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ADJOINT FAMILIES IN TOPOLOGICAL VECTOR SPACES

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Abstract. Given a linear partial differential operator L of order m with c^m-coefficients and a distribution T on an open set Ω of $\mathbb{R}^n$, a necessary and sufficient condition is derived for the existence of a function $f \in L^p(\Omega)$, $1 < p < \infty$, such that $Lf = T$ in the sense of distribution.

1. Introduction

Suppose B is a reflexive Banach space, E a locally convex space and $T : B \to E$ a linear map (continuous or not). We obtain a necessary and sufficient condition so that given $g \in E$, there exists $f \in B$ such that $Tf = g$.

This result is applied to the problem of finding a solution of $f \in L^p(\Omega)$, $1 < p < \infty$ and Ω open in $\mathbb{R}^n$, for the differential equation $Lf = T$ where L is a partial differential operator of order m with c^m-coefficients and T is a distribution defined on Ω.

2. A Preliminary result in a Hilbert space

Proposition 1. Let T be a bounded linear operator on a Hilbert space H. Then given $g \in H$, there exists an $f \in H$ such that $Tf = g$ if and only if $\sup\limits_{\|u\|=1} \dfrac{|(g,u)|}{\|T^* u\|}$ is finite.

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Proof:

1) Suppose $Tf = g$. Then for any $u \in H$, $|(g,u)| = |(Tf,u)| = |(f,T^{\star}u)| \leq \|f\|\|T^{\star}u\|$.

2) Conversely, suppose $|(g,u)| \leq c\|T^{\star}u\|$ for every $u \in H$.

Consider now the linear functional λ defined on Ran $T^{\star}$ as follows:

For $v = T^{\star}u$, $\lambda(v) = (g,u)$. This definition does not depend on the particular choice of u and $|\lambda(v)| \leq c\|T^{\star}u\| = c\|v\|$.

Hence λ is a bounded linear functional on Ran $T^{\star}$ and it can be extended as a continuous linear functional on the whole of H. Let us denote this extension also by λ.

Consequently, there exists an $f \in H$ such that $\lambda(x) = (f,x)$ for every $x \in H$.

In particular, for every $u \in H$, $\lambda(T^{\star}u) = (f,T^{\star}u) = (Tf,u)$. But $T^{\star}u \in$ Ran $T^{\star}$ and hence $\lambda(T^{\star}u) = (g,u)$.

Thus, $(Tf,u) = (g,u)$ for every $u \in H$ and hence $Tf = g$.

3. Adjoint family in topological vector spaces.

Definition 1: Let E_1 and E_2 be two topological vector spaces; E_1' and E_2' are their topological dual spaces. Let $T : E_1 \to E_2$ and $S : E_2' \to E_1'$ be two linear operators (continuous or not) defined on some subspaces of E_1 and E_2' respectively, such that $(Tx,y) = (x,Sy)$ for $x \in$ Dom T and $y \in$ Dom S . Then $(E_1,E_2;T,S)$ is called an adjoint family.

Theorem 1. Let $(B,E;T,S)$ be an adjoint family where B is a reflexive Banach space and E is a topological vector space whose topological dual E' separates points of E. (Examples: $E = l^p, 0 < p < \infty$ or E is any locally convex space). Suppose that Dom T is B and Dom S is a $\sigma(E',E)$-dense subspace in E'.

Then, given $g \in E$, there exists an $f \in B$ such that $Tf = g$ if and only if $|(g,u)| \leq c\|Su\|$ for every $u \in \text{Dom } S$.

When a solution to the equation $Tf = g$ exists, it is unique if and only if Ran S is dense in B'.

Proof:

1) Let $Tf = g$

Then, for $u \in \text{Dom } S$, $|(g,u)| = |(f, Su)| \leq \|f\| \|Su\|$.

2) Suppose now that $g \in E$ is given and $|(g,u)| \leq c\|Su\|$ for every $u \in \text{Dom } S$.

Let $F = \{v \in B'$, where $v = Su$ for some $u \in E'\}$.

Then, as in the proof of Proposition 1, we note that the linear functional $L(v) = (g,u)$ defined on F extends as a bounded linear functional L on the whole of B' (using Hahn-Banach theorem, see Schaefer [1]).

Thus $L \in B'' = B$ since B is reflexive. Identify L with an $f \in B$ to write $L(x) = (f,x)$ for every $x \in B'$.

But, if $u \in \text{Dom } S$, $L(Su) = (g,u)$ since $Su \in F$.

Thus, for every $u \in \text{Dom } S$, $(g,u) = (f,Su) = (Tf,u)$. Since $Tf - g \in E$ vanishes on a dense set of E', $(Tf - g, y) = 0$ for every $y \in E'$ and since E' separates points of E, $Tf - g = 0$ in E.

Uniqueness: Let us suppose now that $Tf = g$ has a solution $f \in B$ for a given $g \in E$. We'll show that the solution is unique if and only if Ran S is dense in B'.

1) Suppose Ran S is not dense in B' .

Then, by Hahn-Banach theorem, there exists $h \in B$, $h \neq 0$ and h (Ran S) $= 0$, i.e. $(h, Su) = 0$ for $u \in \text{Dom } S$. This means that $(Th, u) = 0$ and then as shown earlier $Th = 0$.

Hence, $T(f+h) = Tf = g$ which means that the solution is not unique.

2) Conversely, suppose Ran S is dense in B'.

Suppose $Tf_1 = g = Tf_2$.

Then $(T(f_1 - f_2), u) = 0$ for every $u \in$ Dom S. i.e. $(f_1 - f_2, Su) = 0$. Since Ran S is dense in B', this implies that $f_1 - f_2 = 0$ in B.

4. An application

Let Ω be an open set in $\mathbb{R}^n$, $n \geq 1$. Let $L = \sum_{|k| \leq m} a_k(x)\partial^k$ be a linear partial differential operator of order m, with $a_k(x) \in c^m(\Omega)$.

Let $\mathcal{D}(\Omega)$ be the family of c^∞-functions with compact support in Ω and $\mathcal{D}'(\Omega)$ the space of distributions in Ω. $\mathcal{D}(\Omega)$ is a locally convex space (see Treves [2]).

Let $L^*u = \sum_{|k| \leq m}(-1)^{|k|}\partial^k(a_k(x)u)$ be the adjoint operator; this satisfies the condition $(LT, \varphi) = (T, L^*\varphi)$ for $T \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$.

For the reflexive Banach space $L^p(\Omega)$, $1 < p < \infty$, denote the norm by $\|\cdot\|_p$ and let $\frac{1}{p} + \frac{1}{q} = 1$.

Recall that for $f \in L^p(\Omega)$, $Lf \in \mathcal{D}'(\Omega)$ and that two distributions T and S are said to be equal if and only if $(T, \varphi) = (S, \varphi)$ for every $\varphi \in \mathcal{D}(\Omega)$.

With these notations, the following theorem is an immediate consequence of Theorem 1.

Theorem 2: Let L be a linear partial differential operator of order m with c^m-coefficients defined on an open set Ω in $\mathbb{R}^n$. Then given $T \in \mathcal{D}'(\Omega)$ there exists an $f \in L^p(\Omega)$, $1 < p < \infty$, such that $Lf = T$ if and only if $|T(\varphi)| \leq \|L^*\varphi\|_q$ for every $\varphi \in \mathcal{D}(\Omega)$.

When such a solution exists, it is unique if and only if $L^*(\mathcal{D}(\Omega))$ is dense in $L^q(\Omega)$.

References

[1] H.H. Schaefer: Topological Vector Spaces. Springer-Verlag, Berlin, 1980.

[2] F. Treves: Basic Linear Partial Differential Equations. Academic Press, New York, 1975.

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