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ON SOME CR SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR FIELD IN A COMPLEX SPACE FORM

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Abstract We study CR submanifolds with nonvanishing parallel mean curvature vector field immersed in a complex space form.

Introduction One of typical submanifolds of a Kaehlerian manifold is the so-called *CR submanifolds* which are defined as follows: Let M be a submanifold of a Kaehlerian manifold $\tilde{M}$ with almost complex structure J. If there is a differentiable distribution such that it is invariant and the complementary orthogonal distribution is totally real (cf. [1], [2]). Especially, if each normal space of M is mapped into the tangent space under the action of J, M is called a *generic* submanifold of $\tilde{M}$. Real hypersurface of a Riemannian manifold are the most typical example of the generic submanifold ([13]).

Many subjects for CR submanifold were investigated from various different points of view. In [1, 2, 3, 4, 11] Bejancu, Chen, Kon and Yano studied basic properties of CR submanifolds M in a Kaehlerian manifold. In particular, under the assumptions that the second fundamental forms are commutative with the f-structure induced in the tangent bundle, some characterizations and some classifications of CR submanifolds with parallel mean curvature vector field in a complex space form were obtained (sec [7, 8, 9, 10]).

The purpose of the present paper is to study CR submanifolds of a complex space form with nonvanishing parallel mean curvature vector field under the assumption that the shape operator in the direction of the mean curvature vector field is commutative with the f-structure induced in the tangent bundle.

1. Preliminaries

Let $\tilde{M}$ be a Kaehlerian manifold of real dimension $2m$ equipped with an almost complex structure J and a Hermitian metric tensor G. Then for any vector fields X and Y on $\tilde{M}$, we have

$$J^2X = -X, \quad G(JX, JY) = G(X, Y), \quad \tilde{\nabla} J = 0,$$

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where $\tilde{\nabla}$ denotes the Riemannian connection of $\tilde{M}$.

Let M be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$ and isometrically immersed in $\tilde{M}$ by the immersion $i : M \longrightarrow \tilde{M}$. When the argument is local, M need not be distinguished from $i(M)$ itself. Throughout this paper the indices $i, j, k, \cdots$ run from 1 to n. We represent the immersion i locally by

$$y^A = y^A(x^h), \quad (A = 1, \cdots, n, \cdots, 2m)$$

and put $B_j{}^A = \partial_j y^A$, $(\partial_j = \partial/\partial x^j)$ then $B_j = (B_j{}^A)$ are n-linearly independent local tangent vector fields of M. We choose $2m - n$ mutually orthogonal unit normals $C_x = (C_x{}^A)$ to M. Hereafter the indices $u, v, w, x, \cdots$ run from $n+1$ to $2m$ and the summation convention will be used. The immersion being isometric, the induced Riemannian metric tensor g with components g_{ji} and the metric tensor δ with components δ_{yx} of the normal bundle are respectively obtained:

$$g_{ji} = G(B_j, B_i), \quad \delta_{yx} = G(C_y, C_x).$$

By denoting ∇_j the operator of wan der Waerden-Bortolotti covariant differentiation with respect to g and G, the equations of Gauss and Weingarten for the submanifold M are respectively given by

$$\nabla_j B_i = A_{ji}{}^x C_x, \quad \nabla_j C_x = -A_j{}^h{}_x B_h, \tag{1.1}$$

where $A_{ji}{}^x$ are components of the second fundamental tensors and the shape operator A^x in the direction of C_x are related by

$$A^x = (A_j{}^{hx}) = (A_{jiy} g^{ih} \delta^{yx}), \quad g^{ji} = (g_{ji})^{-1}.$$

A submanifold M of a Kaehlerian manifold $\tilde{M}$ is called CR submanifold of $\tilde{M}$ if there exists a differentiable distribution $D : x \longrightarrow D_x \subset T_x(M)$ on M satisfying the following condition (see [1],[4],[12]):

(1) D is invariant with respect to J, namely, $JD_x = D_x$ for each point x in M, and

(2) the complementary orthogonal distribution $D^\perp : x \longrightarrow D_x^\perp \subset T_x(M)$ is totally real with respect to J, namely, $JD_x^\perp \subset T_x(M)^\perp$ for each point x in M.

We put $\dim D = h$, $\dim D^\perp = p$ and $\operatorname{codim} M = 2m - n = q$. If $p = 0$, then a CR submanifold M is called an invariant submanifold of $\tilde{M}$, and if $h = 0$, then M is called a totally real submanifold of $\tilde{M}$. If $p = q$, then a CR submanifold M is a generic submanifold of $\tilde{M}$ (see [11],[12]).

In the following, we suppose that M is a CR submanifold of a Kaehlerian manifold $\tilde{M}$. Then the transforms of B_i and C_x by J are respectively represented in each coordinte neighborhood as follows:

$$JB_j = f_j{}^h B_h - J_j{}^x C_x, \quad JC_x = J_x{}^h B_h + Q_x{}^y C_y, \tag{1.2}$$

where we have put $f_{ji} = G(JB_j, B_i)$, $J_{jx} = -G(JB_j, C_x)$, $J_{xj} = G(JC_x, B_j)$, $f_j{}^h = f_{ji}g^{ih}$ and $J_j{}^x = J_{jy}\delta^{yx}$. From these definitions we verify that $f_{ji}+f_{ij} = 0$ and $J_{jx} = J_{xj}$.

By the properties of the Kaehlerian structure tensor, it follows from (1.2) that

$$f_j{}^t f_t{}^h = -\delta_j^h + J_j{}^x J_x{}^h, \tag{1.3}$$

$$Q_x{}^y Q_y{}^z = -\delta_x^z + J_x{}^t J_t{}^z, \tag{1.4}$$

$$f_{jt}J^{tz} = 0, \quad J_j{}^x Q_x{}^z = 0, \tag{1.5}$$

$$f^3 + f = 0, \quad Q^3 + Q = 0. \tag{1.6}$$

Equations in (1.6) show that f is an f-structure in M and Q is an f-structure in the normal bundle of M.

Differentiating (1.2) covariantly along M and making use of (1.1) and these equations, we easily find (cf. [10])

$$\nabla_j f_i{}^h = A_{ji}{}^x J_x{}^h - A_j{}^h{}_x J_i{}^x, \tag{1.7}$$

$$\nabla_j J_i{}^x = A_{jt}{}^x f_i{}^t - A_{ji}{}^y Q_y{}^x, \tag{1.8}$$

$$\nabla_j Q_y{}^x = A_{jty} J^{tx} - A_{jt}{}^x J_y{}^t. \tag{1.9}$$

We denote by $\tilde{M}^m(c)$ a 2m-dimensional complex space form of constant holomorphic setional curvature c. Then equations of the Gauss, Codazzi and Ricci of M are given respectively by

$$R_{kjih} = \frac{c}{4}(g_{kh}g_{ji} - g_{jh}g_{ki} + f_{kh}f_{ji} - f_{jh}f_{ki} - 2f_{kj}f_{ih}) + A_{kh}{}^x A_{jix} - A_{jh}{}^x A_{kix}, \tag{1.10}$$

$$\nabla_k A_{ji}{}^x - \nabla_j A_{ki}{}^x = \frac{c}{4}(J_j{}^x f_{ki} - J_k{}^x f_{ji} - 2J_i{}^x f_{kj}), \tag{1.11}$$

$$R_{jiyx} = \frac{c}{4}(J_{jx}J_{iy} - J_{ix}J_{jy} - 2f_{ji}Q_{yx}) + A_{jtx}A_i{}^t{}_y - A_{itx}A_j{}^t{}_y, \tag{1.12}$$

where R_{kjih} and R_{jiyx} are components of the Riemannian curvature tensor of M and those with respect to the connection induced in the normal bundle, respectively.

2. Parallel mean curvature vector field

In this section we prepare some lemmas for later use.

Let M be an n-dimensional CR submanifold in a complex space form $\tilde{M}^{2m}(c)$. A normal vector field $\xi = (\xi^x)$ is called a *parallel section* in the normal bundle if it satisfies $\nabla_j \xi^x = 0$, and furthermore a tensor field T on M is said to be *parallel* in the normal bundle if it is in the normal bundle and $\nabla_j T$ vanishes identically.

In the following, we suppose that the f-structure Q in the normal bundle is parallel. Then (1.9) turns out to be

$$A_{jry}J^{rx} = A_{jr}{}^x J_y{}^r. \tag{2.1}$$

Remark. Notice that Q vanishes identically if M is a generic submanifold of a Kaehlerian manifold $\tilde{M}$. Thus, a generic submanifold of $\tilde{M}$ has always a parallel f-structure in the normal bundle.

Let H be a mean curvature vector field of a CR submanifold M. Namely, it is defined by

$$H = \frac{1}{n} g^{ji} A_{ji}{}^x = \frac{1}{n} h^x C_x,$$

which is independent of the choice of the local field of orthonormal frames $\{C_x\}$.

Suppose that the mean curvature vector field H of M is nonzero and is parallel in the normal bundle. Then we may choose a local field $\{e_x\}$ in such a way that $H = \sigma C_{n+1} = \sigma C_*$, where $\sigma = |H|$ is nonzero constant. Because of the choice of the local field,the parallelism of H yields

$$\begin{cases} h^x = 0, & x \geq n+2 \\ h^* = n|H|. \end{cases} \tag{2.2}$$

Then by (1.10), the Ricci tensor S of M is given by

$$S_{ji} = \frac{c}{4}\{(n+2)g_{ji} - 3J_j{}^z J_{iz}\} + h^* A_{ji}{}^* - A_{jrx}A_i{}^{rx}. \tag{2.3}$$

Since Q is parallel in the normal bundle, the second equations of (1.5), (1.6) and (1.8) imply $h^z Q_z{}^x = 0$, which together with (2.2) gives

$$Q_*{}^x = 0. \tag{2.4}$$

Therefore (1.4) reduces to

$$J_{jx} J^{j*} = \delta_x{}^*. \tag{2.5}$$

Because the mean curvature vector field is assumed to be parallel, the curvature tensor R_{jiyx} of the connection in the normal bundle shows that R_{ji*x} vanishes identically for any index x. Hence the Ricci equation (1.12) yields

$$A_{jr}{}^x A_i{}^{r*} - A_{ir}{}^x A_j{}^{r*} = \frac{c}{4}(J_{j*} J_i{}^x - J_{i*} J_j{}^x) \tag{2.6}$$

by means of (2.4).

In what follows, we suppose that M is an n-dimensional CR submanifold of a complex space form with nonvanishing parallel mean curvature vector field H and parallel f-structure Q in the normal bundle of M. Furthermore, we assume that the shape operator A^* in the direction of H and f-structure f on M is commutative, i.e., $A^* f = f A^*$, which means that

$$A_{jr}{}^* f_i{}^r + A_{ir}{}^* f_j{}^r = 0. \tag{C}$$

The condition (C) is globally defined on M because of (2.2).

By transforming by $f_k{}^i$ and making use of (1.3), we then have

$$A_{ji}{}^* - (A_{jr}{}^* J_z{}^r) J_i{}^z - A_{sr}{}^* f_j{}^r f_i{}^s = 0$$

and consequently $(A_{jr}{}^* J_z{}^r) J_i{}^z - (A_{ir}{}^* J_z{}^r) J_j{}^z = 0$. From this and (1.4) we obtain

$$A_{jr}{}^* J_y{}^r = P_{yz*} J_j{}^z \tag{2.7}$$

because of (2.1) and (2.4), where we have defined

$$P_{yzx} = A_{jix} J_y{}^j J_z{}^i. \tag{2.8}$$

We notice here that P_{yzx} is symmetric for all indices since we have (2.1). Furthermore we have

$$P_{xyz} Q_w^{\ z} = 0. \tag{2.9}$$

Transforming (2.6) by $J_y{}^j J_z{}^i$ and summing for j and i, we find

$$(2.10)\qquad P_{wz*}P_{yx}{}^w - P_{wy*}P_{zx}{}^w = \frac{c}{4}(\delta_{y*}J_z{}^i J_{ix} - \delta_{z*}J_y{}^i J_{ix}),$$

where we have used (2.5), (2.7) and (2.8). Multiplying δ^{yx} to this and summing for y and x, we get

$$(2.11)\qquad P^w P_{wz*} - P_{wy*}P_z{}^{wy} = \frac{c}{4}(1-p)\delta_{z*}$$

because of (2.5), where we have defined $P^x = P^x{}_z{}^z$, which implies that

$$(2.12)\qquad P_{xy*}P^{xy*} - P^z P_{z**} = \frac{c}{4}(p-1).$$

When $z = n+1$ in (2.10) we have

$$P_{w**}P_{yx}{}^w - P_{wy*}P_{x*}{}^w = \frac{c}{4}(\delta_{y*}\delta_{x*} - J_y{}^i J_{ix}),$$

which together with (2.7) gives

$$(2.13)\qquad P_{wy*}P_{x*}{}^w P^{xy*} - P_{w**}P_{xy*}P^{xyw} = \frac{c}{4}(P^* - P_{***}).$$

For the shape operator A^* a function $h_{(m)}$ for any integer $m \geq 2$ is introduced as follows:

$$(2.14)\qquad h_{(m)} = \sum_i (A_{ii}^*)^m.$$

Lemma 2.1. *The second fundamental forms of M satisfy*

$$(2.15)\qquad A^{ji*}A_{jiy} = h^* P_{y**} + \frac{c}{4}(n-1)\delta_{y*},$$

$$(2.16)\qquad h_{(3)} = h^*|P_{z**}|^2 + \frac{c}{4}(n-2)P_{***} + \frac{c}{4}h^*,$$

where A_{jiy} denotes the second fundamental form in the direction of C_y. **Proof.** Differentiating (2.7) covariantly along M and making use of (1.8), we find

$$(\nabla_k A_{jr*})J_y{}^r + A_j{}^r{}_*(A_{ksy}f_r{}^s - A_{kr}{}^z Q_{zy})$$
$$= (\nabla_k P_{yz*})J_j{}^z + P_{yz*}(A_{kr}{}^z f_j{}^r - A_{kj}{}^w Q_w{}^z),$$

from which, taking the skew-symmetric part with respect to indices k and j,

$$A_j{}^r{}_* A_{ksy} f_r{}^s - A_k{}^r{}_* A_{jsy} f_r{}^s - \frac{c}{4}(J_{k*} f_j{}^r - J_{j*} f_{kr} - 2J_r{}^* f_{kj}) J_y{}^r$$
$$= (\nabla_k P_{yz*}) J_j{}^z - (\nabla_j P_{yz*}) J_k{}^z + P_{yz*}(A_{kr}{}^z f_j{}^r - A_{jr}{}^z f_k{}^r),$$

where we have used (1.5), (1.11), (2.4) and (2.9). Because of the condition (C) and (2.5), it follows that we have

$$\text{(2.17)}\quad A_{sr*} A_k{}^s{}_y f_j{}^r - A_{sr*} A_j{}^s{}_y f_k{}^r$$
$$= (\nabla_k P_{yz*}) J_j{}^z - (\nabla_j P_{yz*}) J_k{}^z + P_{yz*}(A_{kr}{}^z f_j{}^r - A_{jr}{}^z f_k{}^r) + \frac{c}{2}\delta_{y*} f_{jk}.$$

Multiplying f^{jk} and summing for j and k, and taking account of (1.3),(1.5),(2.7) and (2.8), we obtain

$$A^{ji*} A_{jiy} - P_{wz*} P^{wz}{}_y = P_{yz*} h^z - P_{yz*} P^z + \frac{c}{4}\delta_{y*}(n-p).$$

By (2.2) and (2.11), we arrive at (2.15).
When $y = n+1$ in (2.17) we see, using the condition (C), that

$$\text{(2.18)}\quad 2A_{sr*} A_k{}^s{}_* f_j{}^r - P_{z**}(A_{kr}{}^z f_j{}^r - A_{jr}{}^z f_k{}^r)$$
$$= (\nabla_k P_{z**}) J_j{}^z - (\nabla_j P_{z**}) J_k{}^z + \frac{c}{2} f_{jk}.$$

On the other hand, we have $A_{kt*} f^{jt} J^{kz} = 0$ by virtue of (1.5) and (2.7). Thus, transforming (2.18) by $A^k{}_{t*} f^{jt}$, we get

$$h_{(3)} - (A_{sr*} J_w{}^r) A^{ks*}(A_{kt*} J^{wt})$$
$$= P_{z**} A_{ji}{}^z A^{ji*} - P_{z**}(A_{kr}{}^z J^{wr})(P_{yw*} J^{ky}) + \frac{c}{4}(h^* - P^*),$$

where we have used (1.3), (2.7) and (2.14). If we take account of (2.8), (2.13) and (2.15), then we obtain (2.16). We have completed the proof of Lemma 2.1.

Now, the mean curvature vector field being parallel in the normal bundle, the restricted Laplacian for A^* is given by

$$\text{(2.18)}\qquad \Delta A_{ji}{}^* = S_{jr} A_i{}^{r*} - R_{kjih} A^{kh*} - \frac{c}{4}\nabla_k(J_{j*} f_i{}^k + 2J_{i*} f_j{}^k).$$

From (2.15) we have

$$\text{(2.19)}\qquad h_{(2)} = h^* P_{***} + \frac{c}{4}(n-1).$$

Lemma 2.2. *$h_{(2)}$ is a harmonic function.* **Proof.** By means of (1.5), (1.11) and (2.5), we find

$$(\nabla_k A_{ji}{}^*)J_*{}^i = J_*{}^i \nabla_i A_{kj}{}^* + \frac{c}{4} f_{kj},$$

which together with (1.3), (2.8) and (C) implies that

$$J_*{}^i(\nabla_k A_{ji}{}^*)A^{kr*} f_r{}^j = \frac{c}{4}(h^* - P^*). \tag{2.20}$$

By definition, we have $P_{***} = A_{ji}{}^* J_*{}^j J_*{}^i$, which implies that

$$\nabla_k P_{***} = (\nabla_k A_{ji}{}^*)J_*{}^j J_*{}^i$$

because of (1.5) and (2.4). Thus, it is seen that

$$\Delta P_{***} = (\Delta A_{ji}{}^*)J_*{}^j J_*{}^i + 2(\nabla_k A_{ji}{}^*)J_*{}^i A_{kr} f^{jr}$$

and hence

$$\begin{aligned}\Delta P_{***} = {} & S_{js} J_*{}^j (A_i{}^{s*} J^{i*}) - R_{kjih} A^{kh*} J_*{}^j J_*{}^i \\ & - \frac{3}{4} J_*{}^j J_*{}^i \nabla_k (J_{j*} f_i{}^k) - \frac{c}{2}(h^* - P^*)\end{aligned} \tag{2.21}$$

with the aid of (2.18) and (2.20).

On the other hand, we have from (2.3) and (2.7)

$$S_{ji} J_*{}^j A_r{}^{i*} J_*{}^r = P_{z**} J_*{}^j J^{zi} \{\frac{c}{4}(n+2) g_{ji} - \frac{3}{4} c J_j{}^w J_{iw} + h^* A_{ji}{}^* - A_j{}^{rx} A_{irx}\},$$

which together with (2.5), (2.7) and (2.8) yields

$$S_{ji} J_*{}^j A_r{}^{i*} J_*{}^r = \frac{c}{4}(n-1) P_{***} + h^* |P_{z**}|^2 - P_{z**} P_{wx*} P^{xzw}. \tag{2.22}$$

By the way, using (1.10) and (2.15) we obtain

$$\begin{aligned}R_{kjih} A^{kh*} = {} & \frac{c}{4}\{h^* g_{ji} + (n-2) A_{ji}{}^* + 3 A^{kh*} f_{jh} f_{ik}\} \\ & + h^* P_{z**} A_{ji}{}^z - A_{jh}{}^x A_{kix} A^{kh*}.\end{aligned} \tag{2.23}$$

Thus, if we take account of (1.5), (2.5), (2.7) and (2.8), then we can get

$$R_{kjih} A^{kh*} J_*{}^j J_*{}^i = \frac{c}{4}\{h^* + (n-2) P_{***}\} + h^* |P_{z**}|^2 - P_{xw*} P_y{}^x{}_* P^{wy*}. \tag{2.24}$$

Substituting (2.22) and (2.24) into (2.21) and making use of (2.13), we see that $\Delta P_{***} = 0$ and hence $\Delta h_{(2)} = 0$ because of (2.19). This completes the proof of Lemma 2.2.

Because of (1.7) and (1.8), we have

$$\begin{aligned} A^{ji*}\nabla_k(J_{j*}f_i{}^k) &= A^{ji*}A_{kr}{}^*f_j{}^rf_i{}^k + A^{ji*}J_{j*}(A_{ir}{}^xJ_x{}^r - h^*J_{i*}) \\ &= A^{ji*}A_{jr}{}^*(\delta_i^r - J_i{}^zJ_z{}^r) + P^zP_{z**} - h^*P_{***} \\ &= h_{(2)} - \frac{c}{4}(p-1) - h^*P_{***}, \end{aligned}$$

where we have used (1.3), (2.7),(2.8),(2,12),(2.14) and (C). Therefore we obtain

$$A^{ji*}\nabla_k(J_{j*}f_i{}^k) = \frac{c}{4}(n-p) \tag{2.25}$$

with the aid of (2.19).

On the other hand, by using (2.6) we have

$$A_j{}^{rx}A_{irx}A^{is*}A_s{}^{j*} = A_j{}^{rx}A^{sj*}\{A_{isx}A_r{}^{i*} + \frac{c}{4}(J_r{}^*J_{sx} - J_s^*J_{rx})\},$$

which joined with (2.5), (2.7), (2.8) and (2.12) yields

$$A_j{}^{rx}A_{irx}A^{is*}A_s{}^{j*} = A_j{}^{rx}A_{isx}A^{js*}A_r{}^{i*} + (\frac{c}{4})^2(p-1). \tag{2.26}$$

Lemma 2.3. *For the shape operator A^* we have*

$$A^{ji*}\Delta A_{ji}{}^* = -\frac{1}{8}c^2(n-p). \tag{2.27}$$

Proof. Multiplying (2.3) with $A^{js*}A_s{}^{i*}$ and making use of (2.5), (2.7) and (2.14), we find

$$S_{ji}A^{js*}A_s{}^{i*} = \frac{c}{4}(n+2)h_{(2)} - \frac{3}{4}cP_{yz*}P^{yz*} + h^*h_{(3)} - A_j{}^{rx}A_{irx}A^{js*}A_s{}^{i*},$$

or, usinng (2.12), (2.16), (2.19) and (2.26)

$$\begin{aligned} S_{ji}A^{js*}A_s{}^{i*} = {} & \frac{c}{2}nh^*P_{***} + (\frac{c}{4})^2(n+2)(n-1) - \frac{3}{4}cP^zP_{z**} - \frac{c^2}{4}(p-1) \\ & + (h^*)^2|P_{z**}|^2 + \frac{c}{4}(h^*)^2 - A_{jr}{}^xA_{isx}A^{js*}A^{ir*}. \end{aligned} \tag{2.28}$$

By (2.23) we have

$$\begin{aligned}R_{kjih}A^{kh*}A^{ji*} &= \frac{c}{4}\{(h^*)^2 + (n-2)h_{(2)} + 3A^{ji*}A^{kh*}f_{jh}f_{ik}\}\\ &\quad + h^*P_{z**}A^{jiz}A_{ji*} - A_{jr}{}^{x}A_{isx}A^{js*}A^{ir*}\\ &= \frac{c}{4}\{(h^*)^2 + (n-2)h_{(2)} + 3(h_{(2)} - P_{yz*}P^{yz*})\}\\ &\quad + (h^*)^2|P_{z**}|^2 + \frac{c}{4}(n-1)h^*P_{***} - A_{jr}{}^{x}A_{isx}A^{js*}A^{ir*},\end{aligned}$$

where we have used (1.3), (2.1), (2.5), (2.15) and (C). Therefore, it follows that we obtain

$$\begin{aligned}(2.29)\quad R_{kjih}A^{kh*}A^{ji*} &= \frac{c}{4}(h^*)^2 + \frac{c}{2}nh^*P_{***} + (\frac{c}{4})^2(n-2)(n-1)\\ &\quad + 3(\frac{c}{4})^2(n-p) - \frac{3}{4}cP^zP_{z**} + (h^*)^2|P_{z**}|^2\\ &\quad - A_{jr}{}^{x}A_{isx}A^{js*}A^{ir*}.\end{aligned}$$

Multiplying (2.18) with A^{ji*} and summing for j and i, and substituting (2.25), (2.28) and (2.29) into this, we arrive at (2.27). Hence, Lemma 2.3 is proved.

3. Theorems

Theorem 3.1. *Let M be an n-dimensional CR submanifold of a complex space form $\tilde{M}^{2m}(c)$ with nonvanishing parallel mean curvature vector field. If the f-structure Q in the normal bundle is parallel, and if $A^*f = fA^*$, then*

$$(3.1)\qquad |\nabla A^*|^2 = \frac{1}{8}c^2(n-p),$$

or equivalently

$$(3.2)\qquad \nabla_k A_{ji}{}^* = -\frac{c}{4}(J_{j*}f_{ki} + J_{i*}f_{kj}),$$

where A^* is the shape operator in the direction of the mean curvature vector field of M.

Proof. Let us put

$$T_{jik} = \nabla_k A_{ji}{}^* + \frac{c}{4}(J_{j*}f_{ki} + J_{i*}f_{kj}).$$

Then we have, by the equation of Codazzi (1.11)

$$|T_{jik}|^2 = |\nabla_k A_{ji}{}^*|^2 - \frac{1}{8}c^2(n-p) \geq 0.$$

Therefore, T vanishes identically if and only if $|\nabla A^*|^2 = \frac{1}{8}c^2(n-p)$.

Generally we have

$$\frac{1}{2}\Delta h_{(2)} = A^{ji*}\Delta A_{ji}{}^* + |\nabla A^*|^2.$$

Thus we have our assertion by Lemma 2.2 and 2.3.

From Remark, we have

Corollary 3.2. *Let M be an n-dimensional generic submanifold of a complex space form $\tilde{M}^{2m}(c)$ with nonvanishing parallel mean curvature vector field. If $A^*f = fA^*$, then we have (3.1) and (3.2).*

Theorem 3.1 and Corollary 3.2 are the generalizations of theorems in [5, 6, 7, 8, 9, 10].

Theorem 3.3. *Under the same assumptions as those in Theorem 3.1, each eigenvalue of A^* is constant.*

Proof. Applying $(A^{ji*})^m$ to (3.2) and using the condition (C), we have $(\nabla_k A_{ji}{}^*)(A^{ji*})^m = 0$ for any integer $m \geq 2$ and consequently $h_{(m)}$=const. on M. Thus all eigenvalues of A^* are constant, which proves the required result.

For any point x in M we can choose a local orthonormal frame field $\{E_i\}$ so that the shape operator A^* is diagonalizable at that point x, say $A_{ji}{}^* = \lambda_j \delta_{ij}$. We denote by σ_{ji} the sectional curvature of M spanned by E_j and E_i. Then by (2.28) and (2.29) we have

$$\sum_{j,i}(\lambda_j - \lambda_i)^2 \sigma_{ji} = \frac{1}{8}c^2(n-p) \geq 0$$

because of (3.1). Thus, if $\sigma_{ji} \leq 0$, then $c(n-p) = 0$, and therefore $\nabla A^* = 0$ by virtue of (3.1). Moreover, we have $c = 0$ or $n = p$. If $n = p$, then $f = 0$ and M is totally real. Thus we have

Theorem 3.4. *Let M be an n-dimensional CR submanifold of a complex space form $\tilde{M}^{2m}(c)$ with nonvanishing parallel mean curvature vector field and parallel f-structure Q in the normal bundle. If $A^*f = fA^*$, and if the sectional curvature of M is nonpositive, then the shape operator A^* in the direction of the mean curvature vector field is parallel. Moreover, M is totally real or $\tilde{M}^{2m}(c)$ is complex Euclidean.*

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