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NUMERICAL SOLUTION OF BOUNDARY VALUE PROBLEMS FOR RETARDED DIFFERENTIAL EQUATIONS WITH A PARAMETER

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One step methods combined with an iterative method are applied to find a numerical solution of boundary value problems for retarded ordinary differential equations with a parameter. This paper deals with the convergence of such methods. Some estimates of errors are given too.

1. Introduction. We consider the system of retarded ordinary differential equations

$$(1)\qquad y'(t) = f(t, y(t), y(\alpha_1(t)), \cdots, y(\alpha_r(t)), \lambda), \quad t \in J = [a, b], \quad a < b,$$

where $f : J \times R^{q(r+1)} \times R^p \to R^q$ and $\alpha_i : J \to R$ are continuous and $\alpha_i(t) < t$, $t \in J$, $i = 1, 2, \cdots, r$. Here $\lambda \in R^p$ is a parameter. We assume that the solution of (1) is given on J_a, so

$$(2)\qquad y(t) = \Psi(t), \quad t \in J_a = [\bar{a}, a], \quad \bar{a} = \inf_{t\in J}\{\alpha_i(t),\ i = 1, 2, \cdots, r\} \quad \Psi \in C^1(J_a, R^q).$$

Here $C^1(J_a, R^q)$ denotes the space of all functions of the class C^1 defined on J_a with a range in R^q. We are interested in the solution of (1-2) that satisfies the nonlinear boundary condition

$$(3)\qquad g(\lambda, y(b)) = \Theta_p, \quad \Theta_p \text{ is zero element in } R^p,$$

where $g : R^p \times R^q \to R^p$. By a solution of (1-3) we mean a function $\varphi \in C^1(J, R^q)$ and a parameter $\lambda \in R^p$ such that (1-3) to be satisfied. Problem (1-3) may also be named as an eigenvalue problem for retarded differential equations or as a problem of terminal control. Sometimes g may be linear with respect to its variables or may depend on λ or $y(b)$ only.

The question of existence and uniqueness of solutions of problems with parameters is alredy investigated (see, for example, [3, 8, 9, 10]). Due to this fact it will be assumed that our problem has the exact solution (φ, λ). A numerical approximation of this solution is a task of this paper.

Notice that φ is a function of λ. It is known that if f has continuous first order partial derivatives with respect to the last $r + 2$ variables, then

$$Y(t; \lambda) \equiv \frac{\partial}{\partial\lambda}\varphi(t; \lambda)$$

is the solution of the problem

$$(4)\qquad \begin{cases} Y'(t;\lambda) = f_0(t, \varphi(t), \varphi(\alpha_1(t)), \cdots, \varphi(\alpha_r(t)), \lambda)Y(t;\lambda) + \\ \qquad + \sum_{i=1}^{r} f_i(t, \varphi(t), \varphi(\alpha_1(t)), \cdots, \varphi(\alpha_r(t)), \lambda)Y(\alpha_i(t);\lambda) + \\ \qquad + f_\lambda(t, \varphi(t), \varphi(\alpha_1(t)), \cdots, \varphi(\alpha_r(t)), \lambda), \quad t \in J, \\ Y(a;\lambda) = 0_{q\times p}. \end{cases}$$

Here f_i denotes the partial derivative of f with respect to the $(i+2)$th variable for $i = 0, 1, \cdots, r$, while f_λ denotes the partial derivative of f with respect to the last variable. Indeed, (φ, λ) is the solution of (1-3) if λ is a fixed point of Φ, where

$$\Phi(\lambda) \equiv g(\lambda, \varphi(b)) = \Theta_p.$$

The value of λ may be obtained by the Newton method, so

$$\lambda_{n+1} = \lambda_n - [\Phi'(\lambda_n)]^{-1} \Phi(\lambda_n), \quad n = 0, 1, \cdots,$$

where

$$\Phi'(\lambda) = g_1(\lambda, \varphi(b)) + g_2(\lambda, \varphi(b)) Y(b; \lambda).$$

Here g_1 and g_2 are partial derivatives of g with respect to the first and second variable, respectively.

Our task is to determine the numerical solution (y_h, λ_{hj}) of $(1-3)$ from the discretization of the above method. The values of y_h will be defined on the set of points t_{hn} which for arbitrary integer N are expressed by $t_{hi} = a + ih$, $i = 0, 1, \cdots, N$ with $h = (b-a)/N$. Let

$$c_i(n) = E\left(\frac{\alpha_i(t_{hn}) - a}{h}\right), \text{ where } E \text{ denotes the integer part,}$$

$$e_i(n) = \frac{\alpha_i(t_{hn}) - a}{h} - c_i(n)$$

for $i = 1, 2, \cdots, r$, $n = 0, 1, \cdots, N$. It is easy to observe that $\alpha_i(t_{hn}) = t_{h, c_i(n))} + he_i(n)$, $i = 1, 2, \cdots, r$, $n = 0, 1, \cdots, N$. Now, we may define the numerical solution (y_h, λ_{hj}) of $(1-3)$ by the following formulas

$$(5) \quad \begin{cases} y_h(t; \lambda_{hj}) = \Psi(t) \ \text{ if } t \in J_a, \\ y_h(t_{hn} + eh; \lambda_{hj}) = y_h(t_{hn}; \lambda_{hj}) + hF(t_{hn}, y_h(t_{hn}; \lambda_{hj}), y_h(t_{h,c_1(n)} + he_1(n); \lambda_{hj}), \\ \qquad , \cdots, y_h(t_{h,c_r(n)} + he_r(n); \lambda_{hj}), \lambda_{hj}, h, e) \ \text{ for } e \in [0,1], \ n = 0, 1, \cdots, N-1, \end{cases}$$

$$(6) \quad \begin{cases} Y_h(t; \lambda_{hj}) = 0_{q \times p} \ \text{ if } t \in J_a, \\ Y_h(t_{hn} + eh; \lambda_{hj}) = [I + hA^j_{hn}(0, e)] Y_h(t_{hn}; \lambda_{hj}) \\ \qquad + h \sum_{i=1}^{r} A^j_{hn}(i, e) Y_h(t_{h,c_i(n)} + he_i(n); \lambda_{hj}) \\ \qquad + h\tilde{A}^j_{hn}(\lambda, e), \quad e \in [0,1], \quad n = 0, 1, \cdots, N-1, \end{cases}$$

and

$$(7) \quad \begin{cases} \lambda_{h0} = \lambda_0 \in R^p, \\ \lambda_{h,j+1} = \lambda_{hj} - \left(B^j_{1h} + B^j_{2h} Y_h(b; \lambda_{hj})\right)^{-1} g(\lambda_{hj}, y_h(b; \lambda_{hj})), \end{cases}$$

defined for $j = 0, 1, \cdots$. Here I is the unit matrix of order q and

$$\begin{aligned} A^j_{hn}(i,e) =& F_i(t_{hn}, y_h(t_{hn};\lambda_{hj}), y_h(t_{h,c_1(n)} + he_1(n); \lambda_{hj}), \\ &, \cdots, y_h(t_{h,c_r(n)} + he_r(n); \lambda_{hj}), \lambda_{hj}, h, e), \\ \tilde{A}^j_{hn}(\lambda,e) =& F_\lambda(t_{hn}, y_h(t_{hn};\lambda_{hj}), y_h(t_{h,c_1(n)} + he_1(n); \lambda_{hj}), \\ &, \cdots, y_h(t_{h,c_r(n)} + he_r(n); \lambda_{hj}), \lambda_{hj}, h, e), \end{aligned}$$

where F_i denotes the partial derivative of F with respect to the $(i+2)$th variable for $i = 0, 1, \cdots, r$ and F_λ is the partial derivative of F with respect to the $(r+3)$th variable. Moreover,

$$B^j_{ih} = g_i(\lambda_{hj}, y_h(b;\lambda_{hj})), \quad i = 1, 2,$$

where g_i denotes the partial derivative of g with respect to the ith variable. It will be assumed that $F(\cdots, 0) = \Theta_q$, $F_\lambda(\cdots, 0) = 0_{q\times p}$, $F_i(\cdots, 0) = 0_{q\times q}$, $i = 0, 1, \cdots, r$. Notice that taking

$$F(t, y_0, y_1, \cdots, y_r, \lambda, h, e) = ef(t, y_0, y_1, \cdots, y_r, \lambda),$$

we obtain the Euler procedure.

If $e = 1$, then (6) yields

$$Y_h(t_{hn};\lambda_{hj}) = \sum_{i=0}^{n-1} \left[\prod_{s=i+1}^{n-1} \left(I + hA^j_{h,n+i-s}(0,1) \right) \right] \bar{B}^j_{hi}, \quad n = 0, 1, \cdots, N,$$

with

$$\bar{B}^j_{hi} = h \left[\sum_{k=1}^{r} A^j_{hi}(k,1) Y_h(t_{h,c_k(i)} + he_k(i); \lambda_{hj}) + \tilde{A}^j_{hi}(\lambda, 1) \right],$$

and $\sum_0^{-1} \cdots = 0_{q\times p}$, $\quad \prod_i^s \cdots = I \quad$ if $i > s$. It is also useful for the case when f does not depend on α_i, $i = 1, 2, \cdots, r$; then $F_i \equiv 0$, $i = 1, 2, \cdots, r$ and F, F_0, F_λ do not depend on the variables from 3rd to (r+2)th and the last one.

Assume for a moment that $p = q$, and

$$g(u,v) = \tilde{M}u + \tilde{N}v - \tilde{K}, \qquad \tilde{K} \in R^p,$$

where $\tilde{M}$ and $\tilde{N}$ are given square matrices of order p. Let the matrix $\tilde{M} + \tilde{N}$ be nonsingular. For this case, we can take $\tilde{M} + \tilde{N}$ instead of $B^j_{1h} + B^j_{2h} Y_h(b;\lambda_{hj})$ and we do not need the elements of Y_h for finding an approximate solution of (1-3). The convergence of the new method (y_h, λ_{hj}) will be guaranteed if among other things one assumes that

$$(\star) \qquad \|(\tilde{M} + \tilde{N})^{-1}\tilde{N}\| \left[1 + \frac{Q_\lambda}{Q}(exp(Q(b-a)) - 1) \right] < 1, \qquad Q = \sum_{i=0}^{r} Q_i \neq 0.$$

Here $Q_0, Q_1, \cdots, Q_r, Q_\lambda$ are Lipschitz constants of F with respect to the variables from the second to the last, respectively. Such methods were considered in [4, 5] both for linear and nonlinear boundary condition (3). The above mentioned condition is not so different from the corresponding results of [1, 7, 11] for problems without retardations and parameters.

The condition similar to ($\star$) can be omitted for convergence of (4-7). In this paper will be formulated such sufficient conditions for convergence of our method (4-7). The estimates of errors will be given too.

2. Definitions, assumptions and lemmas. We introduce the following

Definition 1. *We say that method (5-7) is convergent to the solution (φ, λ) of problem (1-3) if*

$$\lim_{\substack{N\to\infty \\ j\to\infty}} \sup_{t\in J} \|y_h(t;\lambda_{hj}) - \varphi(t)\| = 0,$$

$$\lim_{\substack{h\to 0 \\ j\to\infty}} \|\lambda_{hj} - \lambda\| = 0.$$

Definition 2. *We say that method (5-7) is consistent with problem (1-3) on (φ, λ) if there exists a function $\epsilon : J_h \times H \times [0,1] \to R_+ = [0,\infty)$, $J_h = [a, b-h]$, $H = [0, h^*]$, $h^* > 0$ such that*

$$(i) \qquad \|hF(t,\varphi(t),\varphi(\alpha_1(t)),\cdots,\varphi(\alpha_r(t)),\lambda,h,e) + \varphi(t) - \varphi(t+eh)\| \le \epsilon(t,h,e),$$

$$(ii) \qquad \lim_{h\to 0} \sum_{i=0}^{N-1} \bar\epsilon(t_{hi}, h) = 0, \quad \text{where} \quad \bar\epsilon(t,h) = \sup_{e\in[0,1]} \epsilon(t,h,e).$$

Remark 1. Knowing that φ is a solution of (1-2), condition (i) may be written by

$$\|hF(t,\varphi(t),\varphi(\alpha_1(t)),\cdots,\varphi(\alpha_r(t)),\lambda,h,e) - \int_t^{t+eh} f(\tau,\varphi(\tau),\varphi(\alpha_1(\tau)),\cdots,\varphi(\alpha_r(\tau)),\lambda)d\tau\| \le \epsilon(t,h,e).$$

Notice that condition (ii) will be satisfied if for example $\epsilon(t,h,e) = h^\nu$, $\nu > 1$, $t \in J_h$ and $e \in [0,1]$.

Assumption H. *Assume that*

1^0 *the function $F : J \times R^{q(r+1)} \times R^p \times H \times [0,1] \to R^q$ is continuous and has first order partial derivatives F_i, F_λ with respect to the $(i+2)$th variable for $i = 0,\cdots,r$, where F_λ denotes the partial derivative of F with respect to the $(r+3)$th variable; $\Psi \in C^1(J_a, R^q)$, $\alpha_i \in C(J,[\bar a, b])$, $\alpha_i(t) \le t$ and $F(\cdots,0) = \Theta_q$, $F_\lambda(\cdots,0) = 0_{q\times p}$, $F_i(\cdots,0) = 0_{q\times q}$ for $i = 0,1,\cdots,r$,*

2^0 *$g : R^p \times R^q \to R^p$ is continuous and has first order partial derivatives g_1 and g_2 with respect to the first and second variable, respectively*

3^0 *there exist constants $Q_i, Q^\lambda, L_{si}, L_s^\lambda, M_i, M^\lambda$ for $i = 0,1,\cdots,r$, $s = 0,1,\cdots,r$ such that for $t \in J$, $h \in H$, $e \in [0,1]$, $y_i, \bar y_i \in R^q$, $i = 0,1,\cdots,r$, $\mu, \bar\mu \in R^p$, the conditions*

$$\|F_i(t,y_0,y_1,\cdots,y_r,\mu,h,e)\| \le Q_i, \quad i = 0,1,\cdots,r,$$

$$\|F_\lambda(t, y_0, y_1, \cdots, y_r, \mu, h, e)\| \leq Q_\lambda,$$

and

$$\|F_s(t, y_0, y_1, \cdots, y_r, \mu, h, e) - F_s(t, \bar{y}_0, \bar{y}_1, \cdots, \bar{y}_r, \bar{\mu}, h, e)\| \leq \sum_{i=0}^{r} L_{si}\|y_i - \bar{y}_i\| + L_s^\lambda \|\mu - \bar{\mu}\|, \quad s = 0, 1, \cdots, r,$$

$$\|F_\lambda(t, y_0, y_1, \cdots, y_r, \mu, h, e) - F_\lambda(t, \bar{y}_0, \bar{y}_1, \cdots, \bar{y}_r, \bar{\mu}, h, e)\| \leq \sum_{i=0}^{r} M_i\|y_i - \bar{y}_i\| + M^\lambda \|\mu - \bar{\mu}\|$$

hold,

4^0 *there exist constants* K_{i1}, K_{i2}, $i = 1, 2$ *such that for* $x, \bar{x} \in R^p$, $y, \bar{y} \in R^q$ *we have*

$$\|g_i(x, y) - g_i(\bar{x}, \bar{y}\| \leq K_{i1}\|x - \bar{x}\| + K_{i2}\|y - \bar{y}\|, \quad i = 1, 2.$$

Put

$$V(t) = \|v(t)\|, \quad t \in J, \qquad U_n = \sup_{[a, t_{hn}]} V(t).$$

Then we can formulate the lemma.

Lemma 1. *Assume that* $b_0, b_1, b_2 \geq 0$, $\alpha : J_h \times H \times [0, 1] \to R_+$ *and*

(8) $\quad V(t_{hn} + eh) \leq (1 + hb_0)V(t_{hn}) + hb_1U_n + hb_2 + \alpha(t_{hn}, h, e), \quad n = 0, 1, \cdots, N - 1.$

Then we have

$$\sup_{t \in J} V(t) \leq \left(\bar{b} \sum_{i=0}^{N-1} \bar{\alpha}(t_{hi}, h) + \bar{B}b_2 + \bar{b}V(a) \right) exp(b_1\bar{b}(b - a)), \tag{9}$$

where

$$\bar{b} = exp(b_0(b - a)), \qquad \bar{\alpha}(t, h) = \sup_{e \in [0,1]} \alpha(t, h, e),$$

$$\bar{B} = \begin{cases} \dfrac{\bar{b} - 1}{b_0} & \text{if } b_0 \neq 0, \\ b - a & \text{if } b_0 = 0. \end{cases}$$

Proof. Indeed, for $e = 1$, we get

$$V(t_{h,n+1}) \leq (1 + hb_0)V(t_{hn}) + hb_1U_n + hb_2 + \alpha(t_{hn}, h, 1), \quad n = 0, 1, \cdots, N - 1.$$

It yields the inequality

$$V(t_{hn}) \leq \sum_{i=0}^{n-1} (1 + hb_0)^{n-i-1} \left[hb_1U_i + hb_2 + \alpha(t_{hi}, h, 1)\right] + (1 + hb_0)^n V(a), \quad \sum_{0}^{-1} = 0, \quad n = 0, 1, \cdots, N. \tag{10}$$

Moreover, (8) leads to

$$\sup_{[t_{hn},t_{h,n+1}]} V(t) \le (1+hb_0)V(t_{hn}) + hb_1U_n + hb_2 + \bar{\alpha}(t_{hn},h), \quad n=0,1,\cdots,N-1.$$

Combining this with (10) we arrive at the inequality

$$\begin{aligned}\sup_{[t_{hn},t_{h,n+1}]} V(t) &\le hb_1\sum_{i=0}^{n}(1+hb_0)^{n-i}U_i + \sum_{i=0}^{n}(1+hb_0)^{n-i}[\bar{\alpha}(t_{hi},h)+hb_2]+ \\ &+(1+hb_0)^{n+1}V(a) \\ &\le hb_1\bar{b}\sum_{i=0}^{n}U_i + \bar{b}\sum_{i=0}^{N-1}\bar{\alpha}(t_{hi},h) + \bar{B}b_2 + \bar{b}V(a), \quad n=0,1,\cdots,N-1.\end{aligned}$$

Now, it is easy to prove (by induction with respect to n) that

$$U_n = \max\left(U_{n-1}, \sup_{[t_{h,n-1},t_{hn}]} V(t)\right)$$

satisfies the following inequality

$$U_n \le hb_1\bar{b}\sum_{i=0}^{n-1}U_i + \bar{b}\sum_{i=0}^{N-1}\bar{\alpha}(t_{hi},h) + \bar{B}b_2 + \bar{b}V(a), \quad n=0,1,\cdots,N. \tag{11}$$

Denote the right-hand side of this inequality by β_n. Indeed,

$$\beta_{n+1} - \beta_n = hb_1\bar{b}U_n \le hb_1\bar{b}\beta_n,$$

or

$$\beta_{n+1} \le (1+hb_1\bar{b})\beta_n, \quad n=0,1,\cdots,N-1.$$

Hence we have

$$\beta_n \le (1+hb_1\bar{b})^n\left[\bar{b}\sum_{i=0}^{N-1}\bar{\alpha}(t_{hi},h) + \bar{B}b_2 + \bar{b}V(a)\right], \quad n=0,1,\cdots,N.$$

Combining this with (11) we get the inequality (9). This completes the proof.

Let

$$0 \le z_{n+1} \le D[Az_n^2 + Bz_n + C], \quad A,B,C,D>0, \quad n=0,1,\cdots.$$

Lemma 2([6]). *Assume that there exists a constant d such that*

$$DB < d < 1,$$
$$4\bar{p}^2AC < 1, \quad \textit{where } \bar{p} = \frac{D}{d-DB}.$$

Now, if $z_0 \le \rho = \frac{DC}{1-d} \le \frac{1}{\bar{p}A}$, *then*

$$z_n \le d^n \rho + DC\frac{1-d^n}{1-d}, \quad n = 0, 1, \cdots.$$

3. Convergence of (5-7). Put

$$Q = \sum_{i=1}^{r} Q_i, \quad L_s = \frac{1}{2}\sum_{i=0}^{r} L_{si}, \quad s = 0, 1, \cdots, r,$$

$$L = \sum_{i=0}^{r} L_i, \quad M = \frac{1}{2}\sum_{i=0}^{r} M_i, \quad L^\lambda = M + \frac{1}{2}\sum_{i=0}^{r} L_i^\lambda,$$

$$c = exp(Q_0(b-a)), \quad c_1 = exp(Qc(b-a)),$$

$$B = \begin{cases} \dfrac{c-1}{Q_0} & \text{if } Q_0 \ne 0, \\ b-a & \text{if } Q_0 = 0, \end{cases}$$

$$\bar{K}_{i1} = \frac{1}{2}(K_{i1} + c_1 K_{i2} B Q_\lambda) \quad \bar{K}_{i2} = \frac{1}{2} c_1 c K_{i2}, \quad i = 1, 2,$$

$$\delta_h = \sum_{i=0}^{N-1} \bar{\epsilon}(t_{hi}, h),$$

$$A_1 = c_1 B c(b-a) Q_\lambda \left[L^\lambda + c_1 L B Q_\lambda\right] + \frac{M^\lambda}{2} B,$$

$$B_1(h) = c^2 c_1 (b-a)(2c_1 L B Q_\lambda + L^\lambda)\delta_h,$$

$$C_1(h) = (c^2(b-a) L c_1^2 \delta_h + 1) c \delta_h,$$

$$A_2 = \bar{K}_{11} + c_1 G A_1 + c_1 B Q_\lambda \bar{K}_{21},$$

$$B_2(h) = c_1 G B_1(h) + (\bar{K}_{12} + c_1 c \bar{K}_{21} + c_1 B Q_\lambda \bar{K}_{22})\delta_h,$$

$$C_2(h) = c_1 \left[G C_1(h) + c \bar{K}_{22} \delta_h^2\right].$$

Now we are in a position to establish the main theorem.

Theorem 1. *If Assumption H is satisfied, and*

1^0 *there exists the unique solution* (φ, λ) *of (1-3),*

2^0 *method (5-7) is consistent with problem (1-3) on* (φ, λ),

3^0 *the matrices* $B_{1h}^j + B_{2h}^j Y_h(b; \lambda_{hj})$, $j = 0, 1, \cdots$ are nonsingular and there exists a constant $D > 0$ *such that*

$$\|(B_{1h}^j + B_{2h}^j Y_h(b; \lambda_{hj}))^{-1}\| \le D, \quad j = 0, 1, \cdots,$$

then, for sufficiently small $\bar{h}$, *there exists a constant d such that*

$$(\star\star) \qquad \begin{cases} DB_2(h) < d < 1, \\ 4\bar{p}^2(h) A_2 C_2(h) < 1, \qquad \bar{p}(h) = \dfrac{D}{d - DB_2(h)}, \\ DC_2(h) A_2 \bar{p}(h) + d \le 1 \end{cases}$$

hold for $h \leq \bar{h}$ and method (5-7) is convergent to the solution (φ, λ) of (1-3) provided that

$$\|\lambda_{h0} - \lambda\| \leq u_0(h) = \sup_{0 \leq x \leq \bar{h}} \frac{DC_2(x)}{1-d}, \quad h \leq \bar{h}.$$

Furthermore, the estimates

$$\|\lambda_{hj} - \lambda\| \leq u_j(h), \tag{12}$$

$$\sup_{t \in J} \|y_h(t;\lambda_{hj}) - \varphi(t)\| \leq c_1 B Q_\lambda u_j(h) + c_1 c \sum_{i=0}^{N-1} \bar{\epsilon}(t_{hi}, h), \tag{13}$$

$$\sup_{t \in J} \|Y_h(t;\lambda_{hj})(\lambda_{hj} - \lambda) - y_h(t;\lambda_{hj}) + \varphi(t)\| \leq c_1 \left[A_1 u_j^2(h) + B_1(h) u_j(h) + C_1(h)\right] \tag{14}$$

hold for $h \leq \bar{h}$ and $j = 0, 1, \cdots$, with

$$u_j(h) = d^j u_0(h) + DC_2(h) \frac{1-d^j}{1-d}, \quad j = 0, 1, \cdots.$$

Proof. Put

$$v_h^j(t) = y_h(t;\lambda_{hj}) - \varphi(t), \qquad V_{hn}^j = \|v_h^j(t_{hn})\|, \qquad W_{hn}^j = \sup_{[a,t_{hn}]} \|v_h^j(t)\|,$$

$$z_h^j = \lambda_{hj} - \lambda, \qquad Z_h^j = \|z_h^j\|,$$

$$C_{hn}(e) = hF(t_{hn}, \varphi(t_{hn}), \varphi(\alpha_1(t_{hn})), \cdots, \varphi(\alpha_r(t_{hn}), \lambda, h, e) + \varphi(t_{hn}) - \varphi(t_{hn} + eh),$$

$$\bar{A}_{hn}^j(i,e) = \int_0^1 F_i(t_{hn}, \varphi(t_{hn}) + \tau v_h^j(t_{hn}), \varphi(t_{h,c_1(n)} + he_1(n)) + \tau v_h^j(t_{h,c_1(n)} + he_1(n)),$$
$$, \cdots, \varphi(t_{h,c_r(n)} + he_r(n)) + \tau v_h^j(t_{h,c_r(n)} + he_r(n)), \lambda + \tau z_h^j, h, e) d\tau,$$

$$\tilde{\bar{A}}_{hn}^j(\lambda,e) = \int_0^1 F_\lambda(t_{hn}, \varphi(t_{hn}) + \tau v_h^j(t_{hn}), \varphi(t_{h,c_1(n)} + he_1(n)) + \tau v_h^j(t_{h,c_1(n)} + he_1(n)),$$
$$, \cdots, \varphi(t_{h,c_r(n)} + he_r(n)) + \tau v_h^j(t_{h,c_r(n)} + he_r(n)), \lambda + \tau z_h^j, h, e) d\tau,$$

$$\bar{B}_{ih}^j = \int_0^1 g_i(\lambda + \tau z_h^j, \varphi(b) + \tau v_h^j(b)) d\tau, \quad i = 1, 2.$$

By the definition of y_h and the mean value theorem, we obtain

$$v_h^j(t_{hn} + eh) = y_h(t_{hn};\lambda_{hj}) + hF(t_{hn}, y_h(t_{hn};\lambda_{hj}), y_h(t_{h,c_1(n)} + he_1(n);\lambda_{hj}), \tag{15}$$

$$,\cdots,y_h(t_{h,c_r(n)}+he_r(n);\lambda_{hj}),\lambda_{hj},h,e)-\varphi(t_{hn})+C_{hn}(e)-$$
$$-hF(t_{hn},\varphi(t_{hn}),\varphi(t_{h,c_1(n)}+he_1(n)),\cdots,\varphi(t_{h,c_r(n)}+he_r(n)),\lambda,h,e)$$
$$=\left[I+h\bar{A}^j_{hn}(0,e)\right]v^j_h(t_{hn})+h\sum_{i=1}^{r}\bar{A}^j_{hn}(i,e)v^j_h(t_{h,c_i(n)}+he_i(n))+$$
$$+h\tilde{\bar{A}}^j_{hn}(\lambda,e)z^j_h+C_{hn}(e),\quad n=0,1,\cdots,N-1,\ \ j=0,1,\cdots.$$

It is easy to see that

$$\|v^j_h(t_{hn}+eh)\|\le(1+hQ_0)V^j_{hn}+hQW^j_{hn}+hQ_\lambda Z^j_h+\epsilon(t_{hn},h,e)$$

for $e\in[0,1]$, $n=0,1,\cdots,N-1$, $j=0,1,\cdots$. Now using Lemma 1 we arrive the inequality

$$W^j_{hN}\le c_1c\sum_{i=0}^{N-1}\bar{\epsilon}(t_{hi},h)+c_1BQ_\lambda Z^j_h,\ \ j=0,1,\cdots. \tag{16}$$

Put

$$T^j_h(t)=Y_h(t;\lambda_{hj})z^j_h-v^j_h(t),\qquad \tilde{T}^j_{hn}=\|T^j_h(t_{hn})\|,\qquad X^j_{hn}=\sup_{[a,t_{hn}]}\|T^j_h(t)\|.$$

The definition of Y_h and (15) yield

$$\begin{aligned}(17)\quad T^j_h(t_{hn}+eh)&=\left[I+hA^j_{hn}(0,e)\right]Y_h(t_{hn};\lambda_{hj})z^j_h-v^j_h(t_{hn}+eh)\\&+\left\{h\sum_{i=1}^{r}A^j_{hn}(i,e)Y_h(t_{h,c_i(n)}+he_i(n);\lambda_{hj})+h\tilde{A}^j_{hn}(\lambda,e)\right\}z^j_h\\&=\left[I+hA^j_{hn}(0,e)\right]T^j_h(t_{hn})+h\left[A^j_{hn}(0,e)-\bar{A}^j_{hn}(0,e)\right]v^j_h(t_{hn})+\\&+h\sum_{i=1}^{r}A^j_{hn}(i,e)T^j_h(t_{h,c_i(n)}+he_i(n))+h\left[\tilde{A}^j_{hn}(\lambda,e)-\tilde{\bar{A}}^j_{hn}(\lambda,e)\right]z^j_h+\\&+h\sum_{i=1}^{r}\left[A^j_{hn}(i,e)-\bar{A}^j_{hn}(i,e)\right]v^j_h(t_{h,c_i(n)}+he_i(n))-C_{hn}(e)\end{aligned}$$

for $e\in[0,1]$, $n=0,1,\cdots,N-1$, $j=0,1,\cdots$.

By Assumption H, we get

$$\begin{aligned}\|A^j_{hn}(s,e)-\bar{A}^j_{hn}(s,e)\|&\le\frac{1}{2}\left(L_{s0}V^j_{hn}+\sum_{i=1}^{r}L_{si}\|v^j_h(t_{h,c_i(n)}+he_i(n))\|+L^\lambda_sZ^j_h\right)\\&\le L_sW^j_{hn}+\frac{1}{2}L^\lambda_sZ^j_h,\quad s=0,1,\cdots,r,\\\|\tilde{A}^j_{hn}(\lambda,e)-\tilde{\bar{A}}^j_{hn}(\lambda,e)\|&\le\frac{1}{2}\left(M_0V^j_{hn}+\sum_{i=1}^{r}M_i\|v^j_h(t_{h,c_i(n)}+he_i(n))\|+M^\lambda Z^j_h\right)\\&\le MW^j_{hn}+\frac{1}{2}M^\lambda Z^j_h\end{aligned}$$

for $e\in[0,1]$, $n=0,1,\cdots,N$, $j=0,1,\cdots$.

Combining this with (17), we have

$$\|T_h^j(t_{hn}+eh)\| \le (1+hQ_0)\tilde{T}_{hn}^j + hQX_{hn}^j + \frac{h}{2}M^\lambda\left(Z_h^j\right)^2 + P_{hn}^j(e), \quad n=0,1,\cdots,N-1,$$

$$j=0,1,\cdots$$

for $e \in [0,1]$ with

$$P_{hn}^j(e) = h\left[L\left(W_{hn}^j\right)^2 + L^\lambda Z_h^j W_{hn}^j\right] + \epsilon(t_{hn},h,e).$$

Now, using (16) and Lemma 1, we have the relation

$$X_{hN}^j \le c_1\left[A_1\left(Z_h^j\right)^2 + B_1(h)Z_h^j + C_1(h)\right], \quad j=0,1\cdots. \tag{18}$$

In view of Assumption $H(2^0$ and $4^0)$ and (16), we have

$$\|B_{ih}^j - \bar{B}_{ih}^j\| \le \frac{1}{2}\left(K_{i1}Z_h^j + K_{i2}V_{hN}^j\right) \le \bar{K}_{i1}Z_h^j + \bar{K}_{i2}\sum_{i=0}^{N-1}\bar{\epsilon}(t_{hi},h), \quad i=1,2, \tag{19}$$

$$j=0,1,\cdots.$$

Now we need some relation on z_h^j. By the definition of λ_{hj} and Assumption $H(2^0,4^0)$ we arrive at the inequality

$$\begin{aligned} z_h^{j+1} &= \lambda_{hj} - \left[Q_h^j\right]^{-1} g(\lambda_{hj}, y_h(b;\lambda_{hj})) - \lambda \\ &= \left[Q_h^j\right]^{-1}\left\{Q_h^j(\lambda_{hj}-\lambda) - g(\lambda_{hj}, y_h(b;\lambda_{hj})) + g(\lambda,\varphi(b))\right\} \\ &= \left[Q_h^j\right]^{-1}\left[(B_{1h}^j - \bar{B}_{1h}^j)z_h^j + B_{2h}^j T_h^j(b) + (B_{2h}^j - \bar{B}_{2h}^j)v_h^j(b)\right], \quad j=0,1,\cdots \end{aligned}$$

for

$$Q_h^j = B_{1h}^j + B_{2h}^j Y_h(b;\lambda_{hj}).$$

Combining this with (16, 18, 19) and using condition 3^0 of Theorem 1, we have

$$Z_h^{j+1} \le D\left[A_2\left(Z_h^j\right)^2 + B_2(h)Z_h^j + C_2(h)\right], \quad j=0,1,\cdots.$$

Now the estimates (12-14) follow directly from Lemma 2 and (16, 18).

Remark 2. If

$$\sum_{i=0}^{N-1}\bar{\epsilon}(t_{hi},h) = 0(h^\nu), \quad \nu > 0 \text{ as } h \to 0,$$

then

$$\|\lambda_{hj} - \lambda\| = d^j\|\lambda_0 - \lambda\| + 0(h^\nu),$$

$$\sup_{t\in J}\|y_h(t;\lambda_{hj})-\varphi(t)\| = c_1 BQ_\lambda d^j\|\lambda_0-\lambda\| + 0(h^\nu)$$

as $h\to 0$ and $j\to\infty$.

Now we try to formulate some conditions by which 3^0 of Theorem 1 holds. We have the following lemma.

Lemma 3. *Assume that Assumption H and conditions $1^0 - 2^0$ of Theorem 1 are satisfied, and*

1^0 *there exists a function $\gamma : J_h \times H \times [0,1] \to R_+$ such that*

$$\begin{aligned}&\|[I + hF_0(t,\varphi(t),\varphi(\alpha_1(t)),\cdots,\varphi(\alpha_r(t)),\lambda,h,e)]Y(t;\lambda)+\\ &\quad + h\sum_{i=1}^{r} F_i(t,\varphi(t),\varphi(\alpha_1(t)),\cdots,\varphi(\alpha_r(t)),\lambda,h,e)Y(\alpha_i(t);\lambda)+\\ &\quad + hF_\lambda(t,\varphi(t),\varphi(\alpha_1(t)),\cdots,\varphi(\alpha_r(t)),\lambda,h,e) - Y(t+eh;\lambda)\| \le \gamma(t,h,e),\end{aligned}$$

and

$$\lim_{N\to\infty}\sum_{i=0}^{N-1}\bar{\gamma}(t_{hi},h) = 0 \quad \text{with} \quad \bar{\gamma}(t,h) = \sup_{e\in[0,1]}\gamma(t,h,e),$$

where Y is the solution of (4) and $\|Y\|\le Y_b$,

2^0 *the matrix $Q(\lambda) = g_1(\lambda,\varphi(b)) + g_2(\lambda,\varphi(b))Y(b;\lambda)$ is nonsingular and $\|Q^{-1}(\lambda)\|\le\beta$, then condition 3^0 of Theorem 1 holds if λ_0 is sufficiently close to λ.*

Proof. Let

$$\begin{aligned}&Q_h^j = B_{1h}^j + B_{2h}^j Y_h(b;\lambda_{hj}),\\ &D_{hn}^j(i,e) = F_i(t_{hn},\varphi(t_{hn}),\varphi(t_{h,c_1(n)}+he_1(n)),\cdots,\varphi(t_{h,c_r(n)}+he_r(n)),\lambda,h,e),\\ &\tilde{D}_{hn}^j(\lambda,e) = F_\lambda(t_{hn},\varphi(t_{hn}),\varphi(t_{h,c_1(n)}+he_1(n)),\cdots,\varphi(t_{h,c_r(n)}+he_r(n)),\lambda,h,e).\end{aligned}$$

We see that

$$\begin{aligned}\|Q_h^j - Q(\lambda)\| &\le \|g_1(\lambda_{hj},y_h(b;\lambda_{hj})) - g_1(\lambda,\varphi(b))\|+\\ &+ \|[g_2(\lambda_{hj},y_h(b;\lambda_{hj})) - g_2(\lambda,\varphi(b))]Y(b;\lambda)\|+\\ &+ \|g_2(\lambda_{hj},y_h(b;\lambda_{hj}))[Y_h(b;\lambda_{hj}) - Y(b;\lambda))]\|, \quad j=0,1,\cdots,\end{aligned}$$

and by Assumption H we obtain

(20) $$\|Q_h^j - Q(\lambda)\| \le (K_{11}+K_{21}Y_b)Z_h^j + (K_{12}+K_{22}Y_b)V_{hN}^j + G\|q_h^j(b)\|, \quad j=0,1,\cdots,$$

where

$$q_h^j(t) = Y_h(t;\lambda_{hj}) - Y(t;\lambda).$$

Now we need to have some relation on q_h^j. First we note that

$$\begin{aligned}&\|\tilde{A}_{hn}^j(\lambda,e) - \tilde{D}_{hn}^j(\lambda,e)\| \le 2MW_{hn}^j + M^\lambda Z_h^j,\\ &\|A_{hn}^j(s,e) - D_{hn}^j(s,e)\| \le 2L_s W_{hn}^j + L^\lambda Z_h^j, \quad s=0,1,\cdots,r\end{aligned}$$

for $n = 0, 1, \cdots, N,\ \ j = 0, 1, \cdots$. Using the above inequalities and the relation

$$q_h^j(t_{hn} + eh) = \left[I + hA_{hn}^j(0, e)\right] q_h^j(t_{hn}) + h \sum_{i=1}^{r} A_{hn}^j(i, e) q_h^j(t_{h,c_i(n)} + he_i(n)) +$$
$$+ h\left[\tilde{A}_{hn}^j(\lambda, e) - \tilde{D}_{hn}^j(\lambda, e)\right] + h\left[A_{hn}^j(0, e) - D_{hn}^j(0, e)\right] Y(t_{hn}; \lambda) +$$
$$+ h \sum_{i=1}^{r} \left[A_{hn}^j(i, e) - D_{hn}^j(i, e)\right] Y(t_{h,c_i(n)} + he_i(n); \lambda) +$$
$$+ h\tilde{D}_{hn}^j(\lambda, e) + \left[I + hD_{hn}^j(0, e)\right] Y(t_{hn}; \lambda) + h \sum_{i=1}^{r} D_{hn}^j(i, e) Y(\alpha_i(t_{hn}; \lambda) -$$
$$- Y(t_{hn} + eh; \lambda), \quad n = 0, 1, \cdots, N-1,\ \ j = 0, 1, \cdots,$$

we obtain

$$\|q_h^j(t_{hn} + eh)\| \le (1 + hQ_0)\|q_h^j(t_{hn})\| + hQ\bar{Q}_{hn}^j + h\bar{M}W_{hn}^j + h\bar{\bar{M}}Z_h^j + \gamma(t_{hn}, h, e)$$

for $e \in [0, 1],\ \ n = 0, 1, \cdots, N-1,\ j = 0, 1, \cdots$, where $\bar{M}$ and $\bar{\bar{M}}$ are nonnegative constants and

$$\bar{Q}_{hn}^j = \sup_{[a, t_{hn}]} \|q_h^j(t_{hn})\|.$$

Now applying (16) and Lemma 1 we have

$$\bar{Q}_{hN}^j \le M^* Z_h^j + \xi(h), \quad j = 0, 1, \cdots,$$

where

$$M^* = c_1 B\left[\bar{\bar{M}} + \bar{M} c_1 Q_\lambda B\right], \quad \xi(h) = c_1 c\left[c_1 B\bar{M} \sum_{i=0}^{N-1} \bar{\epsilon}(t_{hi}, h) + \sum_{i=0}^{N-1} \bar{\gamma}(t_{hi}, h),\right].$$

Combining this with (20) and (16) we get

$$\|Q_h^j - Q(h)\| \le \bar{K} Z_h^j + \nu(h), \quad j = 0, 1, \cdots,$$

where

$$\bar{K} = K_{11} + K_{21} Y_b + (K_{12} + K_{22} Y_b) c_1 B Q_\lambda + G M^*,$$
$$\nu(h) = (K_{12} + K_{22} Y_b) c_1 c \sum_{i=0}^{N-1} \bar{\epsilon}(t_{hi}, h) + G\xi(h).$$

Hence

$$p_h^j = \|Q^{-1}(\lambda)[Q_h^j - Q(\lambda)]\| \le \beta[\bar{K} Z_h^j + \nu(h)], \quad j = 0, 1, \cdots.$$

Let

$$Z_h^0 \le \rho = \sup_{x \le \bar{h}} \frac{DC_2(x)}{1 - d} \quad \text{and} \quad \beta \bar{K} \rho < 1,$$

where $\bar{h}$ is sufficiently small such that $(\star\star)$ holds for $h \le \bar{h}$. Because $\nu(h) \to 0$ as $h \to 0$, so there exists α such that

$$\beta[\bar{K}\rho + \nu(h)] \le \alpha < 1$$

holds for sufficiently small h. Now, by Lemma 4.4.14[11], we conclude that the matrix

$$I + Q^{-1}(\lambda)[Q_h^0 - Q(\lambda)]$$

is nonsingular. Hence the matrix

$$Q_h^0 = Q(\lambda)\left\{I + Q^{-1}(\lambda)\left[Q_h^0 - Q(\lambda)\right]\right\}$$

is also nonsingular and

$$\| \left(Q_h^0\right)^{-1} \| \le \frac{\beta}{1-\alpha}.$$

It means that 3^0 of Theorem 1 holds for sufficiently small h and $j = 0$ with $D = \beta/(1-\alpha)$.

Put $u_0(h) = \rho$. Theorem 1 follows $Z_h^1 \le u_1(h) \le \rho$, where u_1 is defined as in Theorem 1. Furthermore,

$$p_h^1 \le \beta(\bar{K}\rho + \nu(h)) \le \alpha < 1,$$

so the matrices

$$I + Q^{-1}(\lambda)[Q_h^1 - Q(\lambda)] \quad \text{and} \quad Q_h^1 = Q(\lambda)[I + Q^{-1}(\lambda)(Q_h^1 - Q(\lambda))]$$

are nonsingular and

$$\| \left(Q_h^1\right)^{-1} \| \le \frac{\beta}{1-\alpha}.$$

Now, by induction with respect to j, we can prove that condition 3^0 of Theorem 1 holds.

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