Nihonkai Math. J.
Vol.6 (1995), 171-175

Conjugacy classes of zero entropy automorphisms on free group factors

Naohito Tanaka

1. Introduction. The entropy $H(\theta)$ of a $*$-automorphism θ on a von Neumann algebra M is defined by Connes - Størmer [4] as an extended version of classical one. The notion of entropy is conjugacy invariant, that is, $H(\theta) = H(\alpha^{-1}\theta\alpha)$ for an automorphism α of M.

Besson[2] gives an example of an uncountable family of automorphisms on the hyperfinite II_1 factor R which have zero entropy but are not pairwize conjugate. An interesting example of II_1-factor which is not hyperfinite is the group von Neumann algebra $L(F_n)$ of the free group F_n on n generators ($n \geq 2$).

The purpose of this paper is to give an alternative version of Besson's result to free group factors. That is, we show :

Theorem. *There exists an uncountable family of automorphisms on $L(F_n)$ which have entropy zero but are pairwize non conjugate.*

The author would like to thank the referee for many valuable comments and pointing out a mistake in the first virsion of this paper.

2. Automorphisms of free group factors. Let G be a countable infinite group and $l^2(G)$ the Hilbert space of all square summable functions on G. For each g in G, let $u(g)$ be the unitary representation of G to $l^2(G)$ defined by

$$(u(g)\xi)(h) = \xi(g^{-1}h) \qquad (\xi \in l^2(G),\, h \in G).$$

The von Neumann algebra on $l^2(G)$ generated by $\{u(g); g \in G\}$ is called the left von Neumann algebra of G and denoted by $L(G)$. It is well known that L(G) is factor if and only if G is an ICC group, that is, every conjugacy class $C_g = \{hgh^{-1}; h \in G\}$ is infinite, except the trivial $\{1\}$. Let $\{\delta(g)\}_{g\in G}$ be an othonomal basis in $l^2(G)$ given by

$$(\delta(g))(h) = \begin{cases} 1 & h = g \\ 0 & \text{otherwise} \end{cases} \qquad (g \in G).$$

The functional τ on $L(G)$ defined by

$$\tau(x) = (x\delta(e)|\delta(e)) \qquad (x \in R(G),\, e \text{ is the unit of } G),$$

is a faithful finite normal trace. For an $x \in L(G)$, put $x(g) = \tau(xu(g^{-1}))$ then x has a unique expansion:

$$x = \sum_{g \in G} x(g)u(g), \quad \text{in the pointwise} \|\cdot\|_2\text{-convergence topology,}$$

and

$$\|x\|_2^2 = \tau(x^*x) = \sum_{g \in G} |x(g)|^2.$$

We fix an integer n and let F_n be the free group on n generators $\{g_1, \cdots, g_n\}$ $(n = 2, 3, \cdots)$. It is obvious that F_n is an ICC group. Each element g in F_n has the expression called a reduced word. For each g in F_n, we shall call the sum of powers of component g_m in the reduced word the *order of* g *with respect to* g_m $(m = 1, 2, \cdots, n)$ and denote it by $O_m(g)$. For an example, let g in F_n be a reduced word

$$g = g_{i_1}^{n_1} g_{i_2}^{n_2} \cdots g_{i_k}^{n_k} \qquad (i_j = 1, 2, \cdots n,\ n_j = \pm 1, \pm 2, \cdots (j = 1, 2, \cdots, k)),$$

then the order $O_m(g)$ of g is $\sum_{j=1}^{k} \delta_{(m,i_j)}\, n_j$. We denote by $\mathrm{Aut}(L(F_n))$ the group of automorphisms of $L(F_n)$.

Put

$$\Gamma = \{\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n);\ \gamma_i \in \mathbb{T}\,(i = 1, 2, \cdots, n)\},$$

where $\mathbb{T}$ be the unit circle in the complex plane. For $\gamma \in \Gamma$, the $\alpha_\gamma \in \mathrm{Aut}(L(F_n))$ is defined by:

$$(*) \qquad \alpha_\gamma(x) = \sum_{g \in F_n} x(g) \prod_{m=1}^{n} \gamma_m^{O_m(g)} u(g) \qquad (x \in L(F_n)).$$

Such automorphisms are treated in [1,3,5]. The following Lemma is well known in the specialists but we denote a proof of it for the sake of completeness.

Lemma 1. *If a sequence* $\{\gamma_i\} \subset \Gamma$ *converges to* $\gamma \in \Gamma$, *then* α_{γ_i} *converges to* α_γ *(in the sense of point wise* $\|\cdot\|_2$ *convergence).*

Proof. Put $\gamma_i = (\gamma_{i_1}, \gamma_{i_2}, \cdots, \gamma_{i_n})$, $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)$. We denote α_{γ_i} (resp. α_γ) by α_i (resp. α).

Let $x \in L(F_n)$. To simplify, we assume $\|x\|_2 = 1$. For a given $\epsilon > 0$, there exists a finite set $K \subset F_n$ such that $\|x - \sum_{g \in K} x(g)u(g)\|_2 < \epsilon/3$. Let

$$M = \max_{g \in K, 1 \le m \le n} |O_m(g)|.$$

Since $\{\gamma_i\} \subset \Gamma$ converges to $\gamma \in \Gamma$, we have an integer r which satisfies that if $i > r$ then $M \cdot n \cdot |\gamma_i - \gamma| < \frac{\epsilon}{3}$. Put

$$y = \sum_{g \in K} x(g)u(g).$$

Then

$$\begin{aligned}\|\alpha_i(y) - \alpha(y)\|_2^2 &\leq \sum_{g \in K} |x(g)|^2 |\prod_{m=1}^{n} \gamma_{im}^{O_m(g)} - \prod_{m=1}^{n} \gamma_m^{O_m(g)}|^2 \\ &\leq \|x\|_2^2 |\prod_{m=1}^{n} \gamma_{im}^{O_m(g)} - \prod_{m=1}^{n} \gamma_m^{O_m(g)}|^2 \\ &\leq M^2 n^2 |\gamma_i - \gamma|^2 < (\frac{\epsilon}{3})^2.\end{aligned}$$

Hence

$$\begin{aligned}\|\alpha_i(x) - \alpha(x)\|_2 &\leq \|\alpha_i(x) - \alpha_i(y)\| + \|\alpha_i(y) - \alpha(y)\|_2 + \|\alpha(y) - \alpha(x)\|_2 \\ &= 2\|x - x_0\|_2 + \|\alpha_i(x_0) - \alpha(x_0)\|_2 \\ &< \epsilon.\end{aligned}$$

□

Two α_1 and $\alpha_2 \in \mathrm{Aut}(L(F_n))$ are said to be conjugate when $\theta^{-1}\alpha_1\theta = \alpha_2$ for some $\theta \in \mathrm{Aut}(L(F_n))$. Put $\gamma_i = (\gamma_{i1}, \gamma_{i2}, \cdots, \gamma_{in}) \in \Gamma$ with $\gamma_{ij} \in \mathbb{T}$ $(i = 1, 2\ , j = 1, 2, \cdots, n)$. Let $\theta \in \mathrm{Aut}(L(F_n))$ satisfy $\theta^{-1}\alpha_{\gamma_1}\theta = \alpha_{\gamma_2}$.

Let

$$\theta(u(g_i)) = \sum_{g \in F_n} x_i(g)u(g),$$

be the Fourier expansion of $\theta(u(g_i))$. Then,

$$\alpha_{\gamma_1} \cdot \theta(u(g_i)) = \sum_{g \in F_n} x_i(g) \prod_{m=1}^{n} \gamma_{1m}^{O_m(g)} u(g)$$

$$= \theta \cdot \alpha_{\gamma_2}(u(g_i)) = \theta(\gamma_{2i}u(g_i)) = \sum_{g \in F_n} \gamma_{2i}x_i(g)u(g) \quad (i = 1, 2, \cdots, n).$$

It follows that

$$x_i(g) \prod_{m=1}^{n} \gamma_{1m}^{O_m(g)} = x_i(g)\gamma_{2i} \quad (i = 1, 2, \cdots, n,\ g \in F_n).$$

Since $\theta(u(g_i))$ is unitary,

$$\sum_{g \in F_n} |x_i(g)|^2 = 1 \qquad (i = 1, 2, \cdots, n).$$

Hence, for each i $(i = 1, 2, \cdots, n)$, there exists h_i in F_n such that $x_i(h_i) \neq 0$, so that

$$(*1) \qquad \prod_{m=1}^{n} \gamma_{1_m}^{O_m(h_i)} = \gamma_{2_i} \quad (i = 1, 2, \cdots, n).$$

From now, we restrict our interest to the case of $n = 2$.

Put $\gamma = (1, \gamma_1)$, $\gamma' = (1, \gamma_1')$ with $\gamma_1, \gamma_1' \in \mathbb{T}$. Suppose that α_γ is conjugate to $\alpha_{\gamma'}$ and that γ_1 is a primitive nth root of 1.

By assumption, there exist an automorphism θ such that $\theta^{-1}\alpha_\gamma\theta = \alpha_{\gamma'}$. Clearly, $\alpha_\gamma^n = id$ by the definition of α_γ (id is the identity automorphism of $L(F_n)$). Therefore,

$$\alpha_{\gamma'}^n = (\theta^{-1}\alpha_\gamma\theta)^n = \theta^{-1}\alpha_\gamma^n\theta = id.$$

Furthermore, if $\alpha_{\gamma'}^m = id$ for some integer m, then $(\theta^{-1}\alpha_\gamma\theta)^m = \theta^{-1}\alpha_\gamma^m\theta = id$. Hence, $\alpha_\gamma^m = \theta \mathrm{I} \theta^{-1} = \mathrm{I}$. Hence, the γ_1 is a primitive nth root of 1 if and only if γ_1' is a primitive nth root of 1. the γ_1 is an irrational if and only if γ_1' is an irrational.

Let γ_1 be irrational. From $(*1)$, there exist a integers j, k such that

$$\gamma_1^j = \gamma_1', \quad \gamma_1'^k = \gamma_1$$

Hence,

$$\gamma_1^{jk} = \gamma_1'^k = \gamma_1.$$

Then $\gamma_1^{jk-1} = 1$. Hence, we give $j = 1$ and $k = 1$, or $j = -1$ and $k = -1$.

Conversely, we suppose that γ_1 and γ_1' are irrational and $\gamma_1 = \gamma_1'^m$ $(m = 1$ or $-1)$. We define an autmorphism θ by

$$\theta(u(g_1)) = u(g_1), \quad \theta(u(g_2)) = u(g_2)^m.$$

This automorphism θ satisfies $\theta^{-1}\alpha_\gamma\theta = \alpha_{\gamma'}$. Then α_γ and α_γ' are conjugate.

Lemma 2. *Put $\gamma = (1, \gamma_1)$, $\gamma' = (1, \gamma_1')$ with $\gamma_1, \gamma_1' \in \mathbb{T}$ and γ_1 is a irrational.Then α_γ and $\alpha_{\gamma'}$ are conjugate if and only if $\gamma_1 = \gamma_1'$ or $\gamma_1^{-1} = \gamma_1'$.*

Proof. Trivial from preceding aurgument.

3. Proof of Theorem. For the sake of simplicity, we show the case of $n = 2$. Another case is proved by a similar method. Let α be an action of $\mathbb{T}^2$ on $\mathrm{Aut}(L(F_n))$ defined by $(*)$. Then α is continuous by Lemma 1. Hence the autmorphism group $\alpha_{\mathbb{T}^2}$ is compact. Besson in [2:Proposition 1.7] proved that an automorphism θ of a finite von Neumann algebra M has entropy zero if θ is contained in a compact group of automorphisms for the topology of pointwise 2-norm convergence on $\mathrm{Aut}(M)$. Therefore $H(\alpha_\gamma) = 0$ for all $\gamma \in \mathbb{T}^2$. A family of uncountable non conjugate automorphisms of $L(F_2)$ is given by case of Lemma 2. □

Remark. J. Phillips gave an example of outer conjugacy classes of automorphisms of $L(F_n)$. His automorphisms have all entropy zero. However his technique to distinguish the automorphisms is not effect for $L(F_n)$ $(n < +\infty)$. Because they are classfied by $\gamma = (1, \gamma_1, \gamma_2, \cdots, \gamma_n, \cdots)$ $(\gamma_i \in \mathbb{T})$ for a group $\{1, \gamma_1, \gamma_2, \cdots, \gamma_n, \cdots\}$.

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Department of Mathematics
Osaka Kyouiku Univercity
Asahigaoka, Kashiwara, 582
Japan

Received February 23, 1995, Revised October 5, 1995