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On the Rational Approximations to $\tan\frac{1}{k}$

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Introduction.

C. S. Davis [1, 2] proved the following theorem: *Let k be a positive integer. Then the inequality*

$$|e^{1/k}-\frac{p}{q}|<\frac{1}{2k}\frac{\log\log q}{q^{2}\log q}$$

has an infinity of solutions in integers p and q. Further, for any $\varepsilon>0$, there exists a number $q'=q'(k,\varepsilon)$ such that

$$|e^{1/k}-\frac{p}{q}|>(\frac{1}{2k}-\varepsilon)\frac{\log\log q}{q^{2}\log q}$$

for all integers p and q with $q\geq q'$.

In this paper, for any positive integer k, we establish Davis' result with $e^{1/k}$ replaced by $\tan\frac{1}{k}$, and give explicit lower bound for q'.

§ 1. The lower estimate for $|\tan 1-\frac{p}{q}|$.

In this section, we assume that p_n/q_n is the n-th convergent of $\tan 1$.
Let N be a positive integer with $N\geq 50$. Let γ_N, δ_m, and γ_N^* be defined by

$$\gamma_N=(2+3/N)\frac{\log(N+3/2)+\log\log((2N+3)/e)}{\log(5(N+3/2)/7)},$$

$$\delta_m=\frac{(2m+3)\log\log q_{2m}}{\log q_{2m}},$$

and

$$\gamma_N^*=\max\{\delta_m|1\leq m<N\},$$

respectively.

LEMMA 1.1. *For all integers p and q with $q \geq q_{2N}$,*

$$|\tan 1 - \frac{p}{q}| > \frac{\log\log q}{\gamma_N q^2 \log q}.$$

PROOF. We may assume that p/q is a convergent of $\tan 1$, since otherwise

$$|\tan 1 - \frac{p}{q}| > \frac{1}{2q^2}$$

(cf. [3] or [7]). The continued fraction of $\tan 1$ is

$$\tan 1 = [a_0, a_1, a_2, a_3, \cdots] = \overline{[1, 2n-1]}_{n=1}^{\infty}$$

(cf. [9]). In other words, $a_{2m} = 1$ and $a_{2m+1} = 2m+1$ for $m \geq 0$.

Case 1: $n = 2m$ $(m \geq N)$. Since $q_{2m+1} = a_{2m+1}q_{2m} + q_{2m-1} = (2m+1)q_{2m} + q_{2m-1} < 2(m+1)q_{2m}$, we have

$$|\tan 1 - \frac{p_{2m}}{q_{2m}}| > \frac{1}{q_{2m}(q_{2m+1} + q_{2m})} > \frac{1}{(2m+3)q_{2m}^2}.$$

Now we must estimate q_{2m}. Since $q_{2m} \geq 2mq_{2m-2} \geq \cdots \geq 2^m m!$, we have

$$\begin{aligned} \log q_{2m} &\geq m\log 2 + \sum_{\nu=1}^{m} \log \nu \geq m\log 2 + m\log m - m + 1 \\ &\geq m\log(2m/e). \end{aligned}$$

Conversely, since $q_{2m} \leq (2m+1)q_{2m-2} \leq \prod_{\nu=1}^{m}(2\nu+1)$, we have

$$\begin{aligned} \log q_{2m} &\leq \sum_{\nu=1}^{m} \log(2\nu+1) \leq (m+3/2)\log(2m+3) - m - (3/2)\log 3 \\ &\leq (m+3/2)\log((2m+3)/e), \end{aligned}$$

$$\log\log q_{2m} \leq \log(m+3/2) + \log\log((2m+3)/e).$$

Since

$$l_1(x) = \frac{\log\log((2x+3)/e)}{\log(x+3/2)} \quad (x \geq 27)$$

and

$$l_2(x) = \frac{\log(x+3/2)}{\log(5(x+3/2)/7)} \quad (x \geq 1)$$

are strictly decreasing functions and $5(x+3/2)/7 \leq 2x/e \ (x \geq 50)$, we have

$$\log\log q_{2m} \leq (1+l_1(N))\log(m+3/2) \leq l_2(N)(1+l_1(N))\log(2m/e).$$

From these inequalities, we find

$$\begin{aligned}\frac{\log\log q_{2m}}{\log q_{2m}} &\leq l_2(N)\cdot\frac{1+l_1(N)}{m} \\ &\leq (2+3/N)\frac{\log(N+3/2)}{\log(5(N+3/2)/7)}(1+\frac{\log\log((2N+3)/e)}{\log(N+3/2)})\cdot\frac{1}{2m+3} \\ &= \frac{\gamma_N}{2m+3}.\end{aligned}$$

Therefore,

$$|\tan 1 - \frac{p_{2m}}{q_{2m}}| > \frac{\log\log q_{2m}}{\gamma_N q_{2m}^2 \log q_{2m}}.$$

Case 2: $n = 2m+1 \ (m \geq N)$. Since $q_{2m+2} = a_{2m+2}q_{2m+1} + q_{2m} = q_{2m+1} + q_{2m}$, we have

$$|\tan 1 - \frac{p_{2m+1}}{q_{2m+1}}| > \frac{1}{q_{2m+1}(q_{2m+2}+q_{2m+1})} > \frac{1}{3q_{2m+1}^2}.$$

As we can see that $\log\log x/\log x \ (x \geq 16)$ is a strictly decreasing function, we have

$$\frac{\log\log q_{2m+1}}{\log q_{2m+1}} \leq \frac{\log\log q_{101}}{\log q_{101}} < \frac{\log\log 16}{\log 16} = 0.36780\cdots < 2/3 < \gamma_N/3,$$

therefore

$$|\tan 1 - \frac{p_{2m+1}}{q_{2m+1}}| > \frac{\log\log q_{2m+1}}{\gamma_N q_{2m+1}^2 \log q_{2m+1}}.$$

This completes the proof.

THEOREM 1.2. *For all integers p and q with $q \geq 2$,*

$$|\tan 1 - \frac{p}{q}| > \frac{\log\log q}{\gamma q^2 \log q},$$

where

$$\gamma \geq \max\{\gamma_N, \gamma_N^*\}$$

for any positive integer $N \geq 50$.

PROOF. It suffices only to consider that p/q is an n-th convergent of $\tan 1$.

Case 1: $n = 2m$ $(m \geq 1)$. From the definition of γ_N^*, we have the following inequalities

$$\begin{aligned} |\tan 1 - \frac{p_{2m}}{q_{2m}}| &> \frac{1}{(2m+3)q_{2m}^2} = \frac{\log\log q_{2m}}{\delta_m q_{2m}^2 \log q_{2m}} \\ &\geq \frac{\log\log q_{2m}}{\gamma_N^* q_{2m}^2 \log q_{2m}} \quad (1 \leq m < N). \end{aligned}$$

And from Lemma 1.1, we have

$$|\tan 1 - \frac{p_{2m}}{q_{2m}}| > \frac{\log\log q_{2m}}{\gamma_N q_{2m}^2 \log q_{2m}} \quad (m \geq N).$$

Case 2: $n = 2m+1$ $(m \geq 1)$. We can see easily that

$$|\tan 1 - \frac{p_{2m+1}}{q_{2m+1}}| > \frac{\log\log q_{2m+1}}{\gamma_N q_{2m+1}^2 \log q_{2m+1}} \quad (m \geq 2).$$

And we have the following inequality

$$|\tan 1 - \frac{p_3}{q_3}| = 0.01402\cdots > \frac{\log\log q_3}{\gamma_N q_3^2 \log q_3}.$$

This completes the proof.

COROLLARY 1.3. *For all integers p and q with $q \geq 2$,*

$$|\tan 1 - \frac{p}{q}| > \frac{\log\log q}{4q^2 \log q}.$$

PROOF. For $N = 50$, we have $\gamma_{50} = 2.98968\cdots$ and $\gamma_{50}^* = \delta_5 = 3.23672\cdots$. Hence we can choose γ so that $\gamma = 4$.

§ 2. The lower estimate for $|\tan \frac{1}{k} - \frac{p}{q}|$ for $k \geq 2$.

In this section, we assume that k is a positive integer with $k \geq 2$, and p_n/q_n is the n-th convergent of $\tan \frac{1}{k}$.

Let N be a positive integer with $N \geq 5$. Let γ_N, δ_m, and γ_N^* be defined by

$$\gamma_N = k(2 + \frac{3}{N-1})(1 + \frac{\log\log(2k(2N+1)/e)}{\log(N+1/2)}),$$

$$\delta_m = \frac{k(2m+1)\log\log q_{2m}}{\log q_{2m}},$$

and

$$\gamma_N^* = \max\{\delta_m | 1 \leq m < N\},$$

respectively.

LEMMA 2.1. *For all integers p and q with $q \geq q_{2N}$,*

$$|\tan\frac{1}{k} - \frac{p}{q}| > \frac{\log\log q}{\gamma_N q^2 \log q}.$$

PROOF. We may assume that p/q is a convergent of $\tan\frac{1}{k}$, since otherwise

$$|\tan\frac{1}{k} - \frac{p}{q}| > \frac{1}{2q^2}.$$

The continued fraction of $\tan\frac{1}{k}$ is

$$\tan\frac{1}{k} = [a_0, a_1, a_2, a_3, \cdots] = [0, k-1, \overline{1, (2n+1)k-2}]_{n=1}^{\infty}$$

(cf. [9]). In other words, $a_0 = 0$, $a_1 = k-1$, and for $m \geq 1$, $a_{2m} = 1$ and $a_{2m+1} = k(2m+1)-2$.

Case 1: $n = 2m$ $(m \geq N)$. Since $q_{2m+1} = a_{2m+1}q_{2m} + q_{2m-1} = (k(2m+1)-2)q_{2m} + q_{2m-1} < (k(2m+1)-1)q_{2m}$, we have

$$|\tan\frac{1}{k} - \frac{p_{2m}}{q_{2m}}| > \frac{1}{q_{2m}(q_{2m+1}+q_{2m})} > \frac{1}{k(2m+1)q_{2m}^2}.$$

Now we must estimate q_{2m}. Since $q_{2m} \geq 2k(m-1)q_{2m-2} \geq \cdots \geq (2k)^m(m-1)!$, we have

$$\begin{aligned}\log q_{2m} &\geq m\log(2k) + \sum_{\nu=1}^{m-1}\log\nu \geq m\log(2k) + (m-1)\log(m-1) - m + 2 \\ &\geq (m-1)\log(2k(m-1)/e).\end{aligned}$$

Conversely, since $q_{2m} \leq k(2m-1)q_{2m-2} \leq k^m\prod_{\nu=1}^{m}(2\nu-1)$, we have

$$\begin{aligned}\log q_{2m} &\leq m\log k + \sum_{\nu=1}^{m}\log(2\nu-1) \leq m\log k + (m+1/2)\log(2m+1) - m \\ &\leq (m+1/2)\log(2k(2m+1)/e),\end{aligned}$$

$$\log\log q_{2m} \leq \log(m+1/2) + \log\log(2k(2m+1)/e).$$

As we can see that

$$l(x) = \frac{\log\log(2k(2x+1)/e)}{\log(x+1/2)} \quad (x \geq 5)$$

is a strictly decreasing function, we have

$$\log\log q_{2m} \leq (1+l(N))\log(m+1/2) \leq (1+l(N))\log(2k(m-1)/e).$$

From these inequalities, we find

$$\begin{aligned}\frac{\log\log q_{2m}}{\log q_{2m}} &\leq \frac{1+l(N)}{m-1}\\ &\leq k(2+\frac{3}{N-1})(1+\frac{\log\log(2k(2N+1)/e)}{\log(N+1/2)})\cdot\frac{1}{k(2m+1)}\\ &= \frac{\gamma_N}{k(2m+1)}.\end{aligned}$$

Therefore,

$$|\tan\frac{1}{k} - \frac{p_{2m}}{q_{2m}}| > \frac{\log\log q_{2m}}{\gamma_N q_{2m}^2 \log q_{2m}}.$$

Case 2: $n = 2m+1$ $(m \geq N)$. Since $q_{2m+2} = a_{2m+2}q_{2m+1} + q_{2m} = q_{2m+1} + q_{2m}$, we have

$$|\tan\frac{1}{k} - \frac{p_{2m+1}}{q_{2m+1}}| > \frac{1}{q_{2m+1}(q_{2m+2}+q_{2m+1})} > \frac{1}{3q_{2m+1}^2}.$$

This completes the proof.

THEOREM 2.2. *For all integers p and q with $q \geq 2$,*

$$|\tan\frac{1}{k} - \frac{p}{q}| > \frac{\log\log q}{\gamma q^2 \log q},$$

where

$$\gamma \geq \max\{\gamma_N, \gamma_N^*\}$$

for any positive integer $N \geq 5$.

PROOF. It suffices only to consider that p/q is a $(2m)$-th convergent of $\tan\frac{1}{k}$. From the definition of γ_N^*, we have the following inequalities

$$|\tan\frac{1}{k} - \frac{p_{2m}}{q_{2m}}| > \frac{1}{k(2m+1)q_{2m}^2} = \frac{\log\log q_{2m}}{\delta_m q_{2m}^2 \log q_{2m}} \geq \frac{\log\log q_{2m}}{\gamma_N^* q_{2m}^2 \log q_{2m}} \quad (1 \leq m < N).$$

And from Lemma 2.1, we have

$$|\tan\frac{1}{k} - \frac{p_{2m}}{q_{2m}}| > \frac{\log\log q_{2m}}{\gamma_N q_{2m}^2 \log q_{2m}} \quad (m \geq N).$$

This completes the proof.

COROLLARY 2.3. *For all integers p and q with $q \geq 2$,*

$$|\tan\frac{1}{2}-\frac{p}{q}|>\frac{\log\log q}{6q^2\log q}.$$

PROOF. For $N=34$, we have $\gamma_{34}=5.98929\cdots$ and $\gamma_{34}^*=\delta_{11}=5.11381\cdots$. Hence we can choose γ so that $\gamma=6$.

COROLLARY 2.4. *For all integers p and q with $q \geq 2$,*

$$|\tan\frac{1}{3}-\frac{p}{q}|>\frac{\log\log q}{9q^2\log q}.$$

PROOF. For $N=41$, we have $\gamma_{41}=8.98303\cdots$ and $\gamma_{41}^*=\delta_{40}=7.02577\cdots$. Hence we can choose γ so that $\gamma=9$.

§ 3. The main thorem.

THEOREM 3.1. *There is an infinity of solutions of the inequality*

$$|\tan\frac{1}{k}-\frac{p}{q}|<\frac{1}{2k}\frac{\log\log q}{q^2\log q}$$

in integers p and q. Further, for any $\varepsilon>0$, there exists a number $q'=q'(k,\varepsilon)$ such that

$$|\tan\frac{1}{k}-\frac{p}{q}|>(\frac{1}{2k}-\varepsilon)\frac{\log\log q}{q^2\log q}$$

for all integers p and q with $q \geq q'$.

PROOF. We prove the first statement. Since $q_{2m+1}>2kmq_{2m}$, we have

$$|\tan\frac{1}{k}-\frac{p_{2m}}{q_{2m}}|<\frac{1}{q_{2m}q_{2m+1}}<\frac{1}{2kmq_{2m}^2}.$$

Now

$$\log q_{2m}=m\log m+O(m)=m\log m\{1+O(1/(\log m))\},$$

so

$$\begin{aligned}\log\log q_{2m} &= \log m+\log\log m+O(1/(\log m))\\ &= \frac{\log q_{2m}}{m}+\log\log m+O(1),\end{aligned}$$

and hence

$$\frac{1}{m} < \frac{\log\log q_{2m}}{\log q_{2m}}$$

for all sufficiently large m. Thus

$$|\tan\frac{1}{k} - \frac{p_{2m}}{q_{2m}}| < \frac{1}{2kmq_{2m}^2} < \frac{\log\log q_{2m}}{2kq_{2m}^2\log q_{2m}}$$

for an infinity of p_{2m} and q_{2m}, as asserted.

The second statement follows immediately from Lemma 1.1 and Lemma 2.1. This completes the proof.

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