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# ON SIX DIMENSIONAL ALMOST HERMITIAN MANIFOLDS WITH POINTWISE CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

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*Dedicated to Professor U-Hang Ki on his 60th birthday*

## 1. INTRODUCTION

Let $M = (M, J, g)$ be a 6-dimensional almost Hermitian manifold. We denote by $\nabla$, $R$, $\rho$ and $\tau$ the Riemannian connection, the curvature tensor, the Ricci tensor and the scalar curvature of $M$, respectively. We assume that the curvature tensor $R$ is given by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$
$$R(X,Y,Z,W) = g(R(X,Y)Z, W)$$

for $X$, $Y$, $Z$, $W \in \mathfrak{X}(M)$. The holomorphic sectional curvature is defined by

$$H(X) = -R(X, JX, X, JX)$$

for $X \in T_pM(p \in M)$ with $g(X,X) = 1$. If $H(X)$ is constant $\mu(p)$ for all $X \in T_pM(p \in M)$ at each point $p$ of $M$, $M$ is said to be of pointwise constant holomorphic sectional curvature. Further, if $\mu$ is constant whole on $M$, then $M$ is said to be of constant holomorphic sectional curvature. It is well known that if a 6-dimensional nearly Kaehler manifold $M$ is of constant holomorphic sectional curvature $\mu$, then

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either $M$ is Kaehlerian, or $M$ is of constant curvature $\mu > 0$([5]). Also, it is well known that any 6-dimensional nearly Kaehler manifold is an Einstein one ([3],[7]) and its curvature tensor $R$ satisfies the following identity([4]):

$$(*) \quad R(X,Y,Z,W) = R(JX,JY,Z,W) + R(JX,Y,JZ,W) + R(JX,Y,Z,JW)$$

for $X$, $Y$, $Z$, $W \in \mathfrak{X}(M)$.

In this paper we want to prove that if a 6-dimensional almost Hermitian manifold $M$ with pointwise constant holomorphic sectional curvature $\mu$ is Einsteinian and the curvature tensor $R$ of $M$ satisfies the identity $(*)$, then either $M$ is Kaehlerian, or $M$ is of constant curvature $\mu$. In a 6-dimensional quasi-Kaehler manifold $M$, we want to have the same conculsion under the assumption that $M$ is locally symmetric and $\tau \neq 0$ (or $\mu \neq 0$) instead of the assumption that $M$ is Einsteinian.

## 2. Preliminaries

Let $M = (M, J, g)$ be a 6-dimensional almost Hermitian manifold. Then we have

$$(\nabla_X J)JY = -J(\nabla_X J)Y,$$
$$g((\nabla_X J)Y, Z) = -g((Y, (\nabla_X J)Z),$$
$$g((\nabla_X J)Y, Y) = 0,$$
$$g((\nabla_X J)Y, JY) = 0,$$

for $X, Y, Z \in \mathfrak{X}(M)$. The Ricci $*$-tensor $\rho^*$ and the $*$-scalar curvature $\tau^*$ are defined respectively by

$$\rho^*(X,Y) = g(Q^*X,Y) = \mathrm{trace}(Z \longmapsto R(X,JZ)JY),$$
$$\tau^* = \mathrm{trace}\, Q^*$$

for all $X$, $Y$, $Z \in T_pM$, $p \in M$. By the definition of $\rho^*$, we get easily

$$\rho^*(X,Y) = \rho^*(JY, JX)$$

for $X, Y \in T_p(M)$, $p \in M$. $M = (M, J, g)$ is said to be a weakly $*$-Einstein manifold if $\rho^* = \frac{\tau^*}{6} g$ holds.

We shall recall the definitions of special kinds of almost Hermitian manifolds. An almost Hermitian manifold $M$ is called Kaehlerian if

$$\nabla_X J = 0$$

for all $X \in \mathfrak{X}(M)$, $M$ is called nearly Kaehlerian if

$$(\nabla_X J)Y + (\nabla_Y J)X = 0$$

for all $X$, $Y \in \mathfrak{X}(M)$ and $M$ is called quasi-Kaehlerian if

$$(\nabla_X J)Y + (\nabla_{JX} J)(JY) = 0$$

for all $X, Y \in \mathfrak{X}(M)$.

We define three linear operators $L_i$, $i = 1, 2, 3$ as the following:

$$\begin{aligned}
(L_1 R)(X,Y,Z,W) &= \frac{1}{2}\{R(JX,JY,Z,W) + R(Y,JZ,JX,W) \\
&\quad + R(JZ,X,JY,W)\}, \\
(L_2 R)(X,Y,Z,W) &= \frac{1}{2}\{R(X,Y,Z,W) + R(JX,JY,Z,W) + R(JX,Y,JZ,W) \\
&\quad + R(JX,Y,Z,JW)\}, \\
(L_3 R)(X,Y,Z,W) &= R(JX,JY,JZ,JW)
\end{aligned}$$

for all $X$, $Y$, $Z$, $W \in \mathfrak{X}(M)$. It is easy to see that curvature identity (*) implies $L_2R = R$ and $L_3R = R$.

For a $(0,2)$ type tensor $S$, we define $\varphi(S)$ and $\psi(S)$ by

$$\begin{aligned}\varphi(S)(X,Y,Z,W) =& g(X,Z)S(Y,W) + g(Y,W)S(X,Z)\\ &- g(X,W)S(Y,Z) - g(Y,Z)S(X,W),\\ \psi(S)(X,Y,Z,W) =& 2g(X,JY)S(Z,JW) + 2g(Z,JW)S(X,JY)\\ &+ g(X,JZ)S(Y,JW) + g(Y,JW)S(X,JZ)\\ &- g(X,JW)S(Y,JZ) - g(Y,JZ)S(X,JW).\end{aligned}$$

Tricerri and Vanhecke proved the following.

**Theorem A([6]).** *Let $M$ be an almost Hermitian manifold with dimension 6 and curvature tensor $R$. Then we have the following identity:*

$$\begin{aligned}(I - L_1)(I + L_2)(I + L_3)R =& -\frac{1}{2}(3\varphi - \psi)\Big\{\rho(R + L_3R) - \rho^*(R + L_3R)\Big\}\\ &+ \frac{1}{4}(\tau - \tau^*)(3\pi_1 - \pi_2),\end{aligned}$$

where

$$\pi_1(X,Y)Z = g(X,Z)Y - g(Y,Z)X,$$

$$\pi_2(X,Y)Z = 2g(JX,Y)JZ + g(JX,Z)JY - g(JY,Z)JX,$$

$$\{\rho(R + L_3R)\}(X,Y) = \mathrm{trace}\big(Z \longmapsto R(Z,X)Y - JR(JZ,JX)JY\big),$$

$$\big\{\rho^*(R + L_3R)\big\}(X,Y) = \mathrm{trace}\big(Z \longmapsto R(X,JZ)JY - JR(JX,Z)Y\big).$$

On the other hand, Gray obtained the following

**Lemma B([1]).** *Let $M$ be a quasi-Kaehler manifold. Then*

$$(2.1)\quad G(X,Y,Z,W)+G(JX,JY,JZ,JW)+G(JX,Y,JZ,W)+G(X,JY,Z,ZW) = -2g((\nabla_{(\nabla_X J)Y-(\nabla_Y J)X}J)Z,W),$$

*where* $G(X,Y,Z,W)=R(X,Y,Z,W)-R(X,Y,JZ,JW)$.

For a quasi-Kaehler manifold $M$ with the curvature identity $(*)$, the equation (2.1) is reduced to

$$(2.2)\qquad G(X,Y,Z,W)=-\frac{1}{2}g((\nabla_{(\nabla_X J)Y-(\nabla_Y J)X}J)Z,W).$$

## 3. Einstein almost Hermitian manifolds with pointwise constant holomorphic sectional curvature

Let $M=(M,J,g)$ be a 6-dimensional almost Hermtian manifold and let the curvature tensor $R$ of $M$ satisfies the identity $(*)$. Then we find, from Theorem A, $L_2R=R$ and $L_3R=R$,

$$\begin{aligned}(3.1)\quad & 6R(X,Y,Z,W)\\ &= 2\{2R(JX,JY,Z,W)-R(JY,JZ,X,W)-R(JZ,JX,Y,W)\}\\ &+2g(X,JY)\{\rho(Z,JW)-\rho^*(Z,JW)\}+2g(Z,JW)\{\rho(X,JY)-\rho^*(X,JY)\}\\ &+g(X,JZ)\{\rho(Y,JW)-\rho^*(Y,JW)\}+g(Y,JW)\{\rho(X,JZ)-\rho^*(X,JZ)\}\\ &-g(X,JW)\{\rho(Y,JZ)-\rho^*(Y,JZ)\}-g(Y,JZ)\{\rho(X,JW)-\rho^*(X,JW)\}\\ &-3[g(X,Z)\{\rho(Y,W)-\rho^*(Y,W)\}+g(Y,W)\{\rho(X,Z)-\rho^*(X,Z)\}\\ &\qquad -g(X,W)\{\rho(Y,Z)-\rho^*(Y,Z)\}-g(Y,Z)\{\rho(X,W)-\rho^*(X,W)\}]\\ &+\frac{3}{4}(\tau-\tau^*)\{g(X,Z)g(Y,W)-g(Y,Z)g(X,W)\}\\ &-\frac{1}{4}(\tau-\tau^*)\{2g(JX,Y)g(JZ,W)+g(JX,Z)g(JY,W)-g(JY,Z)g(JX,W)\}.\end{aligned}$$

Moreover, we assume that $M$ is of pointwise constant holomorphic sectional curvature $\mu$. Then we have

$$
\begin{aligned}
(3.2)\qquad & R(X,Y,Z,W) \\
&= \mu\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W) + g(JX,W)g(JY,Z) \\
&\qquad - g(JX,Z)g(JY,W) - 2g(JX,Y)g(JZ,W)\} \\
&- \{2R(JX,JY,Z,W) - R(JY,JZ,X,W) - R(JZ,JX,Y,W)\}
\end{aligned}
$$

(See Lemma 3.1 in [2]).

From (3.1) and (3.2) we obtain

$$
\begin{aligned}
(3.3)\qquad & 8R(X,Y,Z,W) \\
&= 2g(X,JY)\{\rho(Z,JW) - \rho^*(Z,JW)\} + 2g(Z,JW)\{\rho(X,JY) - \rho^*(X,JY)\} \\
&+ g(X,JZ)\{\rho(Y,JW) - \rho^*(Y,JW)\} + g(Y,JW)\{\rho(X,JZ) - \rho^*(X,JZ)\} \\
&- g(X,JW)\{\rho(Y,JZ) - \rho^*(Y,JZ)\} - g(Y,JZ)\{\rho(X,JW) - \rho^*(X,JW)\} \\
&- 3[g(X,Z)\{\rho(Y,W) - \rho^*(Y,W)\} + g(Y,W)\{\rho(X,Z) - \rho^*(X,Z)\} \\
&\qquad - g(X,W)\{\rho(Y,Z) - \rho^*(Y,Z)\} - g(Y,Z)\{\rho(X,W) - \rho^*(X,W)\}] \\
&+ \{\tfrac{3}{4}(\tau - \tau^*) - 2\mu\}\{g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\} \\
&- \{\tfrac{1}{4}(\tau - \tau^*) + 2\mu\}\{2g(JX,Y)g(JZ,W) + g(JX,Z)g(JY,W) \\
&\qquad - g(JY,Z)g(JX,W)\}.
\end{aligned}
$$

In a 6-dimensional almost Hermitian manifold with pointwise constant holomorphic sectional curvature $\mu$ and with curvature identity $(*)$, we have ([4])

$$(3.4)\qquad \rho(X,Y)+3\rho^*(X,Y)=8\mu g(X,Y),$$

$$\rho(X,Y)=\rho(JX,JY),$$

$$\rho^*(X,Y)=\rho^*(Y,X),$$

$$\rho^*(X,Y)=\rho^*(JX,JY),$$

$$\tau+3\tau^*=48\mu.$$

From (3.3) and (3.4), we find

$$(3.5)\quad R(X,Y,Z,W)$$

$$=\frac{1}{6}\{2g(X,JY)\rho(Z,JW)+2\rho(X,JY)g(Z,JW)+g(X,JZ)\rho(Y,JW)$$

$$+\rho(X,JZ)g(Y,JW)-g(X,JW)\rho(Y,JZ)-\rho(X,JW)g(Y,JZ)\}$$

$$-\frac{1}{2}\{g(X,Z)\rho(Y,W)+g(Y,W)\rho(X,Z)-g(X,W)\rho(Y,Z)-g(Y,Z)\rho(X,W)\}$$

$$+\frac{\tau+2\mu}{8}\{g(X,Z)g(Y,W)-g(Y,Z)g(X,W)\}$$

$$-\frac{\tau+10\mu}{24}\{2g(JX,Y)g(JZ,W)+g(JX,Z)g(JY,W)-g(JY,Z)g(JX,W)\}.$$

Now, we assume that $M$ is Einsteinian (or equivalently, weakly $*$-Einsteinian). Then we have

$$(3.6)\qquad \rho(X,Y)=\frac{\tau}{6}g(X,Y)$$

Substituting (3.6) into (3.5) and using (3.4), we obtain

$$
\begin{aligned}
(3.7)\qquad & R(X,Y,Z,W) \\
&= \left(\frac{\tau}{72} - \frac{5}{12}\mu\right)\{2g(JX,Y)g(JZ,W) + g(JX,Z)g(JY,W) \\
&\qquad - g(JY,Z)g(JX,W)\} \\
&+ \left(-\frac{\tau}{24} + \frac{\mu}{4}\right)\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\}.
\end{aligned}
$$

On the other hand, Tricerri and Vanhecke proved the following

**Theorem C([6]).** *Let $M$ be a connected almost Hermitian manifold with real dimension $2n \geq 6$ and Riemannian curvature tensor $R$ of the following form:*

$$R = f_1\pi_1 + f_2\pi_2$$

*where $f_1$ and $f_2$ are $C^\infty$ functions on $M$ such that $f_2$ is not identical zero. Then $M$ is a complex space form(i.e. a Kaehler manifold with constant holomorphic sectional cuvature).*

In the proof of Theorem C, Tricerri and Vanhecke showed that the functions $f_1$ and $f_2$ are both constant. Thererfore we can conclude that $\frac{\tau}{72} - \frac{5}{12}\mu$ is constant provided that $M$ is connected. So $\mu$ is constant on $M$.

If $\frac{\tau}{72} - \frac{5}{12}\mu = 0$, then we have from (3.7)

$$R(X,Y,Z,W) = \mu\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\}$$

which shows that $M$ is a manifold of constant sectional curvature $\mu$.

If $\frac{\tau}{72} - \frac{5}{12}\mu \neq 0$, then $M$ is a complex space form from Theorem C.

Thus we have the following

**Theorem 1.** *Let $M$ be a six dimensional connected almost Hermitian manifold with pointwise constant holomorphic sectional curvature $\mu$ and with curvature identity $(*)$. If $M$ is Einsteinian or weakly $*$-Einsteinian, then $M$ is one of the following:*

*(a) a manifold of constant sectional curvature $\mu$*

*(b) a complex space form.*

Since a 6-dimensional nearly Kaehlerian manifold is Einsteinian and has the curvature property $(*)$, we have the following

**Corollary 2([5]).** *If $M$ is a 6-dimensional connected nearly Kaehlerian manifold with pointwise constant holomorphic sectional curvature, then $M$ is one of the following:*

*(a) a manifold of constant sectional curvature*

*(b) a complex space form.*

## 4. Locally symmetric almost Hermitian manifolds with pointwise constant holomorphic sectional curvature

Let $M$ be a 6-dimensional almost Hermitain manifold with pointwise constant holomorphic sectional curvature $\mu$ and let its curvature tensor $R$ satisfies the identity $(*)$. Since $\dim M = 6$, it is possible to choose two unit vectors $X$ and $W$ which define orthogonal holomorphic planes $\{X, JX\}$ and $\{W, JW\}$.

We assume that $M$ is locally symmetric and $\tau \neq 0$ (or $\mu \neq 0$). Then we obtain, by the help of (3.5),

$$W(f)JW + 3hg((\nabla_W J)X, JW)JX + \frac{1}{2}\Big[\rho((\nabla_W J)X, JW)JX + \rho(X, W)(\nabla_W J)X + g((\nabla_W J)X, JW)Q(JX)\Big] = 0, \tag{4.1}$$

where $\{X, JX\}$ and $\{W, JW\}$ are arbitrary orthogonal holomorphic planes, $f = \frac{1}{8}(\tau + 2\mu)$, $h = -\frac{\tau + 10\mu}{24}$ and $Q$ is the Ricci tensor of type (1,1).

Moreover, we assume that $M$ is a quasi Kaehler manifold. Then $\mu$ is globally constant on $M$ ([4]) and hence $W(f) = 0$. Thus (4.1) can be rewritten as

$$\begin{aligned} &6hg((\nabla_W J)X, JW)JX + \rho((\nabla_W J)X, JW)JX \\ &+ \rho(X, W)(\nabla_W J)X + g((\nabla_W J)X, JW)Q(JX) = 0. \end{aligned} \tag{4.2}$$

From (4.2), we obtain

$$\rho(X, W)g((\nabla_W J)X, JW) = 0, \tag{4.3}$$

$$\rho(X, W)g(JX, (\nabla_W J)W) = 0, \tag{4.4}$$

$$6hg((\nabla_W J)X, JW) + \rho((\nabla_W J)X, JW) = -g((\nabla_W J)X, JW)\rho(X, X), \tag{4.5}$$

$$6hg((\nabla_W J)W, JX) + \rho((\nabla_W J)W, JX) = -g((\nabla_W J)W, JX)\rho(X, X). \tag{4.6}$$

Substituting (4.5) into (4.2), we have

$$\begin{aligned} &- g((\nabla_W J)X, JW)\rho(X, X)JX \\ &+ \rho(X, W)(\nabla_W J)X + g((\nabla_W J)X, JW)Q(JX) = 0. \end{aligned} \tag{4.7}$$

Multiplying (4.7) with $\rho(X, W)$ and taking account of (4.3), we obtain

$$\rho(X, W)(\nabla_W J)X = 0, \tag{4.8}$$

which and (4.7) imply

(4.9) $$g((\nabla_W J)X, JW)Q(JX) = g((\nabla_W J)X, JW)\rho(X,X)JX.$$

Substituting (4.9) into (4.2), we find

(4.10) $$[6h + \rho(X,X)]g((\nabla_W J)X, JW) = -\rho((\nabla_W J)X, JW),$$

(4.11) $$[6h + \rho(X,X)]g((\nabla_W J)W, JX) = -\rho((\nabla_W J)W, JX).$$

If we interchange $X$ and $W$ respectively in (4.11), then we obtain

$$[6h + \rho(W,W)]g((\nabla_X J)X, JW) = -\rho((\nabla_X J)X, JW),$$

which implies, using $\rho(JW, JW) = \rho(W, W)$ and the fact that $\{W, JW\}$ and $\{JW, J^2W\}$ determine the same holomorphic plane,

(4.12) $$[6h + \rho(W,W)]g((\nabla_X J)X, W) = -\rho((\nabla_X J)X, W).$$

Now, suppose that $M$ is not nearly Kaehleian. Then there exists a unit vector field $X$ in an open neighborhood $U$ of $p \in M$ such that $(\nabla_X J)X \neq 0$. We put

$$X = e_1, \quad JX = e_2, \quad (\nabla_X J)X/\|(\nabla_X J)X\| = e_3, \quad Je_3 = e_4.$$

Then $\{e_1, e_2\}$ and $\{e_3, e_4\}$ are orthogonal holomorphic planes. If we put $W = e_3$ in (4.12), then we obtain

(4.13) $$\rho(e_3, e_3) = \rho(e_4, e_4) = -3h.$$

Next we choose another holomorphic plane $\{e_5, e_6 = Je_5\}$ which is orthogonal to $\{e_1, e_2\}$ and $\{e_3, e_4\}$ respectively.

Since $\left\{\bar{e}_1 = \dfrac{e_1 + e_3}{\sqrt{2}}, J\bar{e}_1\right\}$ and $\left\{\bar{e}_3 = \dfrac{e_1 - e_3}{\sqrt{2}}, J\bar{e}_3\right\}$ are also orthogonal holomorphic planes, we obtain, using (4.8),

$$\rho(\bar{e}_1, \bar{e}_3)(\nabla_{\bar{e}_1} J)\bar{e}_3 = 0,$$

$$\rho(\bar{e}_3, \bar{e}_1)(\nabla_{\bar{e}_3} J)\bar{e}_1 = 0.$$

From these equations, we find

$$\left[\rho(e_1, e_1) - \rho(e_3, e_3)\right]\left[(\nabla_{e_1} J)e_1 - (\nabla_{e_3} J)e_3\right] = 0,$$

which implies, by the help of $g\big((\nabla_{e_3} J)e_3, e_3\big) = 0$,

$$\rho(e_1, e_1) = \rho(e_3, e_3). \tag{4.14}$$

Similarly, for two pairs of orthogonal holomorphic planes $\left\{\dfrac{e_1 + e_5}{\sqrt{2}}, J\dfrac{e_1 + e_5}{\sqrt{2}}\right\}$, $\left\{\dfrac{e_1 - e_5}{\sqrt{2}}, J\dfrac{e_1 - e_5}{\sqrt{2}}\right\}$ and $\left\{\dfrac{e_3 + e_5}{\sqrt{2}}, J\dfrac{e_3 + e_5}{\sqrt{2}}\right\}$, $\left\{\dfrac{e_3 - e_5}{\sqrt{2}}, J\dfrac{e_3 - e_5}{\sqrt{2}}\right\}$, we obtain

$$\left[\rho(e_5, e_5) - \rho(e_1, e_1)\right]\left[(\nabla_{e_1} J)e_1 - (\nabla_{e_5} J)e_5\right] = 0,$$

$$\left[\rho(e_5, e_5) - \rho(e_3, e_3)\right]\left[(\nabla_{e_3} J)e_3 - (\nabla_{e_5} J)e_5\right] = 0.$$

From these equations, we find, by the help of (4.14),

$$\left[\rho(e_5, e_5) - \rho(e_1, e_1)\right]\left[(\nabla_{e_1} J)e_1 - (\nabla_{e_3} J)e_3\right] = 0,$$

which shows that $\rho(e_5, e_5) = \rho(e_1, e_1)$.

Thus we obtain, using (4.13) and (4.14), (4.15) $$\rho(e_i, e_i) = -3h(1 \leq i \leq 6).$$

Since $\sum_{i=1}^{6} \rho(e_i, e_i) = \tau$ and $h = -\dfrac{\tau + 10\mu}{24}$, we have, by the help of (4.15),

(4.16) $$\tau = 30\mu.$$

Since $\tau$ and $\mu$ are constants on $M$, the relation (4.16) holds whole on $M$.

If we put $W = e_5$ and $W = e_6$ respectively in (4.12), then we obtain

(4.17) $$\rho(e_3, e_5) = \rho(e_3, e_6) = \rho(e_4, e_5) = \rho(e_4, e_6) = 0.$$

Since the Ricci tensor of $M$ is parallel, it is easy to check

(4.18) $$\rho(Y, (\nabla_W J)Y) = 0, \quad \rho(JY, (\nabla_W J)Y) = 0,$$
$$\rho(Z, (\nabla_W J)Y) + \rho((\nabla_W J)Z, Y) = 0.$$

From (4.18) and (3.4), we obtain

(4.19) $$\rho(e_1, e_2) = \rho(e_3, e_4) = \rho(e_5, e_6) = \rho(e_1, e_3)$$
$$= \rho(e_1, e_4) = \rho(e_2, e_3) = \rho(e_2, e_4) = 0.$$

Suppose that $\rho(e_1, e_5) \neq 0$ on an open neighborhood $U'(\subset U)$ of $p$. Then we have, using (4.8),

(4.20) $$(\nabla_{e_1} J)e_5 = (\nabla_{e_5} J)e_1 = 0$$

on $U'$. Thus (2.2) and (4.20) imply

(4.21) $$R(e_1, e_5, e_1, e_5) = R(e_1, e_5, e_2, e_6).$$

From (3.5) and (4.21), we find

$$\rho(e_1, e_1) = \frac{\tau}{8} + \frac{\mu}{2},$$

which implies $\tau = 0$ by the help of (4.15) and (4.16). This contradicts to the hypothesis. Therefore we have $\rho(e_1, e_5) = 0$. Similarly, we have $\rho(e_1, e_6) = 0$. From these results, (4.15), (4.17) and (4.19), we can conclude that $Q = \lambda I$ for some function $\lambda$ on $U$.

Now suppose that there exists a point $q \in M$ such that $(\nabla_W J)W = 0$ for any vector field $W$ at $q$. We take arbitrary orthogonal holomorphic planes $\{X, JX\}$ and $\{Y, JY\}$, and assume that $\rho(X, Y) \neq 0$ at $q$. Then we have $(\nabla_X J)Y = (\nabla_Y J)X = 0$ from (4.8) and hence we obtain, by the help of (2.2),

$$R(X, Y, Z, W) - R(X, Y, JZ, JW) = 0 \tag{4.22}$$

for any vector fields $Z$ and $W$ at $q$. If we put $Z = X$ and $W = Y$ in (4.22) and use (3.5), then we find

$$\rho(X, X) + \rho(Y, Y) = \frac{\tau}{4} + \mu. \tag{4.23}$$

If we take another holomorphic plane $\{Z, JZ\}$ which is orthogonal to $\{X, JX\}$ and $\{Y, JY\}$ respectively, then we find from (4.22) and (3.5),

$$\begin{aligned} &\rho(X, JZ)g(Y, JW) - \rho(X, Z)g(Y, W) \\ &- g(X, JW)\rho(Y, JZ) + g(X, W)\rho(Y, Z) = 0 \end{aligned} \tag{4.24}$$

for all $W$. If we put $W = X, Y$ in (4.24) respectively, we have

$$\rho(Y, Z) = \rho(X, Z) = 0. \tag{4.25}$$

For the orthogonal holomorphic planes $\left\{\frac{X+Z}{\sqrt{2}}, J\frac{X+Z}{\sqrt{2}}\right\}$ and $\left\{\frac{X-Z}{\sqrt{2}}, J\frac{X-Z}{\sqrt{2}}\right\}$, we obtain from (4.8)

$$[\rho(X,X)-\rho(Z,Z)]\,[(\nabla_Z J)X-(\nabla_X J)Z]=0.$$

If $\rho(X,X)\neq\rho(Z,Z)$ at $q$, then we have $(\nabla_X J)Z=(\nabla_Z J)X$ at $q$. Since $(\nabla_X J)Z+(\nabla_Z J)X=0$ at $q$, we have $(\nabla_X J)Z=(\nabla_Z J)X=0$ at $q$. By the same arguments as in the preceding paragraph, we have $\rho(X,Y)=0$. This contradicts to the hypothesis. Hence $\rho(X,X)=\rho(Z,Z)$. Similarly, we obtain $\rho(Y,Y)=\rho(Z,Z)$. Therefore we find, by the help of (4.23),

$$\tau=12\mu,$$

which and (4.16) imply $\tau=0$. This is impossible. Hence we can conclude that $\rho(X,Y)=0$ for any orthogonal holomorphic planes $\{X,JX\}$ and $\{Y,JY\}$. Hence $\rho(X,Y)=\rho(X,Z)=\rho(Y,Z)=\cdots=\rho(X,JZ)=\rho(JY,JZ)=0$ for the orthogonal holomorphic planes $\{X,JX\}$, $\{Y,JY\}$ and $\{Z,JZ\}$.

For the orthogonal holomorphic planes $\left\{\frac{X+Y}{\sqrt{2}}, J\frac{X+Y}{\sqrt{2}}\right\}$ and $\left\{\frac{X-Y}{\sqrt{2}}, J\frac{X-Y}{\sqrt{2}}\right\}$, we have $\rho\left(\frac{X+Y}{\sqrt{2}}, J\frac{X-Y}{\sqrt{2}}\right)=0$. Hence we have $\rho(X,X)=\rho(Y,Y)$. Similarly, we obtain $\rho(X,X)=\rho(Z,Z)$. Hence we get

$$\rho(X,X)=\rho(Y,Y)=\rho(Z,Z)=\rho(JX,JX)=\rho(JY,JY)=\rho(JZ,JZ).$$

Therefore, we have $Q=\lambda I$ at $q$.

Summing up, we have $Q=\lambda I$ whole on $M$ and hence $M$ is Einsteinian. From theorem 1 and the hypothesis that $M$ is not nearly Kaehlerian, we can conclude that $M$ is of constant sectional curvature $\mu$.

On the other hand, if $M$ is nearly Kaehlerian, then $M$ is a mainifold of constant sectional curvature or a complex space form by virtue of corollary 2. Thus we have the following

**Theorem 3.** *Let $M$ be a 6-dimensional connected quasi-Kaehler manifold with pointwise constant holomorphic sectional curvature $\mu$ and let the curvature tensor $R$ of $M$ satisfies the identity $(*)$. If $M$ is locally symmetric and $\tau \neq 0$ (or $\mu \neq 0$), then it is one of the following:*

(*a*) *a manifold of constant sectional curvature*

(*b*) *a complex space form.*

## REFERENCES


[1]. A. Gray, *Curvature identities for Hermitian and almost Hermitian manifolds*, Tôhoku Math. J. **28** (1976), 601–612.

[2]. A. Gray and L. Vanhecke, *Almost Hermitian manifolds with constant holomorphic sectional curvature*, Časopis pro pěstování matematiky, roč **104** (1979), 170–179.

[3]. M. Matsumoto, *On 6-dimensional almost Tachibana spaces*, Tensor, N.S. **23** (1972), 250–252.

[4]. T. Sato, *On some almost Hermitian manifolds with constant holomorphic sectional curvature*, Kyungpook Math. Jour. **29** (1989), 11–25.

[5]. S. Tanno, *Constancy of holomorphic sectional curvature in almost Hermitian manifolds*, Kōdai Math. Sem. Rep. **25** (1973), 190–201.

[6]. F. Tricerri and L. Vanhecke, *Curvature tensors on almost Hermitian manifolds*, Trans. Amer. Math. Soc. **267** (1981), 365–398.

[7]. K. Yano and M. Kon, Structures on manifolds, *World Scientific Publishing Co.*, 1984. (Singapore)



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