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Pareto Optimum in a Cooperative Dynkin's Stopping Problem [1]

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Abstract. We consider two-person cooperative stopping gameof Dynkin's type and find ε–Pareto optimal pairs of stopping times by three methods. The first is the so–called scalarization and the corresponding optimal value process is characterized by a recursive relation. In the second we find an ε–Pareto optimal pair nearest to a goal, which two players desire but may not be able to achieve. We select thirdly a Pareto optimal pair which dominates a conservative value for each player. The set of such Pareto optimal pairs is called core. We finally apply them to a Markov model and give simple examples.

Key words. stopping game, cooperative game, Pareto optimal, martingale, core

1. Introduction.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_n)_{n\in N}$ an increasing family of sub-σ-fields of $\mathcal{F}$, where $N = \{0,1,2,\ldots\}$ is a time space. Let $\mathcal{W}$ denote the family of all sequences of random variables $X = (X_n)_{n\in N}$ defined on $(\Omega, \mathcal{F}, P)$ and adapted to $(\mathcal{F}_n)$ such that random variable $\sup_n X_n^+$ is integrable and sequence (X_n^-) is uniformly integrable, where $x^+ = \max(0,x)$ and $x^- = (-x)^+$. For each $n \in N$, we also denote by Λ_n the class of pairs of $(\mathcal{F}_n)$–stopping times (τ, σ) such that $n \leqq \tau \wedge \sigma < \infty$ a. s. , where $\tau \wedge \sigma = \min(\tau, \sigma)$.

For six random sequences X^i, Y^i and W^i $(i = 1,2)$ in $\mathcal{W}$, we consider the following cooperative stopping game with a finite constraint. There are two players and the first and the second players choose stopping times τ_1 and τ_2, respectively, such that (τ_1, τ_2) is in Λ_0. Then the ith player $(i = 1,2)$ gets the reward

$$J_i(\tau_1,\tau_2) = X^i_{\tau_i} I_{(\tau_i<\tau_j)} + Y^i_{\tau_j} I_{(\tau_j<\tau_i)} + W^i_{\tau_i} I_{(\tau_i=\tau_j)}, \quad j = 1,2,\ j \neq i,$$

where I_A is the indicator function of a set A in $\mathcal{F}$. The aim of the ith player is to maximize the expected gain $E[J_i(\tau_1,\tau_2)]$ with respect to τ_i, cooperating with another player, if necessary. However, the stopping time chosen by one of them generally depends upon one decided by another, even if they cooperate. Thus we shall use the concept of Pareto optimality as in the usual cooperative game or the multiobjective problem.

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Before giving the definition of Pareto optimality, we define partial orders in the two-dimensional Euclidean space as follows: for two vectors $x = (x_1, x_2)$ and $y = (y_1, y_2)$, $x > y$ if $x_i > y_i, i = 1,2$; $x \geqq y$ if $x_i \geqq y_i, i = 1,2$; $x = y$ if $x_i = y_i, i = 1,2$; $x \geq y$ if $x \geqq y$ and $x \neq y$. We also define a conditional expected reward for each player by $G_n^i(\tau,\sigma)= E[J_i(\tau,\sigma) \mid \mathcal{F}_n]$, $i = 1,2$, and a vector by $G_n^*(\tau,\sigma) = (G_n^1(\tau,\sigma), G_n^2(\tau,\sigma))$ and let $e = (1,1)$. For the sake of simplicity, without further comments we assume that all inequalities and equalities between random variables hold in the sense of "almost surely".

For $n \in N$and a nonnegative real number ε, we say that a pair $(\tau_\varepsilon, \sigma_\varepsilon)$ in Λ_n is *ε-weak* (resp. *ε-strong*) *Pareto optimal at n* if there is no pair (τ,σ) in Λ_n such that

$$G_n^*(\tau,\sigma) > G_n^*(\tau_\varepsilon,\sigma_\varepsilon) + \varepsilon e \quad (\text{resp. } G_n^*(\tau,\sigma) \geq G_n^*(\tau_\varepsilon,\sigma_\varepsilon) + \varepsilon e).$$

We shall simply call a 0–weak (resp. 0–strong) Pareto optimal pair a weak (resp. strong) Pareto optimal one.

In §2 we consider an optimal stopping problem with the reward

$$J(\tau,\sigma) \ = X_\tau I_{(\tau<\sigma)} + Y_\sigma I_{(\sigma<\tau)} + W_\tau I_{(\tau=\sigma)}$$

to maximize the expected gain $E[J(\tau,\sigma)]$ with respect to pair (τ,σ) in Λ_0, and we give fundamental results for the problem. These play an important role though this paper. We also define shadow optimal value process, which is optimal value process for one player, and obtain a martingale property. The shadow optimal value processes are used to select a Pareto optimal pair in §§ 4 and 5. We find Pareto optimal pairs by a method of the so-called scalarization in §3, and introduce a concept of goal programming and find a Pareto optimal pair nearest to a given goal in §4. We in §5 investigate core which is a set of Pareto optimal pairs dominating a conservative value for each player, and in §6 consider a Markov model as a special case.

The two–person zero–sum stopping game with a discrete time space was first introduced by Dynkin [6], who proved that the game has a value and constructed ε–saddle point, and it was developed by Neveu [16] and Elbakidze [7]. Continuous time analogue of such a game problem was studied by Lepeltier and Maingueneau [11], Morimoto [13], Stettner [22] and many others. Ohtsubo [17] also investigated a zero–sum stopping game with a finite constraint, that is, in the game pairs of any stopping times τ, σ satisfy $\tau \wedge \sigma < \infty$ a. s. , and he developed the problem in [18,19,21].

Such a stopping game was extended to a nonzero–sum case and studied in many literature (for example, Bensoussan and Friedman [2], Morimoto [14], Nagai [15], Ohtsubo [20] and Cattiaux and Lepeltier [3]), in order to find ε–Nash equilibrium point. A non–Dynkin's type of nonzero–sum stopping game was considered in Mamer [12] for discrete time and in Huang and Li [10] for continuous one.

All these are noncooperative game. In the present paper we deal with a cooperative Dynkin's stopping game and find ε–Pareto optimal pairs. On the other hand the cooperative game is anologous to multiobjective optimal stopping problem in the sense of using the concept of Pateto optimality. Hisano [9] formulated a multiobjective stopping problem and found an optimal stopping time, and Gugerli [8] investigated such a problem for the class of randomized stopping times.

2. Shadow optimum and fundamental lemmas.

In this section we give fundamental results, in order to obtain properties of shadow optimum and to use these results in the remaining sections. We first define shadow optimum α^i for the reward $J_i\ (\tau,\sigma)$ as follows:

$$\alpha_n^i = \operatorname*{ess\ sup}_{(\tau,\sigma)\in\Lambda_n} G_n^i(\tau,\sigma), \quad n\in N,\ i=1,2.$$

In multiobjective programming, the shadow optima are also called "ideal solution". If there exists a pair (τ^*,σ^*) in Λ_n such that $\alpha_n^i = G_n^i(\tau^*,\sigma^*)$ for every $i=1,2$, then the pair is not only Pareto optimal at n but also best optimal in the sense that $G_n^i(\tau^*,\sigma^*) \geqq G_n^i(\tau,\sigma)$ for all (τ,σ) in Λ_n and every $i=1,2$. However such a case seldom occurs. The shadow optima are useful to select a Pareto optimal pair.

Now, to obtain constructive property of the shadow optima, we generally consider an optimal stopping problem so as to maximize the expected reward $G_n(\tau,\sigma)= E[J(\tau,\sigma)\mid\mathcal{F}_n]$ with respect to (τ,σ) in Λ_n, where

$$J(\tau,\sigma) = X_\tau I_{(\tau<\sigma)} + Y_\sigma I_{(\sigma<\tau)} + W_\tau I_{(\tau=\sigma)}$$

for X,Y and W in $\mathcal{W}$. The optimal value process $\beta=(\beta_n)_{n\in N}$ is defined by

$$\beta_n = \operatorname*{ess\ sup}_{(\tau,\sigma)\in\Lambda_n} G_n(\tau,\sigma), \quad n\in N.$$

For $n\in N$and $\varepsilon \geqq 0$, we say that a pair $(\tau_\varepsilon,\sigma_\varepsilon)$ in Λ_n is (ε,β)*-optimal at* n if $\beta_n \leqq G_n(\tau_\varepsilon,\sigma_\varepsilon) +\varepsilon$.

LEMMA 2.1. (i) *The process* $\beta=(\beta_n)$ *satisfies the recursive relation:*

$$\beta_n = \max(X_n,Y_n,W_n,E[\beta_{n+1}\mid\mathcal{F}_n]), \quad n\in N. \tag{1}$$

(ii) β *is the smallest supermartingale dominating the process* $(\max(X_n,Y_n,W_n))_{n\in N}$.
(iii) $\limsup_n \beta_n = \liminf_n \max(X_n,Y_n,W_n)$.

PROOF. The lemma is easily proved as in the classical optimal stopping problem (cf. Chow, Robbins and Siegmund [4] or Neveu [16]). □

From this lemma it is easy to see that the process β coincides with an optimal value process $\hat\beta=(\hat\beta_n)$ in an optimal stopping problem with a reward $\hat Z_n=\max(X_n,Y_n,W_n)$ of time n, i. e.

$$\hat\beta_n = \operatorname*{ess\ sup}_{n\leq\tau<\infty} E[\hat Z_\tau\mid\mathcal{F}_n].$$

Hence $\beta = \hat{\beta}$ is constructive by the method of the backward induction as in [4].

For each $n \in N$and $\varepsilon \geqq 0$, define stopping times $\tau_n^\varepsilon \equiv \tau_n^\varepsilon(\beta)$ and $\sigma_n^\varepsilon \equiv \sigma_n^\varepsilon(\beta)$ by

$$\tau_n^\varepsilon = \inf\{k \geqq n \mid \beta_k \leqq \max(X_k, W_k) + \varepsilon\}$$

$$\sigma_n^\varepsilon = \inf\{k \geqq n \mid X_k + \varepsilon < \beta_k \leqq \max(Y_k, W_k) + \varepsilon\}$$

where $\inf(\phi) = +\infty$.

LEMMA 2.2. *Let $n \in N$be arbitrary.*

(i) *For each $\varepsilon > 0$, the pair $(\tau_n^\varepsilon, \sigma_n^\varepsilon)$ is (ε, β)-optimal at n.*

(ii) *The stopping time $\tau_n^0 \wedge \sigma_n^0$ is a. s. finite, the pair (τ_n^0, σ_n^0) is $(0, \beta)$-optimal at n.*

PROOF. When ε is positive, it follows from Lemma 2.1 (iii) that the stopping time $\tau_n^\varepsilon \wedge \sigma_n^\varepsilon$ is a. s. finite. Thus, for $\varepsilon \geqq 0$, it suffices to show that inequality $\beta_n \leqq G_n(\tau_n^\varepsilon, \sigma_n^\varepsilon) + \varepsilon$ holds for each $n \in N$. From Lemma 2.1 (i) and the optional sampling theorem, we have $\beta_n = E[\beta_{\tau_n^\varepsilon \wedge \sigma_n^\varepsilon} \mid \mathcal{F}_n]$. Furthermore, since $\beta_k \leqq X_k + \varepsilon$ on $\{\tau_n^\varepsilon = k < \sigma_n^\varepsilon\}$, $\beta_k \leqq Y_k + \varepsilon$ on $\{\sigma_n^\varepsilon = k < \tau_n^\varepsilon\}$ and $\beta_k \leqq W_k + \varepsilon$ on $\{\tau_n^\varepsilon = \sigma_n^\varepsilon = k\}$, we have the desired inequality $\beta_n \leqq G_n(\tau_n^\varepsilon, \sigma_n^\varepsilon) + \varepsilon$. □

REMARK 2.1. In the above thorem, it remains true even if the pair $(\tau_n^\varepsilon, \sigma_n^\varepsilon)$ is replaced by pairs $(\hat{\tau}_n^\varepsilon, \hat{\sigma}_n^\varepsilon)$ or $(\hat{\tau}_n^\varepsilon, \sigma_n^\varepsilon)$, where

$$\hat{\tau}_n^\varepsilon = \inf\{k \geqq n \mid Y_k + \varepsilon < \beta_k \leqq \max(X_k, W_k) + \varepsilon\}$$

$$\hat{\sigma}_n^\varepsilon = \inf\{k \geqq n \mid \beta_k \leqq \max(Y_k, W_k) + \varepsilon\}.$$

However other pair $(\tau_n^\varepsilon, \hat{\sigma}_n^\varepsilon)$ is not necessarily (ε, β)–optimal.

3. Scalarization and Pareto optima.

In this section we find Pareto optimal pairs by the method of the well–known scalarization.

Let S denote the set of vectors $\lambda = (\lambda_1, \lambda_2)$ in $\mathbf{R}^2$ satisfying $\lambda \geq 0$ and $\lambda_1 + \lambda_2 = 1$, and S_0 the set of λ in S such that $\lambda > 0$. For given X^i, Y^i, W^i in $\mathcal{W}$ and λ in S, we define sequences of random variables by

$$X_n(\lambda) = \lambda_1 X_n^1 + \lambda_2 Y_n^2, \quad Y_n(\lambda) = \lambda_1 Y_n^1 + \lambda_2 X_n^2, \quad W_n(\lambda) = \lambda_1 W_n^1 + \lambda_2 W_n^2,$$

random variables by,

$$J(\tau, \sigma; \lambda) = \lambda_1 J_1(\tau, \sigma) + \lambda_2 J_2(\tau, \sigma), \quad G_n(\tau, \sigma; \lambda) = E[J(\tau, \sigma; \lambda) \mid \mathcal{F}_n],$$

and a maximum value process by

$$V_n(\lambda) = \operatorname*{ess\,sup}_{(\tau,\sigma) \in \Lambda_n} G_n(\tau, \sigma; \lambda), \quad n \in N.$$

Then we have relations

$$J(\tau,\sigma;\lambda) = X_\tau(\lambda)I_{(\tau<\sigma)} + Y_\sigma(\lambda)I_{(\sigma<\tau)} + W_\tau(\lambda)I_{(\tau=\sigma)},$$
$$G_n(\tau,\sigma;\lambda) = \lambda_1 G_n^1(\tau,\sigma) + \lambda_2 G_n^2(\tau,\sigma).$$

We also define stopping times for the process $V(\lambda) = (V_n(\lambda))$ as follows:

$$\tau_n^\varepsilon \equiv \tau_n^\varepsilon(\lambda) = \inf\{k \geqq n \mid V_k(\lambda) \leqq \max(X_k(\lambda), W_k(\lambda)) + \varepsilon\}$$

$$\sigma_n^\varepsilon \equiv \sigma_n^\varepsilon(\lambda) = \inf\{k \geqq n \mid X_k(\lambda) + \varepsilon < V_k(\lambda) \leqq \max(Y_k(\lambda), W_k(\lambda)) + \varepsilon\}$$

for $n \in N$ and $\varepsilon \geqq 0$. The following theorems are immediate results of Lemmas 2.1 and 2.2.

THEOREM 3.1. *Let λ in S be arbitrary.*
(i) *The process $V(\lambda) = (V_n(\lambda))$ satisfies the recursive relation:*

$$V_n(\lambda) = \max(X_n(\lambda), Y_n(\lambda), W_n(\lambda), E[V_{n+1}(\lambda) \mid \mathcal{F}_n]), \quad n \in N. \tag{2}$$

(ii) *$V(\lambda)$ is the smallest supermartingale dominating the process $Z(\lambda) = (Z_n(\lambda))$,* where $Z_n(\lambda) = \max(X_n(\lambda),\ Y_n(\lambda),\ W_n(\lambda))$.

THEOREM 3.2. *Let $n \in N$ and $\lambda \in S$ be arbitrary.*
(i) *For each $\varepsilon > 0$, the pair $(\tau_n^\varepsilon, \sigma_n^\varepsilon)$ is $(\varepsilon, V(\lambda))$-optimal at n in the sense that $V_n(\lambda) \leqq G_n(\tau_n^\varepsilon, \sigma_n^\varepsilon; \lambda) + \varepsilon$.*
(ii) *If the stopping time $\tau_n^0 \wedge \sigma_n^0$ is a. s. finite, the pair (τ_n^0, σ_n^0) is $(0, V(\lambda))$-optimal at n.*

The general lemma below is a well-known result in multiobjective problem.

LEMMA 3.1. *Let $n \in N$, $\varepsilon \geqq 0$ and $\lambda \in S$ be arbitrary. If a pair $(\tau_\varepsilon, \sigma_\varepsilon)$ in Λ_n satisfies inequality $V_n(\lambda) \leqq G_n(\tau_\varepsilon, \sigma_\varepsilon; \lambda) + \varepsilon$, then the pair $(\tau_\varepsilon, \sigma_\varepsilon)$ is ε-weak Pareto optimal at n. Furthermore when λ is in S_0, the pair $(\tau_\varepsilon, \sigma_\varepsilon)$ is ε-strong Pareto optimal at n.*

PROOF. We suppose that the pair $(\tau_\varepsilon, \sigma_\varepsilon)$ is not ε-weak Pareto optimal. There then exists a pair (τ, σ) in Λ_n such that $G_n^*(\tau,\sigma) > G_n^*(\tau_\varepsilon, \sigma_\varepsilon) + \varepsilon e$, that is, $G_n^i(\tau,\sigma) > G_n^i(\tau_\varepsilon,\sigma_\varepsilon) + \varepsilon$ for every $i = 1, 2$. Thus we have

$$\begin{aligned} G_n(\tau,\sigma;\lambda) &= \lambda_1 G_n^1(\tau,\sigma) + \lambda_2 G_n^2(\tau,\sigma) \\ &> \lambda_1 G_n^1(\tau_\varepsilon,\sigma_\varepsilon) + \lambda_2 G_n^2(\tau_\varepsilon,\sigma_\varepsilon) + \varepsilon \\ &= G_n(\tau_\varepsilon,\sigma_\varepsilon;\lambda) + \varepsilon, \end{aligned}$$

so that $V_n(\lambda) > G_n(\tau_\varepsilon, \sigma_\varepsilon; \lambda) + \varepsilon$, which is a contradiction. Hence the pair $(\tau_\varepsilon, \sigma_\varepsilon)$ is ε-weak Pareto optimal. Similarly, the statement for $\lambda > 0$ is proved. □

For each $n \in N$, let a subset $\mathcal{G}_n$ in R^2 denote the set of all vectors $G_n^*(\tau,\sigma)$ satisfying $(\tau,\sigma) \in \Lambda_n$. Then the set $\mathcal{G}_n$ is not in general convex in our problem. If $\mathcal{G}_n$ is a convex set, we can discuss the converse of Lemma 3.1, that is, if a pair $(\tau_\varepsilon, \sigma_\varepsilon)$ is ε–weak Pareto optimal at n there is a vector λ in S such that $V_n(\lambda) \leqq G_n(\tau_\varepsilon, \sigma_\varepsilon; \lambda) + \varepsilon$ (cf. Aubin [1]).

Theorem 3.2 and Lemma 3.1 immediately imply the following theorem.

THEOREM 3.3. *Let $n \in N$ and $\lambda \in S$ be arbitrary.*

(i) *For each $\varepsilon > 0$, the pair $(\tau_n^\varepsilon, \sigma_n^\varepsilon)$ is ε–weak Pareto optimal at n ; if in addition λ is in S_0 then the pair $(\tau_n^\varepsilon, \sigma_n^\varepsilon)$ is ε–strong Pareto optimal at n.*

(ii) *If the stopping time $\tau_n^0 \wedge \sigma_n^0$ is a. s. finite, the pair (τ_n^0, σ_n^0) is weak Pareto optimal at n ; if in addition λ is in S_0 then the pair (τ_n^0, σ_n^0) is strong Pareto optimal at n.*

4. Pareto optimum nearest to a goal.

In order that each player selects a convincible Pareto optimal pair, we introduce in this section a concept of goal programming in multiobjective problem (cf. Cohon [5]).

Let two processes $M^i = (M_n^i), i = 1,2$, be in $\mathcal{W}$, and we define a vector process $M = (M_n)$ by $M_n = (M_n^1, M_n^2)$. This M_n^i is a goal (target value) for the ith player when he starts the game at time n, but it may be impossible that he achieves the goal. For $n \in N$, $(\tau,\sigma) \in \Lambda_n$ and $\mu = (\mu_1, \mu_2) \geq 0$, a distance from M_n is defined by

$$D_n^p(\tau,\sigma) = \| M_n - G_n^*(\tau,\sigma) \|_{(\mu,p)},$$

where $\| x \|_{(\mu,p)} = (\sum_{i=1}^2 \mu_i \mid x_i \mid^p)^{1/p}$ for $x = (x_1, x_2)$ and $1 \leqq p < \infty$, and $\| x \|_{(\mu,\infty)} = \max_{i=1,2}(\mu_i \mid x_i \mid)$ for $p = \infty$, which are called Minkowski's norm, and a minimum value process $D^p = (D_n^p)$ for the distance is defined by

$$D_n^p \equiv D_n^p(\mu, M) = \operatorname*{ess\,inf}_{(\tau,\sigma)\in\Lambda_n} D_n^p(\tau,\sigma).$$

For each $n \in N$, $\mu \geq 0, 1 \leqq p \leqq \infty$ and $\varepsilon \geqq 0$, we say that a pair $(\tau_\varepsilon, \sigma_\varepsilon)$ in Λ_n is $(\varepsilon; p)$–*optimal at n* if $D_n^p \geqq D_n^p(\tau_\varepsilon, \sigma_\varepsilon) - \varepsilon$.

LEMMA 4.1. *Suppose that $M \geqq \alpha$, i. e. $M_n^i \geqq \alpha_n^i$ for each $n \in N$ and every $i = 1,2$, and let $n \in N$ and $\varepsilon \geqq 0$ be arbitrary.*

(i) *Let μ be in S (resp. S_0). If a pair $(\tau_\varepsilon, \sigma_\varepsilon)$ is $(\varepsilon; 1)$–optimal at n, then the pair $(\tau_\varepsilon, \sigma_\varepsilon)$ is ε–weak (resp. ε–strong) Pareto optimal at n.*

(ii) *Let $\mu = (1,1)$. If a pair $(\tau_\varepsilon, \sigma_\varepsilon)$ is $(\varepsilon; \infty)$–optimal at n, then the pair $(\tau_\varepsilon, \sigma_\varepsilon)$ is ε–weak Pareto optimal at n.*

(iii) *Let μ be in S (resp. S_0) and $1 < p < \infty$. If a pair (τ_0, σ_0) is $(0; p)$–optimal at n, then the pair (τ_0, σ_0) is weak (resp. strong) Pareto optimal at n.*

PROOF. We shall prove only statement (ii). Other statements are similarly proved. We suppose that the pair $(\tau_\varepsilon, \sigma_\varepsilon)$ is not ε-weak Pareto optimal, so that there is a pair (τ, σ) in Λ_n satisfying $G_n^*(\tau,\sigma) > G_n^*(\tau_\varepsilon,\sigma_\varepsilon) + \varepsilon e$. Thus we have

$$\begin{aligned} D_n^\infty(\tau,\sigma) &= \max_{i=1,2}\{M_n^i - G_n^i(\tau,\sigma)\} \\ &< \max_{i=1,2}\{M_n^i - G_n^i(\tau_\varepsilon,\sigma_\varepsilon) - \varepsilon\} \\ &= D_n^\infty(\tau_\varepsilon,\sigma_\varepsilon) - \varepsilon. \end{aligned}$$

Hence $D_n^\infty < D_n^\infty(\tau_\varepsilon,\sigma_\varepsilon) - \varepsilon$, which is contrary to $(\varepsilon;\infty)$–optimality of $(\tau_\varepsilon,\sigma_\varepsilon)$. □

We shall consider a case $p = 1$ in the remaining part of this section, characterize the minimum value process $D^1 = (D_n^1)$ and select ε–Pareto optimal pair by using the process D^1. For $\mu \geq 0$, $k \geqq n$ and M^i $(i = 1,2)$ in $\mathcal{W}$ such that $M \geqq \alpha$, we define sequences of random variables by

$$\begin{aligned} \hat{X}_k(n) &= \mu_1(M_n^1 - X_k^1) + \mu_2(M_n^2 - Y_k^2) = \mu_1 M_n^1 + \mu_2 M_n^2 - X_k(\mu), \\ \hat{Y}_k(n) &= \mu_1(M_n^1 - Y_k^1) + \mu_2(M_n^2 - X_k^2) = \mu_1 M_n^1 + \mu_2 M_n^2 - Y_k(\mu), \\ \hat{W}_k(n) &= \mu_1(M_n^1 - W_k^1) + \mu_2(M_n^2 - W_k^2) = \mu_1 M_n^1 + \mu_2 M_n^2 - W_k(\mu), \end{aligned}$$

where $X_k(\mu), Y_k(\mu)$ and $W_k(\mu)$ are given in §3 by replacing λ by μ, as well as $G_n(\tau,\sigma;\mu)$ and $V_n(\mu)$ below. It then follows that for $(\tau,\sigma) \in \Lambda_n$,

$$\begin{aligned} D_n^1(\tau,\sigma) &= E[\hat{X}_\tau(n) I_{(\tau<\sigma)} + \hat{Y}_\sigma(n) I_{(\sigma<\tau)} + \hat{W}_\tau(n) I_{(\tau=\sigma)} \mid \mathcal{F}_n] \\ &= \mu_1 M_n^1 + \mu_2 M_n^2 - \mu_1 G_n^1(\tau,\sigma) - \mu_2 G_n^2(\tau,\sigma) \\ &= \mu_1 M_n^1 + \mu_2 M_n^2 - G_n(\tau,\sigma;\mu). \end{aligned} \tag{3}$$

THEOREM 4.1. *Suppose that $M^i (i = 1,2)$ in $\mathcal{W}$ are martingales and $M \geqq \alpha$, and let $\mu \geq 0$.*

(i) *The process $D^1 = (D_n^1)$ satisfies the recursive relation :*

$$D_n^1 = \min(\hat{X}_n(n), \hat{Y}_n(n), \hat{W}_n(n), E[D_{n+1}^1 \mid \mathcal{F}_n]), \quad n \in N. \tag{4}$$

(ii) *D^1 is the largest submartingale dominated by the process $\hat{Z} = (\hat{Z}_n)$*, where $\hat{Z}_n = \min(\hat{X}_n(n), \hat{Y}_n(n), \hat{W}_n(n))$.

PROOF. From (3) we have

$$D_n^1 = \mu_1 M_n^1 + \mu_2 M_n^2 - \operatorname*{ess\,sup}_{(\tau,\sigma)\in\Lambda_n} G_n(\tau,\sigma;\mu) = \mu_1 M_n^1 + \mu_2 M_n^2 - V_n(\mu).$$

Since M^i are martingales, we also have

$$E[D_{n+1}^1 \mid \mathcal{F}_n] = \mu_1 M_n^1 + \mu_2 M_n^2 - E[V_{n+1}(\mu) \mid \mathcal{F}_n].$$

Thus the relation (4) is equivalent to

$$V_n(\mu) = \max(X_n(\mu), Y_n(\mu), W_n(\mu), E[V_{n+1}(\mu) \mid \mathcal{F}_n]).$$

Hence the proof of this theorem is reduced to that of Theorem 3.1 or Lemma 2.1. □

REMARK 4.1. If the shadow optima α^i $(i = 1,2)$ are martingales, we may be take α^i as M^i. Also constant processes $M^i = E[\sup_{n\in N}\{\max(X_n^i, Y_n^i, W_n^i)\}^+]$, $i = 1,2$, satisfy conditions in Theorem 4.1.

We define stopping times $\tau_n^\varepsilon \equiv \tau_n^\varepsilon(\mu, M)$ and $\sigma_n^\varepsilon \equiv \sigma_n^\varepsilon(\mu, M)$ for the process D^1 by

$$\tau_n^\varepsilon = \inf\{k \geqq n \mid D_k^1 \geqq \min(\hat{X}_k(k), \hat{W}_k(k)) - \varepsilon\},$$

$$\sigma_n^\varepsilon = \inf\{k \geqq n \mid \hat{X}_k(k) - \varepsilon > D_k^1 \geqq \min(\hat{Y}_k(k), \hat{W}_k(k)) - \varepsilon\}.$$

Using Lemma 4.1 and Theorem 4.1 and making argument analogous to Theorem 3.3 or Lemma 2.2, we have another main theorem in this section.

THEOREM 4.2. *Suppose that M^i $(i = 1,2)$ in $\mathcal{W}$ are martingales and $M \geqq \alpha$, and let $n \in N$ and $\mu \in S$ be arbitrary.*

(i) *For each $\varepsilon > 0$, the pair $(\tau_n^\varepsilon, \sigma_n^\varepsilon)$ is ε–weak Pareto optimal at n ; if in addition μ is in S_0 then the pair $(\tau_n^\varepsilon, \sigma_n^\varepsilon)$ is ε–strong Pareto optimal at n.*

(ii) *If the stopping time $\tau_n^0 \wedge \sigma_n^0$ is a. s. finite, the pair (τ_n^0, σ_n^0) is weak Pareto optimal at n ; if in addition μ is in S_0 then the pair (τ_n^0, σ_n^0) is strong Pareto optimal at n.*

5. Core.

In this section, we introduce a core which is a subset of Pareto optimal pairs. For given $Z^i = (Z_n^i), i = 1,2$, in $\mathcal{W}$ and $\varepsilon \geqq 0$, we define ε–*core* $C_n^\varepsilon(Z)$ at time n by the class of all pairs $(\tau_\varepsilon, \sigma_\varepsilon)$ in Λ_n such that $(\tau_\varepsilon, \sigma_\varepsilon)$ is ε–weak Pareto optimal at n and inequality

$$G_n^*(\tau_\varepsilon, \sigma_\varepsilon) \geqq Z_n - \varepsilon e \tag{5}$$

holds, where $Z_n = (Z_n^1, Z_n^2)$. This Z^i is one called threat functional and is interpreted as a minimum value which the ith player is able to compromise with. By the definition of core, if a pair $(\tau_n^\varepsilon, \sigma_n^\varepsilon)$ in Theorems 3.3 or 4.2 satisfies inequality (5), then the pair $(\tau_n^\varepsilon, \sigma_n^\varepsilon)$ is an element of ε–core $C_n^\varepsilon(Z)$.

In general ε–core $C_n^\varepsilon(Z)$ may be empty, even if ε is positive and $Z_n^i \leqq \alpha_n^i$ $(i = 1,2)$. For example, when $X_n^i = W_n^i = a$, $Y_n^i = b$ and $Z_n^i = c$ $(i = 1,2,\ n \in N)$ for constants a, b and c satisfying $a < c < b$, we have $\mathcal{G}_n = \{(a,b),(b,a),(a,a)\}$, $n \in N$, and vectors (a,b) and (b,a) correspond weak (and strong) Pareto optimal pairs. However, since there is no pair $(\tau_\varepsilon, \sigma_\varepsilon)$ satisfying (5) for sufficiently small ε, ε–core $C_n^\varepsilon(Z)$ is empty.

In this section we first give necessary and sufficient conditions for ε-core $C_n^\varepsilon(Z)$ $(\varepsilon > 0)$ to be nonempty. To end this, for given other processes M^i $(i = 1, 2)$ in $\mathcal{W}$ and a pair (τ, σ) in Λ_n, we define random variables by, if these exist,

$$\gamma_n^i(\tau,\sigma) = \frac{M_n^i - G_n^i(\tau,\sigma)}{M_n^i - Z_n^i}, \quad i = 1, 2,$$
$$\gamma_n(\tau,\sigma) = \max\left(\gamma_n^1(\tau,\sigma), \gamma_n^2(\tau,\sigma)\right),$$

and a minimum value process $\gamma^* = (\gamma_n^*)$ by

$$\gamma_n^* \equiv \gamma_n^*(M, Z) = \operatorname*{ess\,inf}_{(\tau,\sigma)\in\Lambda_n} \gamma_n(\tau,\sigma).$$

Here M^i may be goals as in §4. The following assumption is natural in our problem.

ASSUMPTION 5.1. $M \geqq \alpha \geqq Z$ and $M > Z$.

If Assumption 5.1 is satisfied, γ_n^* is nonnegative, but it is not necessarily less than or equal to 1. Indeed, in the above example, letting $M_n^i = b$, $i = 1, 2$, $n \in N$, we have $\gamma_n^* = (b - a)/(b - c) > 1$, $n \in N$.

ASSUMPTION 5.2. Processes $M^i - Z^i$ $(i = 1, 2)$ are bounded from above, that is, there is a constant L such that $M_n^i - Z_n^i \leqq L$ for all $i = 1, 2$ and all $n \in N$.

THEOREM 5.1. *Suppose Assumptions 5.1 and 5.2 are satisfied. For each $n \in N$, the following conditions are equivalent :*

(a) *For each $\varepsilon > 0$, ε-core $C_n^\varepsilon(Z)$ is nonempty.*

(b) *For each $\varepsilon > 0$, there exists a pair $(\tau_\varepsilon, \sigma_\varepsilon)$ in Λ_n satisfying inequality (5).*

(c) $\gamma_n^* \leqq 1$.

Furthermore, if one of conditions (a),(b) *and* (c) *is satisfied, a pair $(\hat{\tau}_\varepsilon, \hat{\sigma}_\varepsilon)$ in Λ_n such that $\gamma_n^* \geqq \gamma_n(\hat{\tau}_\varepsilon, \hat{\sigma}_\varepsilon) - \varepsilon/L$ is an element of $C_n^\varepsilon(Z)$ for each $n \in N$ and every $\varepsilon > 0$.*

PROOF. By the definition of ε-core $C_n^\varepsilon(Z)$, the implication (a) $\Rightarrow$ (b) is immediate.
(b) $\Rightarrow$ (c). From inequality (5), we have $\gamma_n^i(\tau_\varepsilon, \sigma_\varepsilon) \leqq 1 + \varepsilon(M_n^i - Z_n^i)^{-1}$ for every $i = 1, 2$, so that

$$\gamma_n^* \leqq \gamma_n(\tau_\varepsilon, \sigma_\varepsilon) \leqq 1 + \varepsilon \max_{i=1,2}(M_n^i - Z_n^i)^{-1}.$$

Letting as $\varepsilon \downarrow 0$, we have the desired inequality $\gamma_n^* \leqq 1$.

(c) $\Rightarrow$ (a). By the definition of γ_n^*, there is a pair $(\hat{\tau}_\varepsilon, \hat{\sigma}_\varepsilon)$ in Λ_n such that $\gamma_n^* \geqq \gamma_n(\hat{\tau}_\varepsilon, \hat{\sigma}_\varepsilon) - \varepsilon/L$. Thus since $\gamma_n^* \leqq 1$, we have

$$G_n^i(\hat{\tau}_\varepsilon, \hat{\sigma}_\varepsilon) \geqq Z_n^i - \varepsilon(M_n^i - Z_n^i)/L \geqq Z_n^i - \varepsilon, \quad i = 1, 2,$$

which implies that the pair $(\hat{\tau}_\varepsilon, \hat{\sigma}_\varepsilon)$ satisfies inequality (5). Next assume that this pair $(\hat{\tau}_\varepsilon, \hat{\sigma}_\varepsilon)$ is not ε–weak Pareto optimal at n, that is, there exists a pair (τ, σ) in Λ_n satisfying $G_n^i(\tau, \sigma) > G_n^i(\hat{\tau}_\varepsilon, \hat{\sigma}_\varepsilon) + \varepsilon$ for every $i = 1, 2$. Then we have

$$\gamma_n^i(\tau, \sigma) < \gamma_n^i(\hat{\tau}_\varepsilon, \hat{\sigma}_\varepsilon) - \varepsilon/L \leqq \gamma_n(\hat{\tau}_\varepsilon, \hat{\sigma}_\varepsilon) - \varepsilon/L, \quad i = 1, 2,$$

so that

$$\gamma_n(\tau, \sigma) < \gamma_n(\hat{\tau}_\varepsilon, \hat{\sigma}_\varepsilon) - \varepsilon/L \leqq \gamma_n^*,$$

which is contrary to the fact that in general $\gamma_n(\tau, \sigma) \geqq \gamma_n^*$. Hence the pair $(\hat{\tau}_\varepsilon, \hat{\sigma}_\varepsilon)$ is ε–weak Pareto optimal at n, and it is in $C_n^\varepsilon(Z)$. Therefore ε–core $C_n^\varepsilon(Z)$ is nonempty.

The proof of the second statement is given in that of the implication (c) $\Rightarrow$ (a). □

In the following theorem, we give a characterization of an element in $C_n^0(Z)$.

THEOREM 5.2. *Suppose that Assumption* 5.1 *is satisfied and that* $\gamma_n^* \leqq 1$ *for every* $n \in N$. *For each* $n \in N$, *a pair* (τ^*, σ^*) *in* Λ_n *satisfies* $\gamma_n^* = \gamma_n(\tau^*, \sigma^*)$ *if and only if*

$$G_n^i(\tau^*, \sigma^*) \geqq (1 - \gamma_n^*) M_n^i + \gamma_n^* Z_n^i, \quad i = 1, 2, \tag{6}$$

where the equality holds for at least one i. *Furthermore, such a pair* (τ^*, σ^*) *is in* $C_n^0(Z)$.

PROOF. If $\gamma_n^* = \gamma_n(\tau^*, \sigma^*)$, we have

$$\gamma_n^* \geqq \gamma_n^i(\tau^*, \sigma^*), \quad i = 1, 2,$$

where at least one i have equality (as well as in the inequality below), and hence

$$G_n^i(\tau^*, \sigma^*) \geqq M_n^i - \gamma_n^*(M_n^i - Z_n^i) = (1 - \gamma_n^*) M_n^i + \gamma_n^* Z_n^i, \quad i = 1, 2.$$

Conservely, if a pair (τ^*, σ^*) satisfies (6), it is clear that $\gamma_n^* = \gamma_n(\tau^*, \sigma^*)$. Next, by argument analogous to the proof of Theorem 5.1 it is easy to see without Assumption 5.2 that the pair (τ^*, σ^*) is in $C_n^0(Z)$. □

Now the threat functionals Z^i are important when we consider core. Hence Z^i must be significant and we hope that Z^i are able to analyze. We shall take minimax or maximin value processes for each player in a zero-sum stopping game as one of threat functionals. These value processes $\bar{V}^i$ and $\underline{V}^i$ $(i = 1, 2)$ in our problem are defined by

$$\bar{V}_n^i = \underset{(\tau_1,\tau_2)\in\Lambda_n}{\operatorname*{ess\,inf}_{\tau_j}\ \operatorname*{ess\,sup}_{\tau_i}} G_n^i(\tau_1, \tau_2), \quad \underline{V}_n^i = \underset{(\tau_1,\tau_2)\in\Lambda_n}{\operatorname*{ess\,sup}_{\tau_i}\ \operatorname*{ess\,inf}_{\tau_j}} G_n^i(\tau_1, \tau_2), \quad j = 1, 2, \ j \neq i.$$

Such a zero-sum stopping game with a finite constraint was investigated in [17,18,19,21]. The processes $\bar{V}^i$ and $\underline{V}^i$ are also called conservative value for the ith player. We shall recall results for $\bar{V}^i$ and $\underline{V}^i$, and give sufficient conditions for ε–core $C_n^\varepsilon(Z)$ to be nonempty, when threat functionals Z^i are $\bar{V}^i$ or $\underline{V}^i$.

The following assumptions are necessary when we consider the zero-sum stopping game.

ASSUMPTION 5.3. Random variables $\sup_n (Y_n^i)^-$ $(i = 1,2)$ are integrable, and inequalities $X_n^i \leqq W_n^i \leqq Y_n^i$ are satisfied for each $n \in N$ and every $i = 1,2$.

ASSUMPTION 5.4. $\limsup_n X_n^i \geqq \liminf_n Y_n^i$.

PROPOSITION 5.1. *Suppose Assumption* 5.3 *is satisfied. Then*

(i) *For each* $n \in N$ *and every* $i = 1,2$, *inequalities* $\bar{V}_n^i \leqq \underline{V}_n^i$ *hold, and both* $\bar{V}^i$ *and* $\underline{V}^i$ *satisfy the recursive relation* :

$$V_n = \mathrm{med}(X_n^i, Y_n^i, E[V_{n+1} \mid \mathcal{F}_n]), \quad n \in N, \tag{7}$$

where $\mathrm{med}(a,b,c)$ *is median for real numbers* a, b *and* c.

(ii) *For each* $n \in N$ *and every* $\varepsilon > 0$, *stopping times* $\hat{\tau}_n^\varepsilon$ *and* $\hat{\sigma}_n^\varepsilon$ *defined by*

$$\hat{\tau}_n^\varepsilon = \inf\{k \geqq n \mid \bar{V}_k^1 \leqq X_k^1 + \varepsilon\}, \quad \hat{\sigma}_n^\varepsilon = \inf\{k \geqq n \mid \bar{V}_k^2 \leqq X_k^2 + \varepsilon\}$$

are a. s. finite and satisfy inequalities

$$\bar{V}_n^1 \leqq G_n^1(\hat{\tau}_n^\varepsilon, \sigma) + \varepsilon, \quad \bar{V}_n^2 \leqq G_n^2(\tau, \hat{\sigma}_n^\varepsilon) + \varepsilon, \tag{8}$$

for all stopping times τ *and* σ.

(iii) *If Assumption* 5.4 *is satisfied for some* i, *then* $\bar{V}^i = \underline{V}^i$ $(= V^i$, *say) for the* i *and* V^i *is the unique process satisfying the recursive relation* (7).

We can find the proof of this proposition in Proposition 2.1 of [17] for statement (i), in p.597 of [17] or Lemma 3.3 of [19] for (ii), and in Theorem 2 of [21] (or [18]) for (iii).

COROLLARY 5.1. *Suppose Assumption* 5.3 *is satisfied. For each* $n \in N$ *and every* $\varepsilon > 0$, ε–*core* $C_n^\varepsilon(Z)$ *is nonempty, if one of the following conditions is satisfied* :

(a) *Assumptions* 5.1 *and* 5.2 *for* $Z^i = \bar{V}^i$ $(i = 1,2)$ *hold.*

(b) *Assumptions* 5.1, 5.2 *for* $Z^i = \underline{V}^i$ $(i = 1,2)$ *and Assumption* 5.4 *for every* $i = 1,2$ *hold.*

PROOF. Under condition (a), inequalities (8) imply condition (b) in Theorem 5.1. Hence ε–core $C_n^\varepsilon(Z)$ is nonempty. From Proposition 5.1 (iii) and condition (b), we have $\bar{V}^i = \underline{V}^i$ $(i = 1,2)$, to which condition (a) is applied. □

6. Markov model.

In this section we consider a Markov model as a special case, and give simple examples. Let $(X_n, \mathcal{F}_n, P_x)$ be a time-homogeneous Markov process on phase space $(E, \mathcal{B})$, and let

$B(E)$ be the class of bounded $\mathcal{B}$-measurable functions on $(E,\mathcal{B})$. We give six functions f_i, g_i and h_i ($i=1,2$) in $B(E)$. For a pair of stopping times (τ_1,τ_2) in Λ_0, the reward function of the ith player ($i=1,2$) is

$$G_x^i(\tau_1,\tau_2)=E_x[f_i(X_{\tau_i})I_{(\tau_i<\tau_j)}+g_i(X_{\tau_j})I_{(\tau_j<\tau_i)}+h_i(X_{\tau_i})I_{(\tau_i=\tau_j)}],\ j=1,2,\ j\neq i,\ x\in E,$$

where E_x denotes the expectation operator with respect to P_x. For $\varepsilon \geqq 0$, we say that a pair $(\tau_\varepsilon,\sigma_\varepsilon)$ in Λ_0 is *ε-weak* (resp. *ε-strong*) *Pareto optimal* if, for any $x\in E$, there is no pair (τ,σ) in Λ_0 such that

$$G_x^*(\tau,\sigma)>G_x^*(\tau_\varepsilon,\sigma_\varepsilon)+\varepsilon e \quad (\text{resp. } G_x^*(\tau,\sigma)\geq G_x^*(\tau_\varepsilon,\sigma_\varepsilon)+\varepsilon e),$$

where $G_x^*(\tau,\sigma)=(G_x^1(\tau,\sigma),G_x^2(\tau,\sigma))$. We define the shadow optima as follows:

$$\alpha_i(x)=\operatorname*{ess\,sup}_{(\tau,\sigma)\in\Lambda_0} G_x^i(\tau,\sigma),\quad i=1,2,\ x\in E,$$

and we then have the recursive equation for α_i ($i=1,2$) which corresponds to the relation(1):

$$\alpha_i(x)=\max(f_i(x),g_i(x),h_i(x),T\alpha_i(x)),\quad x\in E,$$

where operator T is a semigroup, that is, $Tf(x)=E_x[f(X_1)]$ for $x\in E$ and $f\in B(E)$. For given $\lambda,\mu\in S$, $1\leqq p\leqq\infty$ and $M_i,Z_i\in B(E), i=1,2$, we can similarly define $f_\lambda(x)$, $g_\lambda(x)$, $h_\lambda(x)$, $G_x^i(\tau,\sigma;\lambda)$, $V_\lambda(x)$, $\hat f_k(x)$, $\hat g_k(x)$, $\hat h_k(x)$, $D_p(x)\equiv D_p(x;\mu,M)$, $\gamma_x(\tau,\sigma)$ and $\gamma^*(x)\equiv\gamma^*(x;M,Z)$, which correspond to $X_n(\lambda)$, $Y_n(\lambda)$, $W_n(\lambda)$, $G_n^i(\tau,\sigma;\lambda)$, $V_n(\lambda)$, $\hat X_k(n)$, $\hat Y_k(n)$, $\hat W_k(n)$, $D_n^p\equiv D_n^p(\mu,M)$, $\gamma_n(\tau,\sigma)$ and $\gamma_n^*\equiv\gamma_n^*(M,Z)$, respectively, where $M=(M_1,M_2)$ and $Z=(Z_1,Z_2)$.

For any $Z=(Z_1,Z_2)$ satisfying $Z_i\in B(E)$ ($i=1,2$), we next define a core $C_0(Z)$ by the class of all pairs (τ,σ) in Λ_0 such that (τ,σ) is 0–weak Pareto optimal and inequality $G_x^*(\tau,\sigma)\ \geqq Z(x)$ holds on E. We also define minimax and maximin values for the ith player by

$$\bar v_i(x)=\inf_{\tau_j}\sup_{\tau_i}{}_{(\tau_1,\tau_2)\in\Lambda_0} G_x^i(\tau_1,\tau_2),\quad \underline{v}_i(x)=\sup_{\tau_i}\inf_{\tau_j}{}_{(\tau_1,\tau_2)\in\Lambda_0} G_x^i(\tau_1,\tau_2),\quad j=1,2,\ j\neq i.$$

We then have the following recursive equations and inequalities corresponding to the relations (2), (4), (7) and (6), respectively,

$$\begin{aligned}
V_\lambda(x)&=\max(f_\lambda(x),g_\lambda(x),h_\lambda(x),TV_\lambda(x)),\\
D_1(x)&=\min(\hat f_0(x),\hat g_0(x),\hat h_0(x),TD_1(x)),\\
v(x)&=\operatorname{med}(f_i(x),g_i(x),Tv(x)), &(9)\\
G_x^i(\tau^*,\sigma^*)&\geqq(1-\gamma^*(x))M_i(x)+\gamma^*(x)Z_i(x). &(10)
\end{aligned}$$

Also we can give Markov versions of Theorems 3.3, 4.2, 5.1 and so on.

We shall next give simple examples for the Markov model.

EXAMPLE 6.1. Let the state space E consist of three states $1, 2$ and 3, and the transition probability matrix P be given as follows:

$$P = \begin{bmatrix} 2/3 & 1/3 & 0 \\ 1/2 & 1/2 & 0 \\ 1/2 & 1/3 & 1/6 \end{bmatrix}.$$

The set of states $\{1,2\}$ is then a recurrent class and state 3 is transient. We also give the functions f_i, g_i and h_i by

$$\begin{array}{llllll} f_1(1)=2, & f_1(2)=-2, & f_1(3)=-3, & f_2(1)=-2, & f_2(2)=0, & f_2(3)=-3, \\ g_1(1)=4, & g_1(2)=0, & g_1(3)=2, & g_2(1)=2, & g_2(2)=4, & g_2(3)=6/5, \\ h_1(1)=2, & h_1(2)=-1, & h_1(3)=-1, & h_2(1)=-2, & h_2(2)=0, & h_2(3)=-2, \end{array}$$

and, for $x \in E$, let a subset $\mathcal{G}(x)$ in R^2 denote the set of all vectors $G_x^*(\tau,\sigma)$ satisfying $(\tau,\sigma) \in \Lambda_0$. It then follows that $\mathcal{G}(1) = \mathcal{G}(2)$ and it is a subset of convex hull of S_1, and $\mathcal{G}(3)$ is a subset of convex hull of S_2, where S_1 is the set of six vectors $(2,2), (-2,4), (4,-2), (2,-2), (-1,0)$ and $(0,0)$ and S_2 is the union of S_1 and $\{(-3,6/5), (2,-3), (-1,-2)\}$ (see Figure 1). Note that the covex hull of S_1 coincides with the closure of $\mathcal{G}(1) = \mathcal{G}(2)$.

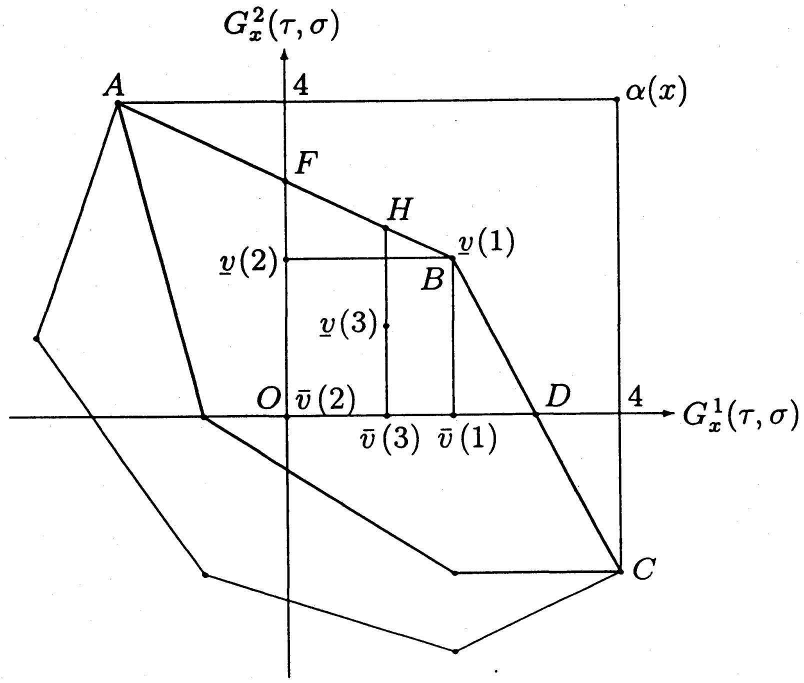

FIGURE 1

Since $f_i \leqq h_i \leqq g_i$ on E, the shadow optimum $\alpha(x) = (\alpha_1(x), \alpha_2(x))$ is minimum solution to the recursive equations $\alpha_i(x) = \max(g_i(x), T\alpha_i(x))$, $x \in E$, so that we have $\alpha(1) = \alpha(2) = \alpha(3) = (4,4)$. We easily see from Figure 1 that 0–weak (and 0–strong) Pareto optimal pairs correspond vectors on segments AB and BC, where $A = (-2,4), B = (2,2)$ and $C = (4,-2)$. For example, a weak Pareto optimal stopping pair (τ_A, σ_A) corresponding the vector A is represented by $\tau_A = \inf\{k \geqq 0 \mid X_k = 2\}$ and $\sigma_A = +\infty$. Also, we can check that when $\lambda = (\lambda_1, \lambda_2) \in S$ satisfies $2\lambda_1 < \lambda_2$ the pair (τ_A, σ_A) coincides with $(\tau_0(\lambda), \sigma_0(\lambda))$, where

$$\tau_0(\lambda) = \inf\{k \geqq 0 \mid V_\lambda(X_k) = \max(f_\lambda(X_k), h_\lambda(X_k))\}$$

$$\sigma_0(\lambda) = \inf\{k \geqq 0 \mid f_\lambda(X_k) < V_\lambda(X_k) = \max(g_\lambda(X_k), h_\lambda(X_k))\}.$$

Indeed, we have $f_\lambda(x) > \max(g_\lambda(x), h_\lambda(x))$ and $V_\lambda(x) = f_\lambda(2) = 4 - 6\lambda_1$, $x \in E$. Similarly, when a goal M is a constant vector (c,c) with $c > 4$ and $\mu = (\mu_1, 1-\mu_1)$ satisfies $1/3 < \mu_1 < 2/3$, $(0;1)$–optimal pair $(\tau_0(\mu, M), \sigma_0(\mu, M))$ corresponds to the vector B.

Next let us consider core, when we take the minimax value $\bar{v}_i$ or the maximin $\underline{v}_i$ as threat functins Z_i. Since f_i, g_i and h_i $(i = 1,2)$ satisfy Assumption 5.3, value functions $\bar{v}_i$ and $\underline{v}_i$ satisfy the recursive equation (9). Also, since Assumption 5.4 for $i = 1$ holds, *i.e.* $\limsup_n f_1(X_n) = 2 > \liminf_n g_1(X_n) = 0$ a. s., it follows from Proposition 5.1(iii) that $\bar{v}_1(x) = \underline{v}_1(x)$ ($\equiv v_1(x)$, say) for all $x \in E$ and it is the unique solution to (9), so that $v_1(1) = 2, v_1(2) = 0$ and $v_1(3) = 6/5$. On the other hand, Assumption 5.4 for $i = 2$ is not satisfied, as $\limsup_n f_2(X_n) = 0$ and $\liminf_n g_2(X_n) = 2$ a. s.. Hence $\bar{v}_2(x)$ (resp. $\underline{v}_2(x)$) is the smallest (resp. largest) solution to the equation (9), and we have $\bar{v}_2(x) = 0, x \in E$, $\underline{v}_2(1) = \underline{v}_2(2) = 2$ and $\underline{v}_2(3) = 6/5$. Let $\bar{v}(x) = (\bar{v}_1(x), \bar{v}_2(x))$ and $\underline{v}(x) = (\underline{v}_1(x), \underline{v}_2(x))$, that is, $\bar{v}(1) = (2,0)$, $\underline{v}(1) = (2,2)$ and so on. We first take $\bar{v}$ as a threat function Z. Then the core $C_0(\bar{v})$ is the set of all (τ, σ) for which $G_x^*(\tau, \sigma)$ is on segment DB if $x = 1$, on DB or BF if $x = 2$, and on DB or BH if $x = 3$, where $D = (3,0), F = (0,3)$ and $H = (6/5, 12/5)$. Also, if $M = \alpha$, it is easy to see that $\gamma^*(1) = 3/4$, $\gamma^*(2) = 1/2$ and $\gamma^*(3) = 5/8$. For example, we define a pair (τ^*, σ^*) as follows :

when $X_0 = 1$,

$$\tau^* = \begin{cases} \tau_1 & \text{off } \Omega_1 \\ +\infty & \text{on } \Omega_1, \end{cases} \qquad \sigma^* = \begin{cases} \tau_1 & \text{on } \Omega_1 \\ +\infty & \text{off } \Omega_1, \end{cases}$$

when $X_0 = 2$,

$$\tau^* = \inf\{k \geqq 0 \mid X_k = 1\}, \quad \sigma^* = +\infty,$$

when $X_0 = 3$,

$$\tau^* = \begin{cases} \tau_1 & \text{off } \Omega_2 \\ +\infty & \text{on } \Omega_2, \end{cases} \qquad \sigma^* = \begin{cases} \tau_1 & \text{on } \Omega_2 \\ +\infty & \text{off } \Omega_2, \end{cases}$$

where

$$\tau_1 = \inf\{k \geqq \tau_2 \mid X_k = 1\}, \qquad \tau_2 = \inf\{k \geqq 0 \mid X_k = 2\},$$
$$\Omega_1 = \{X_{\tau_2+1} = 2, X_{\tau_2+2} = 1\}, \quad \Omega_2 = \{X_{\tau_2+1} = X_{\tau_2+2} = 2, X_{\tau_2+3} = 1\}.$$

Then we can easily check that $G_x^*(\tau^*, \sigma^*) = (5/2, 1)$ if $x = 1$, $= (2,2)$ if $x = 2$, and $= (9/4, 3/2)$ if $x = 3$. Thus, since inequalities (10) (as a matter of fact, equalities) are satisfied, Theorem 5.2 implies that $\gamma^*(x) = \gamma_x(\tau^*, \sigma^*)$, $x \in E$, and the pair (τ^*, σ^*) is in $C_0(\bar{v})$. Secondly, let $Z = \underline{v}$ and $M = \alpha$. We similarly see that the core $C_0(\underline{v})$ is nonempty, and $\gamma^*(1) = 1$, $\gamma^*(2) = 3/4$ and $\gamma^*(3) = 5/7$, though Assumption 5.4 for $i = 2$ is not satisfied.

In the following example, core $C_0(\underline{v})$ is empty.

EXAMPLE 6.2. Let $E = \{1, 2\}$ and let the transition probability matrix $P = (p_{ij})$ be $p_{ij} = 0$ if $i = j$ and $p_{ij} = 1$ if $i \neq j$. Also let f_i, g_i and h_i be satisfy $f_i(x) = h_i(x) = a$ and $g_i(x) = b$, $x \in E$, for constants a and b $(a < b)$. Then we have $\mathcal{G}(x) = \{(a, b), (b, a), (a, a)\}$ and $\underline{v}(x) = (b, b), x \in E$. Thus, since there is no pair (τ, σ) in Λ_0 such that $G_x^*(\tau, \sigma) \geqq \underline{v}(x)$ for all $x \in E$, the core $C_0(\underline{v})$ is empty.

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