus,

$$\begin{aligned}
&(I - T^{*^{n(1)-1}} T^{n(1)-1})(I - TT^*) \\
&= \{E_1^{(1)} \oplus (O_{12}^{(2)} \oplus E_{11}^{(2)}) \oplus \cdots \oplus (O_{12}^{(k)} \oplus E_{11}^{(k)}) \oplus \cdots\} \\
&\qquad \cdot\{E_1^{(1)} \oplus E_1^{(2)} \oplus E_1^{(3)} \oplus \cdots \oplus E_1^{(k)} \oplus \cdots\} \\
&= E_1^{(1)}
\end{aligned}$$

is an element of $M(T)$. By applying a similar argument for

$$I - T^{*^{n(1)-2}} T^{n(1)-2} \quad \text{and} \quad I - T^2 T^{*^2},$$

we can show that $E_1^{(1)} + E_2^{(1)}$ is an element of $M(T)$ and so $E_2^{(1)}$ is also an element of $M(T)$ where $E_s^{(k)}$ is the projection on

$$O_0^{(k)} \oplus \cdots \oplus O_0^{(k)} \oplus \underbrace{K_0^{(k)}}_{s} \oplus O_0^{(k)} \oplus \cdots \oplus O_0^{(k)} \quad (k = 1,\ 2,\ 3,\ \cdots).$$

Continuing this process, we can show that $E_1^{(1)},\ E_2^{(1)}, \cdots, E_{n(1)}^{(1)}$ are elements of $M(T)$ and so $E^{(1)}$ is an element of $M(T)$. Hence, $E^{(1)}$ is a central element of $M(T)$. Next, applying the above process for

$$I - T^{*^{n(2)-1}} T^{n(2)-1} \quad \text{and} \quad I - TT^*,$$

$$I - T^{*^{n(2)-2}} T^{n(2)-2} \quad \text{and} \quad I - T^2 T^{*^2}, \quad \cdots$$

since $E_1^{(1)},\ E_2^{(1)}, \cdots, E_{n(1)}^{(1)}$ are elements of $M(T)$ and $E^{(1)}$ is a central element of $M(T)$, we can show that $E_1^{(2)},\ E_2^{(2)}, \cdots, E_{n(2)}^{(2)}$ are elements of $M(T)$ and so

$E^{(2)}$ is a central element of $M(T)$. Continuing this process, we can show that $E_1^{(k)}, E_2^{(k)}, \cdots, E_{n(k)}^{(k)}$ are elements of $M(T)$ and so $E^{(k)}$ is an element of $M(T)$ $(k=1, 2, \cdots)$. Therefore, since every $U_{n(k)}$ is a truncated shift, $M(U_{n(k)})$ is a von Neumann algebra of type $\mathrm{I}_{n(k)}$ and so $M(T)$ is a von Neumann algebra of type I. Thus, we have the complete proof of Theorem 4.

Theorem 5. *Let A be the adjacency operator of a graph G. If A is a partial isometry, then the von Neumann algebra $M(A)$ generated by A is of type* I.

Proof. By Lemma 2 and the proof of Proposition 3, we can assume that A has a representation of the direct sum in the following ;

$$A = \sum_{n=1}^{6} \oplus U_n \quad \text{on} \quad H = \sum_{n=1}^{6} \oplus H_n$$

such that

(1) $U_1 = 0$ on H_1,

(2) $U_2 =$ the identity on H_2,

(3) U_3 is the finite or infinite direct sum of truncated shifts with the different indices on H_3,

(4) H_4 and H_5 are the infinite dimensional subspaces and U_4 (resp. U_5) is a unilateral shift (resp. a backward shift) on H_4 (resp. H_5),

(5) U_6 is a unitary operator on H_6.

Let E_n be the projection of H onto H_n. Then, since each H_n reduces A, E_n is an element of the commutant $M(A)'$ of $M(A)$. Furthermore, we show that each E_n is an element of $M(A)$ and so E_n is a central element of $M(A)$.

Since

$$E_1 = I - \sum_{n=2}^{6} \oplus U_n^* U_n \quad \text{and} \quad E_2 = (I - \sum_{n=3}^{6} \oplus U_n^* U_n) - E_1,$$

E_1 and E_2 are elements of $M(A)$.

Now, since $U_3^{*m} U_3^m$ and $U_5^{*m} U_5^m$ (resp. $U_4^{*m} U_4^m$ and $U_6^{*m} U_6^m$) converge to 0 as $m \to \infty$ in the weak topology (resp. equal to the identity for every $m = 1, 2, \cdots$ on H_4 and H_6 respectively), $E_4 + E_6$ is an element of $M(A)$ and so $E_3 + E_5$ is also an element of $M(A)$.

Furthermore, $U_5^m U_5^{*m} = E_5$ for every $m = 1, 2, \cdots$ and $U_3^m U_3^{*m} \to 0$ as $m \to \infty$ in the weak topology. Thus, E_5 and so E_3 are elements of $M(A)$. By a similar argument, since $U_6^m U_6^{*m} = E_6$ for every $m = 1, 2, \cdots$ and $U_4^m U_4^{*m} \to 0$ as $m \to \infty$ in the weak topology. Thus, E_6 and so E_4 are elements of $M(A)$.

Since $M(AE_1) = \mathbb{C}E_1$, $M(AE_2) = \mathbb{C}E_2$ and $M(AE_6)$ acting on H_1, H_2 and H_6 respectively are abelian, these von Neumann algebras are of type I where $\mathbb{C}$ is the complex number field. Furthermore, since U_4 and U_5 are unilateral shift, $M(AE_4)$ and $M(AE_5)$ are von Neumann algebras of type I on H_4 and H_5 respectively. And furthermore, the von Neumann algebra $M(AE_3)$ acting on H_3 is of type I by Theorem 4. Therefore,

$$M(A) = \sum_{n=1}^{6} \oplus M(AE_n)$$

is a von Neumann algebra of type I. We get the complete proof of Theorem 5.

3. Remarks and Example.

We showed in the previous section the following ; if the adjacency operator A of a graph is a partial isometry, then the von Neumann algebra generated by A is of type I. Thus, we have a following remark.

Remark 1. Saito showed in [4] that for a certain von Neumann algebra M the following properties are equivalent ;

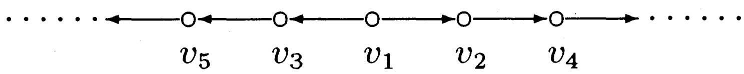
(a) M has a single generator.

(b) M is generated by one partial isometry.

Precisely certain von Neumann algebra mentioned above means an AF-von Neumann algebra, of type II_1, of type II_∞ or type III. But, if we consider a generator by the restriction in the adjacency operators of a graph, then we can't get a similar result to Saito's result by our Theorem 5.

Next, we shall give an example in which we consider a generator of von Neumann algebra of type I except the partial isometries.

Example. We consider an example of graph $G = (V, E)$ as below ;



Put e_n the element e_{v_n} of the Hilbert space $H = \ell^2(V)$, then the adjacency operator $A = A(G)$ with respect to G is determined by the following ;

$$e_1 \to e_2 + e_3, \quad e_{2n} \to e_{2(n+1)}, \quad e_{2n+1} \to e_{2(n+1)+1} \quad (n \geqq 1).$$

Thus, the operators A^*A and AA^* are of the following form ;

$$A^*A \ : \ e_1 \to 2e_1, \quad e_n \to e_n \quad (n \geqq 2)$$

and

$$AA^* \ : \ e_1 \to 0, \quad e_2 \to e_2 + e_3, \quad e_3 \to e_2 + e_3, \quad e_n \to e_n \quad (n \geqq 4).$$

And so A is not a partial isometry.

Now, define the projections $P_{[e_n]}$, $P_{[e_n+e_{n+1}]}$, $P_{[e_n,e_{n+1}]}$, $P_{[e_n,e_{n+1},e_{n+2},\cdots]}$ (resp.) on the one dimensional subspace $[e_n]$ and $[e_n + e_{n+1}]$, the two dimensional subspace $[e_n, e_{n+1}]$ and the infinite dimensional subspace $[e_n, e_{n+1}, e_{n+2}, \cdots]$ (resp.). Then $A^*A = 2P_{[e_1]} + P_{[e_2,e_3,e_4,\cdots]}$ and $AA^* = P_{[e_2,e_3]} + P_{[e_4,e_5,e_6,\cdots]}$. Furthermore, since $(A^*A)^2 = 4P_{[e_1]} + P_{[e_2,e_3,e_4,\cdots]}$, $P_{[e_1]}$ and $P_{[e_2,e_3,e_4,\cdots]}$ are elements of $M(A)$. By a similar way, we can show that $P_{[e_2,e_3]}$ and $P_{[e_4,e_5,e_6,\cdots]}$ are elements of $M(A)$.

On the other hand, since

$$\left(\frac{1}{\sqrt{2}}AP_{[e_1]}\right)^* \left(\frac{1}{\sqrt{2}}AP_{[e_1]}\right) = \frac{1}{2}AP_{[e_1]}A^* = P_{[e_2+e_3]},$$

$P_{[e_2+e_3]}$ is an element of $M(A)$ and $\dfrac{1}{\sqrt{2}}AP_{[e_1]}$ is a partial isometry with the initial projection $P_{[e_1]}$ and the final projection $P_{[e_2+e_3]}$. Thus, the projection $P_{[e_2-e_3]}$ is also an element of $M(A)$ because $P_{[e_2,e_3]}$ and $P_{[e_2+e_3]}$ are elements of $M(A)$. Now, since we can show by an elementary computation that $P_{[e_2+e_3]}$ is not a central element of the von Neumann algebra $P_{[e_2,e_3]}M(A)P_{[e_2,e_3]}$, $M(A)$ contains the matrix units with respect to $\{P_{[e_2+e_3]}, P_{[e_2-e_3]}\}$ and so with respect to $\{P_{[e_2]}, P_{[e_3]}\}$. Hence, $M(A)$ contains the matrix units with respect to $\{P_{[e_1]}, P_{[e_2]}, P_{[e_3]}\}$. By repeating a similar argument, we can show that $M(A)$ contains the matrix units with respect to $\{P_{[e_1]}, P_{[e_2]}, P_{[e_3]}, P_{[e_4]}, P_{[e_5]}\}$, $\cdots\cdots$, $\{P_{[e_1]}, \cdots, P_{[e_{2n}]}, P_{[e_{2n+1}]}\}$. Thus, we can get the property $M(A) = B(H)$ and so the conclusion of this example.

Remark 2. We showed the generator of von Neumann algebra of type I in the previous part. But we don't know whether an adjacency operator generates a von Neumann algebra of type II or type III. This is a problem that we must consider after this. In general case, Pearcy [3] firstly showed that there exists a partial isometry V (resp. W) generating a von Neumann algebra of type II_1 (resp. II_∞). But, if we consider this Pearcy's result in the adjacency operators, we can't expect a similar result by considering our previous result (i.e. Theorem 5).

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