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Doubly warped product and the periodicity of geodesics

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§0. Introduction.

A Riemannian manifold (M,g) is called a C_L-manifold (and g called a C_L-metric) if every geodesic is a closed curve with the same length L. The known examples are Zoll surface of revolution ([3], [10]), C_L-surface studied by Guillemin ([5], see also Kiyohara [7]), n-dimensional sphere S^n with Zoll metric generalized by Weinstein ([3]), RP^n, $\mathbb{C}P^n$, HP^n, $\mathbb{C}ay\ P^2$ and S^n with the canonical metrics.

The induced metric on Zoll surfaces of revolution is a warped product metric

(0.1) $\quad ds^2 = dr^2 + v(r)^2 d\theta^2,$

where r denotes the distance from the north pole and v(r) the Euclidean distance from the axis of rotation. We demand certain conditions at 0 and L/2 for v to give a smooth metric. Furthermore, in [3], the necessary and sufficient condition for v to be a C_L-metric was given (cf. §3).

In this paper, we shall consider a metric on a complex 2-dimensional complex projective space $\mathbb{C}P^2$, which corresponds to the metric (0.1) on S^2. Our construction of the metric g on

$\mathbb{C}P^2$ is as follows. Let M^* be the 4-dimensional manifold obtained by removing a ball centered at an arbitrarily fixed point p_0 from $\mathbb{C}P^2$. It is well-known that M^* is diffeomorphic to the disc bundle associated to the Hopf bundle $\pi : S^3 \longrightarrow S^2$. The submanifold S_∞ consisting of points at infinity is regarded as the base manifold. We note that the metric on ∂M^* induced from the canonical metric g_{can} on $\mathbb{C}P^2$ is obtained by rescaling the canonical metric g_0 of S^3 on the fiber direction $\mathcal{V}$ with factor $\sin^2 r \cos^2 r$ and on its orthogonal complement $\mathcal{H}$ with factor $\cos^2 r$, where r denotes the distance (with respect to g_{can}) from S_∞. So, it is very natural that we consider a metric obtained by replacing $\sin^2 r \cos^2 r$ and $\cos^2 r$ by $v^2(r)$ and $h^2(r)$ respectively which satisfy some conditions to define a smooth metric (cf. (1.6)). Thus the metric can be given by

$$g = dr^2 + v(r)^2 g_0|_{\mathcal{V}} + h(r)^2 g_0|_{\mathcal{H}}, \tag{0.2}$$

which was used to make examples with various geometric conditions (for instance, see [1], [2], [4], [8], [9]).

Let g be a metric of the form (0.2) on $\mathbb{C}P^2$. The purpose of this paper is to study the periodicity of geodesics. In virtue of Green's theorem (cf. [3]), we can easily see that if g is a C_L-metric, then $g = g_{can}$ (cf. Cor. 4.5). However, it is significant to study how many geodesics are closed under what conditions on h and v. If $g = g_{can}$ on M^* obtained by removing a small ball from $\mathbb{C}P^2$, it is trivial that all complete geodesics in M^* are closed curves with the same length. We do not intend to study such metrics.

In §1, we explain the metric g of the form (0.2). In §2, we calculate the Cristoffel symbols of g and give the equation of geodesics. From this equation we derive Clairaut's first integral analogous to one of geodesics on a surface of revolution. We also prove that S_∞ is a totally geodesic submanifold of constant sectional curvature and, for each $q \in S_\infty$, geodesics emanating from p_0 to q form a totally geodesic submanifold S_q. The induced metric on S_q is of the form (0.1). In §3, we explain the Zoll metric on S^2. Moreover, we give a sufficient condition for a geodesic passing through S_∞ to be closed (Th. 3.4). In §4, we deal with geodesics which are contained in a geodesic sphere centered at p_0 and geodesics which do not pass through S_∞ and are not contained in any geodesic sphere. Our main result is stated in Theorem 4.7.

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§1. Doubly warped product.

Let I be the closed interval $[0,\pi/2]$. Let $\pi : S^3 \longrightarrow S^2$ be the Hopf fibering which is defined by

(1.1) $$\pi(z_1,z_2) = (z_1\bar{z}_2, \frac{1}{2}(|z_1|^2-|z_2|^2)) \in S^2 \subset \mathbb{C} \times \mathbb{R}$$

for $(z_1,z_2) \in S^3 \subset \mathbb{C} \times \mathbb{C}$. The complex projective space $\mathbb{C}P^2$ of complex dimension 2 is topologically equivalent to $((I \times S^3)/A)/\sim$, where $A = \{\pi/2\} \times S^3$ and $(s,x) \sim (t,y)$ if and only if $s = t$ and

$$x = y \qquad (s = t \neq 0),$$

$$\pi(x) = \pi(y) \quad (s = t = 0).$$

Namely, A collapses to a point and $\{0\} \times S^3$ fiberwise to S^2.

We introduce coordinates $(r,\theta,\varphi,\psi) \in (0,\pi/2) \times (0,2\pi) \times (0,\pi/2) \times (0,\pi)$ on an open subset $(0,\pi/2) \times U$, where U is an open subset in S^3 such that $\overline{U} = S^3$. We note that the fiber through $(z_1,z_2) \in S^3(1)$ is the subset $\{(z_1e^{i\theta},z_2e^{i\theta}) : \theta \in [0,2\pi)\}$ in $S^3(1)$. If we put

$$z_1 = \cos\varphi\, e^{i\varphi_1}, \qquad z_2 = \sin\varphi\, e^{i\varphi_2},$$

then

$$\pi(z_1,z_2) = (\tfrac{1}{2}\sin 2\varphi\, e^{i(\varphi_1-\varphi_2)}, \tfrac{1}{2}\cos 2\varphi).$$

Putting $2\psi = \varphi_1 - \varphi_2$, we see that

$$(1.2) \qquad \pi(z_1e^{i\theta},z_2e^{i\theta}) = \tfrac{1}{2}(\sin 2\varphi\cos 2\psi, \sin 2\varphi\sin 2\psi, \cos 2\varphi) \in \mathbf{R}^3.$$

Therefore the image $\pi(S^3(1))$ is a sphere $S^2(1/2)$ with radius $1/2$. The equation (1.2) means that 2φ (resp. 2ψ) denotes the angle between $\pi(z_1e^{i\theta},z_2e^{i\theta})$ and $(0,0,1/2)$ (resp. $(1/2,0,0)$). So we let U be the maximal open subset where the coordinates (θ,φ,ψ) are valid. We can write

$$(1.3) \qquad x = (\cos\varphi\, e^{i(\psi+\theta)}, \sin\varphi\, e^{i(-\psi+\theta)})$$

for arbitrary $x \in U$.

The Fubini Study metric g_{can} on $\mathbb{C}P^2$ is a metric such that Hopf fibering $S^5 \longrightarrow \mathbb{C}P^2$ is a Riemannian submersion (cf. [2]). If the metric on S^5 is of constant sectional curvature 1, then the holomorphic sectional curvature of $(\mathbb{C}P^2,g_{can})$ is equal to 4. In the sequel, p_0 will be an arbitrarily fixed point in $\mathbb{C}P^2$. The cut-locus of p_0 with respect to g_{can} will be denoted by S_∞, which is isometric to $S^2(1/2)$. We note that the distance from p_0 to S_∞ is equal to $\pi/2$. The first component r in the coordinates $\{r,\theta,\varphi.\psi\}$ denotes the distance measured from S_∞.

Let $S^3(1)$ be the unit sphere in the tangent space $T_{p_0}\mathbb{C}P^2$ at p_0. The map sending $X \in S^3(1)$ to $\exp_{p_0} \frac{\pi}{2} X \in S_\infty$ coincides with $\pi : S^3(1) \longrightarrow S^2(1/2)$ ((1.2)). For each $q \in S_\infty$, geodesics from p_0 to q form a submanifold S_q which is isometric to $S^2(1/2)$ and the intersection $T_{p_0}S_q \cap S^3(1)$ is the fiber $\pi^{-1}(q)$. Therefore the topological description of $\mathbb{C}P^2$ stated above is explained as follows. The boundary ∂D_r of the disc bundle D_r of S_∞, which is the geodesic sphere with center p_0 and radius $\pi/2 - r$, is diffeomorphic to $\{r\} \times S^3$ for each $r \in (0,\pi/2)$ and collapses to p_0 as $r \longrightarrow \pi/2$. On the other hand, if $r \longrightarrow 0$, then $S_q \cap \partial D_r$ (= the boundary of the fiber of D_r over q) collapses to q. The <u>vertical line</u> $\mathcal{V}_p$ at $p \in \partial D_r$ $(r \in (0,\pi/2))$ will be the tangent line of $S_q \cap \partial D_r$, where, if γ is the minimal geodesic from p_0 to p, then $q = \gamma(\pi/2)$. The <u>horizontal plane</u> $\mathcal{H}_p$ at p will be the orthogonal complement of $\mathcal{V}_p$ in the tangent space $T_p\partial D_r$.

Let g_0 be the canonical metric on $S^3(\approx \partial D_r)$ of constant curvature 1. It is easily verified that

(1.4) $$g_{can} = dr^2 + \sin^2 r \cos^2 r \, g_0|_{\mathcal{V}} + \cos^2 r \, g_0|_{\mathcal{H}} ,$$

where $g_0|_{\mathcal{V}}$ (resp. $g_0|_{\mathcal{H}}$) denotes the restriction of g_0 to the vertical line $\mathcal{V}$ (resp. horizontal plane $\mathcal{H}$). In the subsequent sections, we shall study geodesics of the following metric on $\mathbb{C}P^2$:

(1.5) $$g = dr^2 + v(r)^2 g_0|_{\mathcal{V}} + h(r)^2 g_0|_{\mathcal{H}} .$$

We need some conditions for v and h in order that g is a smooth complete metric on $\mathbb{C}P^2$. We shall assume that v and h are smooth functions defined on $[0,\pi/2]$ and satisfy (cf. [3][6])

$$v > 0, \quad h > 0 \quad \text{on } (0,\pi/2),$$

$$
\begin{aligned}
(1.6)\qquad & v(0) = 0, \quad v'(0) = 1, \quad v(\tfrac{\pi}{2}) = 0, \quad v'(\tfrac{\pi}{2}) = -1, \\
& v^{(2k)}(0) = v^{(2k)}(\tfrac{\pi}{2}) = 0 \quad \text{for positive integer } k, \\
& h(0) = 1, \quad h'(0) = 0, \quad h(\tfrac{\pi}{2}) = 0, \quad h'(\tfrac{\pi}{2}) = -1, \\
& h^{(2k+1)}(0) = h^{(2k)}(\tfrac{\pi}{2}) = 0 \quad \text{for positive integer } k.
\end{aligned}
$$

§2. Equation of geodesics.

We shall give the equation of geodesics of the metric (1.5), from which we derive two Clairaut's first integrals. We number the coordinates $\{r,\theta,\varphi,\psi\}$ as

$$x^1 = r, \quad x^2 = \theta, \quad x^3 = \varphi, \quad x^4 = \psi.$$

We note that the vertical line $\mathcal{V}$ is spanned by $\partial/\partial\theta$ and horizontal plane $\mathcal{H}$ by $\partial/\partial\varphi$ and $-\cos 2\varphi\ \partial/\partial\theta + \partial/\partial\psi$. It is easy to calculate the components of the metric g.

Lemma 2.1. *The components* $g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j)$ $(i,j=1,\dots,4)$ *are given by*

$$g_{11} = 1, \quad g_{22} = v^2, \quad g_{24} = g_{42} = v^2 \cos 2\varphi, \quad g_{33} = h^2,$$
$$g_{44} = v^2\cos^2 2\varphi + h^2 \sin^2 2\varphi, \quad g_{ij} = 0 \quad (i,j : \text{the others}).$$

The contravariant components are given by

$$g^{11} = 1, \quad g^{22} = \frac{1}{v^2} + \frac{1}{h^2}\cot^2 2\varphi, \quad g^{24} = g^{42} = -\frac{\cos 2\varphi}{h^2 \sin^2 2\varphi},$$
$$g^{33} = \frac{1}{h^2}, \quad g^{44} = \frac{1}{h^2 \sin^2 2\varphi}, \quad g^{ij} = 0 \quad (i,j : \text{the others}).$$

Since the Christoffel's symbols $\left\{ {i \atop j\ k} \right\}$ $(i,j,k : 1,\dots,4)$ are given by the formula :

$$\left\{ {i \atop j\ k} \right\} = \frac{1}{2} g^{i\ell}(\partial_j g_{\ell k} + \partial_k g_{j\ell} - \partial_\ell g_{jk}),$$

where we use the Einstein's summation convention, we have, from Lemma 2.1,

<u>Lemma</u> 2.2. <u>The Christoffel symbols of the metric</u> g <u>are given by</u>

$$\left\{{}_{2}{}^{1}{}_{2}\right\} = - vv', \quad \left\{{}_{2}{}^{1}{}_{4}\right\} = \left\{{}_{4}{}^{1}{}_{2}\right\} = - vv' \cos 2\varphi,$$

$$\left\{{}_{3}{}^{1}{}_{3}\right\} = - hh', \quad \left\{{}_{4}{}^{1}{}_{4}\right\} = - (vv' \cos^2 2\varphi + hh' \sin^2 2\varphi),$$

$$\left\{{}_{1}{}^{2}{}_{2}\right\} = \left\{{}_{2}{}^{2}{}_{1}\right\} = \frac{v'}{v}, \quad \left\{{}_{1}{}^{2}{}_{4}\right\} = \left\{{}_{4}{}^{2}{}_{1}\right\} = \left(\frac{v'}{v} - \frac{h'}{h}\right) \cos 2\varphi,$$

$$\left\{{}_{2}{}^{2}{}_{3}\right\} = \left\{{}_{3}{}^{2}{}_{2}\right\} = \frac{v^2}{h^2} \cot 2\varphi,$$

$$\left\{{}_{3}{}^{2}{}_{4}\right\} = \left\{{}_{4}{}^{2}{}_{3}\right\} = - \left\{\frac{1}{\sin 2\varphi} + \left(1 - \frac{v^2}{h^2}\right) \frac{\cos^2 2\varphi}{\sin 2\varphi}\right\},$$

$$\left\{{}_{1}{}^{3}{}_{3}\right\} = \left\{{}_{3}{}^{3}{}_{1}\right\} = \frac{h'}{h}, \quad \left\{{}_{2}{}^{3}{}_{4}\right\} = \left\{{}_{4}{}^{3}{}_{2}\right\} = \frac{v^2}{h^2} \sin 2\varphi,$$

$$\left\{{}_{4}{}^{3}{}_{4}\right\} = - \left(1 - \frac{v^2}{h^2}\right) \sin 4\varphi, \quad \left\{{}_{1}{}^{4}{}_{4}\right\} = \left\{{}_{4}{}^{4}{}_{1}\right\} = \frac{h'}{h},$$

$$\left\{{}_{2}{}^{4}{}_{3}\right\} = \left\{{}_{3}{}^{4}{}_{2}\right\} = - \frac{v^2}{h^2} \frac{1}{\sin 2\varphi},$$

$$\left\{{}_{3}{}^{4}{}_{4}\right\} = \left\{{}_{4}{}^{4}{}_{3}\right\} = \left(2 - \frac{v^2}{h^2}\right) \cot 2\varphi,$$

$\left\{{}_{j}{}^{i}{}_{k}\right\} = 0$ (i,j,k : <u>the others</u>).

Let ∇ be the covariant differential operator with respect to g. For each $q \in S_\infty$, the tangent spaces of the submanifold S_q are spanned by $\partial/\partial r$ and $\partial/\partial\theta$. Since

$$\nabla_{\partial/\partial r}\partial/\partial r = \left\{{}_{1}{}^{i}{}_{1}\right\}\partial/\partial x^i = 0,$$

$$\nabla_{\partial/\partial r}\partial/\partial\theta = \left\{{}_{1}{}^{i}{}_{2}\right\}\partial/\partial x^i = \frac{v'}{v}\partial/\partial\theta,$$

$$\nabla_{\partial/\partial\theta}\partial/\partial\theta = \left\{{}_{2}{}^{i}{}_{2}\right\}\partial/\partial x^i = - vv'\partial/\partial r$$

because of Lemma 2.2, we see that every S_q is totally geodesic.

Proposition 2.3.

(1) Every geodesic issuing from p_0 with respect to g_0 is also a geodesic with respect to g.

(2) The submanifold S_q is totally geodesic for each $q \in S_\infty$.

(3) The submanifold S_∞ is totally geodesic and isometric to $S^2(1/2)$ of constant sectional curvature 4.

Proof. We sketch the proof of (3). Since the coordinates $\{r, \theta, \varphi, \psi\}$ is not valid on S_∞, we introduce

$$u^1 = r \cos \theta, \quad u^2 = r \sin \theta, \quad u^3 = \varphi, \quad u^4 = \psi$$

on a neighbourhood of S_∞. The calculation of components of g and the Christoffel symbols with respect to this coordinates is routine. As a result, we see that, as $r \longrightarrow 0$, they contain only the first and second derivatives of v and h. Since functions v and h satisfy (1.6), they coincide with those of g_{can} as $r \longrightarrow 0$. The fact that (3) is true for g_{can} is well known. Q.E.D.

The equation of geodesics is given by

$$\frac{d^2x^i}{ds^2} + \left\{ {i \atop j\ k} \right\} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0 \qquad (i = 1, \ldots, 4), \tag{2.1}$$

where s is the arclength parameter. By Lemma 2.2, we have

Lemma 2.4. Let $r = r(s)$, $\theta = \theta(s)$, $\varphi = \varphi(s)$, $\psi = \psi(s)$ be a geodesic. Then (2.1) becomes

$$\frac{d^2r}{ds^2} - vv'\left(\frac{d\theta}{ds}\right)^2 - 2vv' \cos 2\varphi \frac{d\theta}{ds}\frac{d\psi}{ds} - hh'\left(\frac{d\varphi}{ds}\right)^2 - (vv' \cos^2 2\varphi + hh' \sin^2 2\varphi)\left(\frac{d\psi}{ds}\right)^2 = 0, \tag{2.2}$$

(2.3) $$\frac{d^2\theta}{ds^2} + 2\frac{v'}{v}\frac{dr}{ds}\frac{d\theta}{ds} + 2\left(\frac{v'}{v} - \frac{h'}{h}\right)\cos 2\varphi \frac{dr}{ds}\frac{d\psi}{ds}$$
$$+ 2\frac{v^2}{h^2}\cot 2\varphi \frac{d\theta}{ds}\frac{d\varphi}{ds}$$
$$- 2\left\{\frac{1}{\sin 2\varphi} + \left(1 - \frac{v^2}{h^2}\right)\frac{\cos^2 2\varphi}{\sin 2\varphi}\right\}\frac{d\varphi}{ds}\frac{d\psi}{ds} = 0,$$

(2.4) $$\frac{d^2\varphi}{ds^2} + 2\frac{h'}{h}\frac{dr}{ds}\frac{d\varphi}{ds} + 2\frac{v^2}{h^2}\sin 2\varphi \frac{d\theta}{ds}\frac{d\psi}{ds}$$
$$- \left(1 - \frac{v^2}{h^2}\right)\sin 4\varphi \left(\frac{d\psi}{ds}\right)^2 = 0,$$

(2.5) $$\frac{d^2\psi}{ds^2} + 2\frac{h'}{h}\frac{dr}{ds}\frac{d\psi}{ds} - 2\frac{v^2}{h^2}\frac{1}{\sin 2\varphi}\frac{d\theta}{ds}\frac{d\varphi}{ds}$$
$$+ 2\left(2 - \frac{v^2}{h^2}\right)\cot 2\varphi \frac{d\varphi}{ds}\frac{d\psi}{ds} = 0.$$

Since s is the arclength parameter, we have

(2.6) $$\left(\frac{dr}{ds}\right)^2 + v^2\left(\frac{d\theta}{ds} + \cos 2\varphi \frac{d\psi}{ds}\right)^2$$
$$+ h^2\left\{\left(\frac{d\varphi}{ds}\right)^2 + \sin^2 2\varphi \left(\frac{d\psi}{ds}\right)^2\right\} = 1.$$

Lemma 2.5. *Along any geodesic, the following c and d are constant* :

$$c = v^2\left(\frac{d\theta}{ds} + \cos 2\varphi \frac{d\psi}{ds}\right), \qquad d = h^2\left\{\left(\frac{d\varphi}{ds}\right)^2 + \sin^2 2\varphi \left(\frac{d\psi}{ds}\right)^2\right\}^{1/2}.$$

Proof. Differentiating c, we get

$$\frac{dc}{ds} = v^2\frac{d^2\theta}{ds^2} + 2vv'\frac{dr}{ds}\frac{d\theta}{ds} + v^2\cos 2\varphi \frac{d^2\psi}{ds^2}$$
$$+ 2vv'\cos 2\varphi \frac{dr}{ds}\frac{d\psi}{ds} - 2v^2\sin 2\varphi \frac{d\varphi}{ds}\frac{d\psi}{ds}.$$

The right hand side of this equation is equal to

$$v^2\,(\text{L.H.S. of (2.3)}) + v^2\cos 2\varphi\,(\text{L.H.S. of (2.5)}).$$

Similarly, we differentiate d^2. Then we see that the derivative of d^2 is equal to

$$2h^4\frac{d\varphi}{ds}\,(\text{L.H.S. of (2.4)}) + 2h^4\sin^2 2\varphi \frac{d\psi}{ds}\,(\text{L.H.S. of (2.5)}).$$

Q.E.D.

The geometric interpretation of the constants c and d is as follows. Let α (resp. β) be the angle between the unit tangent vector $\dot{\gamma}$ of a geodesic γ and the vertical line $\mathcal{V}$ (resp. horizontal plane $\mathcal{H}$). Let $X_1,\ldots,X_4$ be orthonormal vectors defined by

$$X_1 = \frac{\partial}{\partial r}, \quad X_2 = \frac{1}{v}\frac{\partial}{\partial \theta}, \quad X_3 = \frac{1}{h}\frac{\partial}{\partial \varphi},$$
$$X_4 = \frac{1}{h \sin 2\varphi}\left(\frac{\partial}{\partial \psi} - \cos 2\varphi \frac{\partial}{\partial \theta}\right).$$

Then we can write

$$\dot{\gamma} = \frac{dr}{ds} X_1 + v\left(\frac{d\theta}{ds} + \cos 2\varphi \frac{d\psi}{ds}\right)X_2 + h \frac{d\varphi}{ds} X_3 + h \sin 2\varphi \frac{d\psi}{ds} X_4$$

because of Lemma 2.1. Since $\mathcal{V}$ is spanned by X_2 and $\mathcal{H}$ is spanned by $\{X_3,X_4\}$, we have

$$\cos \alpha = v\left|\frac{d\theta}{ds} + \cos 2\varphi \frac{d\psi}{ds}\right|,$$
$$\cos \beta = h\left\{\left(\frac{d\varphi}{ds}\right)^2 + \sin^2 2\varphi \left(\frac{d\psi}{ds}\right)^2\right\}^{1/2}.$$

It follows that

$$|c| = v \cos \alpha, \qquad d = h \cos \beta.$$

These equations are analogous to the Clairant's relation of geodesics on surfaces of revolution.

§3. Special geodesics.

In the present and next sections, we shall study the periodicity of geodesics of the metric g. In particular, it is shown that the Fubini Study metric g_{can} is the only C_π-metric in the class of metrics of the form (1.5). We have already seen that the length of geodesics passing through p_0 or tangent to S_∞ is equal to π. Now let us suppose that g is a C_π-metric.

Then, by Proposition 2.3 (2), S_q must be a C_π-manifold with the induced metric $dr^2 + v(r)^2 d\theta^2$ for each $q \in S_\infty$. C_π-metric of this type on a 2-dimensional sphere (Zoll surface) was studied in [3,10]. The following explanation of Zoll metric

(3.1) $$ds^2 = dr^2 + v(r)^2 d\theta^2$$

on a surface of revolution M (differomorphic to S^2) is owing to [3], where r denotes the distance from the north pole and v(r) the distance from the axis. For each geodesic γ on M, we have Clairaut's relation (cf. Lemma 2.5) that

(3.2) $$c = v^2 \frac{d\theta}{ds}$$

is constant on γ. Taking (2.6) into account, we see that $|c| \le v(r)$ and $v(r_0) = |c|$ if and only if $\frac{dr}{ds} = 0$ at s_0, where $r_0 = r(s_0)$. Let $v'(r_1) = 0$. Then the latitude $r = r_1$ is a geodesic. Using the second variation formula for an one parameter family of geodesics passing through a point on this latitude, we can show $v''(r_1) < 0$. Therefore v is monotone increasing on $[0, r_1)$ and decreasing on $(r_1, \pi/2]$. It follows that γ lies between the latitudes $r = r_0$ and $r = r_0^*$, where $v(r_0) = v(r_0^*) = |c|$ and $0 \le r_0 \le r_0^* \le \pi/2$. In particular, $r_0 = r_0^* = r_1$ if and only if γ is the equator (latitude $r = r_1$). Also we note that $c = 0$ if and only if γ is a meridian. Let $c \ne 0$. If γ is not the equator, then γ is tangent to latitudes $r = r_0$ and $r = r_0^*$ at one point respectively. The rotation angle between these two tangent points is equal to π. Therefore we have, from (2.6) and (3.2),

(3.3) $$\int_{r_0}^{r_0^*} \frac{|c|}{v\sqrt{v^2 - c^2}}\, dr = \pi.$$

Next we change the parameter r for x defined by

(3.4) $v(r) = \frac{1}{2} \sin x$

where $x \in [0,\pi]$ and we note that the maximum $v(r_1)$ of v is equal to $\frac{1}{2}$ since the length of the equator is equal to π. If we put

(3.5) $|c| = \frac{1}{2} \sin x_0$,

then (3.3) becomes

(3.6) $$\int_{x_0}^{\pi - x_0} \frac{\sin x_0 \, f_1(\cos x)}{\sin x \sqrt{\sin^2 x - \sin^2 x_0}} \, dx = \pi$$

for some positive function f_1 on $[-1,1]$ such that $f_1(-1) = f_1(1) = 1$. It is shown in p.p. 103 ~ 104 in [3] that (3.6) holds for every $x_0 \in (0,\pi/2]$ if and only if a function $\eta_1 : [-1,1] \longrightarrow (-1,1)$ defined by $\eta_1(t) = f_1(t) - 1$ is an odd function which satisfies $\eta_1(-1) = \eta_1(1) = 0$. Conversely, if (3.6) holds for every $x_0 \in (0,\pi/2]$, then (3.1) is a C_π-metric on S^2. In this way, v must be a function which gives a Zoll metric on S_q for each $q \in S_\infty$. We note that geodesics lie on S_q for some $q \in S_\infty$ if and only if $d = 0$.

Next we deal with geodesics satisfying $c = 0$, $d \neq 0$. In the following Lemmas 3.1(2), 3.2, 3.5 and Proposition 3.3, we shall assume that g is a C_π-metric. Under this assumption, we have $h(r) = \cos r$ (Lemma 3.5). Since (r,φ)-surface is diffeomorphic to RP^2 and totally geodesic as shown in the proof of Lemma 3.1(2), the induced metric $dr^2 + h(r)^2 d\varphi^2$ on this surface gives a C_π-metric on RP^2. Thus by L.W. Green's theorem (cf. [3]), it must be the canonical metric and hence we can conclude that $h(r) = \cos r$. However, to be self-contained and to complete this paper, we shall repeat this proof by the method written in [3].

Lemma 3.1. (1) Let $S(s_0)$ be the geodesic sphere with center p_0 and radius $s_0 \in (0,\pi/2)$. The submanifold $S(s_0)$ is totally geodesic if and only if $v'(r_0) = h'(r_0) = 0$, where $r_0 = \pi/2 - s_0$.

(2) If the metric g is a C_π-metric, then h is monotone decreasing, i.e., $h'(r) < 0$ on $(0,\pi/2]$.

Proof. (1) : Let $U^* = \{\exp_{p_0} sX : s \in (0,\pi/2),\ X \in U\}$. Since $\{\theta,\varphi,\psi\}$ is a local coordinates defined on $U^* \cap S(s_0)$ and $\partial/\partial r$ is the unit normal vector of $S(s_0)$, $S(s_0)$ is totally geodesic if and only if $\left\{ {}_{j}{}^{1}{}_{k} \right\} = 0$ for $j,k = 2,3,4$. The assertion follows from Lemma 2.2.

(2) : Firstly we show that (r,φ)-surface (θ,ψ : constant) is totally geodesic. Vectors $\partial/\partial\theta$ and $\partial/\partial\psi$ are normal to the surface. The tangential components of $\nabla_{\partial/\partial r}\partial/\partial\theta$, $\nabla_{\partial/\partial r}\partial/\partial\psi$, $\nabla_{\partial/\partial\varphi}\partial/\partial\theta$ and $\nabla_{\partial/\partial\varphi}\partial/\partial\psi$ vanish because of Lemma 2.2. Using Weingarten equation for the surface in $(\mathbb{C}P^2,g)$, we see that (r,φ)-surface is totally geodesic. Assume that $h'(r_0) = 0$ for some $r_0 \in (0,\pi/2)$. Then φ-curve ($r = r_0$, θ, ψ : constant) is a geodesic since equations (2.2) ~ (2.5) reduce to $d^2\varphi/ds^2 = 0$ and (2.6) implies that $|d\varphi/ds| = 1/h(r_0)$. Let p be a fixed point in U^* such that $r(p) = r_0$. Consider the (r,φ)-surface M containing p. Let γ be the geodesic through p which coincides with φ-curve on M and J be the Jacobi field along γ induced from variation consisting of geodesics which pass through p and lie on M in the domain U^*. Then J is proportional to X_1

$(= \partial/\partial r)$. Since, using Lemma 2.2,

$$R(X_1, X_3)X_3 = -\frac{h''(r_0)}{h(r_0)},$$

where $X_3 = (\partial/\partial\varphi)/h$ and R denotes the curvature tensor, the Jacobi equation becomes

$$\nabla_{\dot\gamma}^{\ 2} J = \frac{h''(r_0)}{h(r_0)}\, J.$$

Since $J(0) = J(\pi) = 0$, we obtain

$$0 \geq -\int_0^\pi \|\nabla_{\dot\gamma} J\|^2\, ds = \frac{h''(r_0)}{h(r_0)} \int_0^\pi \|J\|^2\, ds.$$

We have proved $h''(r_0) < 0$. The length of γ is equal to $\pi h(r_0)$. Hence $h(r_0) = 1$. Taking account of $h(0) = 1$, we have a contradiction. Q.E.D.

Lemma 3.2. Assume that g is a C_π-metric. Let γ be a geodesic such that $\gamma(0) \in \mathbb{C}P^2 \setminus (\{p_0\} \cup S_\infty)$, $c = 0$ and $d \neq 0$. Then γ transversally intersects S_∞. Moreover, there exists $r_0 \in (0, \pi/2)$ such that $r \leq r_0$ along γ and $\dot\gamma(s_0)$ is contained in the horizontal space, where $r(s_0) = r_0$.

Proof. There uniquely exists $r_0 \in (0,\pi/2)$ such that $h(r_0) = d$ for $0 < d \leq 1$. Since $h(r) \geq d$ along γ, $r(s) \leq r_0$ along γ. Assume that γ is tangent to a geodesic sphere $S(\pi/2 - r_1)$. Then $dr/ds = 0$ at the tangent point. Since $c = 0$, we have $\cos\beta = 1$ at the tangent point and hence $h(r_1) = d$. Therefore $r_1 = r_0$, because h is monotone decreasing. If γ (or a segment of γ) is contained in $S(\pi/2 - r_0)$, then (2.2) reduces to

$$hh'\{(\frac{d\varphi}{ds})^2 + \sin^2 2\varphi\ (\frac{d\psi}{ds})^2\} = 0$$

on $S(\pi/2 - r_0)$, where we have used the assumption $c = 0$ and

Lemma 2.5, and hence $d = 0$ which is a contradiction. From these facts, we can easily prove the assertion. Q.E.D.

__Remark__. We see from the above proof that each connected component in $\mathbb{C}P^2 \setminus S_\infty$ of a geodesic γ with $c = 0$ and $d \neq 0$ is tangent to a geodesic sphere $S(\pi/2 - r_0)$ at only one point.

__Proposition__ 3.3. __Assume that__ g __is a__ C_π__-metric__. __Then__ h __satisfies__

$$(3.7) \qquad 2\int_0^{r_0} \frac{h}{\sqrt{h^2 - d^2}}\, dr = \pi,$$

$$(3.8) \qquad 2\int_0^{r_0} \frac{d}{h\sqrt{h^2 - d^2}}\, dr = \pi,$$

__for any__ $0 < d < 1$, __where__ $h(r_0) = d$.

Proof. We first show that the projection to S_∞ of a geodesic γ with $c = 0$ and $d \neq 0$ by the map $P : \mathbb{C}P^2 \setminus \{p_0\} \longrightarrow S_\infty$ defined by $P(\exp_{p_0} sx) = \exp_{p_0} \frac{\pi}{2} X$ is a great circle. For γ with $c = 0$ and $d \neq 0$, equations of geodesics (2.2) ~ (2.5) reduce to

$$(3.9) \qquad \frac{d^2r}{ds^2} - \frac{d^2h'}{h^3} = 0,$$

$$(3.10) \qquad \frac{d\theta}{ds} + \cos 2\varphi \frac{d\psi}{ds} = 0,$$

$$(3.11) \qquad \frac{d^2\varphi}{ds^2} + 2\frac{h'}{h}\frac{dr}{ds}\frac{d\varphi}{ds} - \sin 4\varphi \left(\frac{d\psi}{ds}\right)^2 = 0,$$

$$(3.12) \qquad \frac{d^2\psi}{ds^2} + 2\frac{h'}{h}\frac{dr}{ds}\frac{d\psi}{ds} + 4\cot 2\varphi \frac{d\varphi}{ds}\frac{d\psi}{ds} = 0.$$

Moreover, (2.6) reduces to

$$(3.13) \qquad \left(\frac{dr}{ds}\right)^2 + \frac{d^2}{h^2} = 1$$

which implies (3.9). We change the parameter s for t defined by

$$t = \int^{s} \frac{d}{h^2(r(s))} ds$$

on each connected component of $\gamma([0,\pi]) \setminus S_\infty$ ($\gamma(0) \in S_\infty$). Then we get, from (3.11) and (3.12),

$$(3.14) \qquad \frac{d^2\varphi}{dt^2} - \sin 4\varphi \left(\frac{d\psi}{dt}\right)^2 = 0,$$

$$(3.15) \qquad \frac{d^2\psi}{dt^2} + 4 \cot 2\varphi \frac{d\varphi}{dt}\frac{d\psi}{dt} = 0.$$

The definition of the parameter t implies

$$(3.16) \qquad \left(\frac{d\varphi}{dt}\right)^2 + \sin^2 2\varphi \left(\frac{d\psi}{dt}\right)^2 = 1.$$

It follows that the image of each connected component of $\gamma([0,\pi]) \setminus S_\infty$ by the map P is a segment of a great circle on S_∞ ($= S^2(1/2)$) with the induced metric $ds^2 = d\varphi^2 + \sin^2 2\varphi\, d\psi^2$ and t is the arclength parameter of the great circle. Since γ is a closed curve with length π, we have

$$2 \int_0^{r_0} \frac{h}{\sqrt{h^2 - d^2}} dr = \frac{\pi}{m}$$

for some positive integer m, where we have used (3.13). Since γ approaches to a geodesic passing through p_0 when $d \longrightarrow 0$, we must have $m = 1$ because of the continuity. Hence we see that $\gamma([0,\pi]) \cap S_\infty$ consists of one point and $P(\gamma)$ is a whole great circle. Therefore

$$2 \int_0^{r_0} \frac{d}{h\sqrt{h^2 - d^2}} dr = \pi,$$

which is the length of $P(\gamma)$. Q.E.D.

<u>Theorem</u> 3.4. <u>Let</u> g <u>be a metric of the form</u> (1.5) <u>on</u> $\mathbb{C}P^2$, <u>where</u> h <u>and</u> v <u>satisfy</u> (1.6). <u>If</u> h <u>is a monotone decreasing</u>

function on $[0,\pi/2]$ and satisfies

$$(3.17)\qquad 2\int_0^{r_0} \frac{h}{\sqrt{h^2 - d^2}}\, dr = L < \infty$$

and (3.8) for an arbitrarily given d such that $0 < d < 1$, where $h(r_0) = d$, then every geodesic γ whose Clairaut's first integrals are equal to 0 (= c) and d is a closed curve with length L.

Proof. Let r_0 satisfy $h(r_0) = d$. Then $r \le r_0$ along γ. Let $\gamma(0) \in \mathbb{C}P^2 \setminus (S_\infty \cup \{p\})$. By argument similar to the proof of Lemma 3.2, we see that γ is not contained in any geodesic sphere centered at p_0. We may assume that $dr/ds > 0$ at $s = 0$. If $dr/ds > 0$ on $(0,\infty)$, then there exists r_1 ($\le r_0$) such that $r(s) \longrightarrow r_1$ and $dr/ds \longrightarrow 0$ as $s \longrightarrow \infty$. From (3.13), we have $d = h(r_1)$, so that $r_1 = r_0$. Therefore we have

$$\int_0^{r_0} \frac{h}{\sqrt{h^2 - d^2}}\, dr = \infty$$

which contradicts to (3.17). Hence $dr/ds = 0$ at some s_0 where $r(s_0) = r_0$. Equation (3.9) shows $d^2r/ds^2 < 0$ on each connected component of $\gamma((-\infty,\infty)) \setminus S_\infty$. It follows that γ is tangent to $S(\pi/2 - r_0)$ only one time and transversally intersects S_∞ at $s = s_0 + L/2$ and $s = L/2 - s_0$. We put $q = \gamma(L/2 - s_0)$ and $q' = \gamma(s_0 + L/2)$.

Next we prove $q = q'$. Reparametrize γ by arclength parameter in such a way that $\gamma(-L/2) = q$, $\gamma(L/2) = q'$ and $r(0) = r_0$. Equations (3.14) ~ (3.16) hold for γ with $c = 0$ and $d \ne 0$. Thus the curve $P(\gamma)$ projected to S_∞ is a whole great circle because of (3.8). Since $ds/dt > 0$ along γ, the projection P is one to one. It follows that $q = q'$.

We prove $\dot{\gamma}(L/2 - 0) = \dot{\gamma}(- L/2 + 0)$ to show that γ is a closed curve. The vector $d\varphi/ds\, \partial/\partial\varphi + d\psi/ds\, \partial/\partial\psi$ is the tangent vector of the great circle $P(\gamma)$. So we have $d\varphi/ds(L/2 - 0) = d\varphi/ds(- L/2 + 0)$ and $d\psi/ds(L/2 - 0) = d\psi/ds(- L/2 + 0)$. Put $u^1 = r \cos\theta$ and $u^2 = r \sin\theta$ on a neighborhood of q (cf. Proof of Prop. 2.3). Then

$$\frac{du^1}{ds} = \frac{dr}{ds}\cos\theta - r\frac{d\theta}{ds}\sin\theta, \qquad \frac{du^2}{ds} = \frac{dr}{ds}\sin\theta + r\frac{d\theta}{ds}\cos\theta.$$

Since $d\theta/ds$ is bounded on $(-L/2, L/2)$ because of (3.10), we have

$$\frac{du^1}{ds}(\mp L/2 \pm 0) = \frac{dr}{ds}(\mp L/2 \pm 0)\cos\theta\,(\mp L/2 \pm 0),$$

(3.18)

$$\frac{du^2}{ds}(\mp L/2 \pm 0) = \frac{dr}{ds}(\mp L/2 \pm 0)\sin\theta\,(\mp L/2 \pm 0).$$

By a suitable coordinate change on S^2, we may assume that the great circle $P(\gamma)$ is defined by $2\varphi = \pi/2$, $\psi = t + \pi/2$ $(-\pi/2 \le t < \pi/2)$. Then θ is constant on $(-L/2, L/2)$ because of (3.10). In the normal plane at q of S_∞, θ is the angle measured from the tangent vector $\dot{\sigma}_X(\pi/2)$ of the geodesic $\sigma_X : s \longrightarrow \exp_{p_0} sX$, where $X = (\cos\varphi e^{i\psi}, \sin\varphi e^{-i\psi}) \in S^3(1) \subset T_{p_0}\mathbb{C}P^2$. Since $\dot{\sigma}_{-X}(\pi/2) = -\dot{\sigma}_X(\pi/2)$ and $dr/ds(\mp L/2 \pm 0) = \pm\sqrt{1 - d^2}$, we see from (3.18) that $\dot{\gamma}(L/2 - 0) = \dot{\gamma}(- L/2 + 0)$. Q.E.D.

Let us consider the condition (3.8). If g is a C_π-metric, then (3.8) holds for every d such that $0 < d < 1$.

Lemma 3.5. *Let* g *be a metric of the form* (1.5) *on* $\mathbb{C}P^2$. *If it is a* C_π-*metric, then* $h(r) = \cos r$.

Proof. The idea is similar to 4.14 in p. 103 [3]. We change the parameter r for y defined by y = arccos h(r) ($0 \le y \le \pi/2$). We easily see that $h''(0) \le 0$, and so there exists $y'(0 + 0)$, which is equal to $\sqrt{-h''(0)}$. Define a monotone increasing function $\nu : [0,1] \longrightarrow [0,\pi/2]$ by $\nu(t) = h^{-1}\left(\sqrt{1 - t^2}\right)$ for $0 \le t \le 1$. Hence we have $\nu(\sin y) = r$ for $y \in [0,\pi/2]$ and

$$(3.19) \qquad dr = \frac{-\sin y}{h'(\nu(\sin y))}\, dy.$$

We put

$$f_2(t) = -\frac{t}{h'(\nu(t))}$$

for $0 \le t \le 1$. Then f_2 is a positive function on [0,1]. It is smooth on (0,1), satisfies $f_2(0 + 0) = 1 / \sqrt{-h''(0)}$ and $f_2(1) = 1$ which is derived from (1.6). Using (3.19), we can rewrite (3.8) as

$$(3.20) \qquad 2d \int_0^{y_0} \frac{f_2(\sin y)}{\cos y \sqrt{\cos^2 y - \cos^2 y_0}}\, dy = \pi,$$

where $0 < y_0 < \pi/2$ and $d = \cos y_0$. Since

$$2d \int_0^{y_0} \frac{dy}{\cos y \sqrt{\cos^2 y - \cos^2 y_0}} = \pi,$$

(3.20) is equivalent to

$$(3.21) \qquad \int_0^{y_0} \frac{\eta_2(\sin y)}{\cos y \sqrt{\cos^2 y - \cos^2 y_0}} = 0$$

for the function η_2 defined by $\eta_2(t) = f_2(t) - 1$. Here we note that η_2 should satisfy

$$\eta_2(0 + 0) = \frac{1}{\sqrt{-h''(0)}} - 1, \qquad \eta_2(1) = 0.$$

We show that η_2 satisfies (3.21) for every $y_0 \in (0,\pi/2)$ if and only if $\eta_2 \equiv 0$. We put, for $a \in [0,\pi/2]$,

$$I(a) = \int_0^a \frac{\sin z \cos z\, H(z)}{\sqrt{\cos^2 z - \cos^2 a}}\, dz,$$

where

$$H(z) = \int_0^z \frac{\eta_2(\sin y)}{\cos y \sqrt{\cos^2 y - \cos^2 z}}\, dy.$$

Let T_a denote the domain $\{(y,z) \in \mathbf{R}^2 : z \geq y,\ 0 \leq y \leq a\}$. Then we have

$$(3.22) \quad I(a)$$
$$= \iint_{T_a} \frac{\sin z \cos z\, \eta_2(\sin y)}{\cos y \sqrt{\cos^2 z - \cos^2 a} \sqrt{\cos^2 y - \cos^2 z}}\, dydz$$

$$= \int_0^a \frac{\eta_2(\sin y)}{\cos y} \left(\int_y^a \frac{\sin z \cos z}{\sqrt{\cos^2 z - \cos^2 a} \sqrt{\cos^2 y - \cos^2 z}}\, dz\right) dy$$

because of Fubini's theorem. Substituting

$$x = \sqrt{\cos^2 z - \cos^2 a} \,/\, \sqrt{\cos^2 y - \cos^2 z}$$

into (3.22), we have

$$I(a) = \int_0^a \frac{\eta_2(\sin y)}{\cos y}\, dy \int_0^\infty \frac{1}{1 + x^2}\, dx = \frac{\pi}{2} \int_0^a \frac{\eta_2(\sin y)}{\cos y}\, dy.$$

If (3.21) holds for every $y_0 \in (0,\pi/2)$, then $I(a) = 0$ for every $a \in (0,\pi/2)$ and hence $\eta_2 \equiv 0$. We easily see that $\eta_2 \equiv 0$ implies $h(r) = \cos r$. Q.E.D.

If we examine the above proof, we have

<u>Theorem</u> 3.6. <u>Let</u> h <u>be a monotone decreasing function on</u> $[0,\pi/2]$ <u>satisfying</u> (1.6) <u>and</u> $h''(0) = -1$. <u>For an arbitrarily fixed</u> $\tilde{r}$ $(0 < \tilde{r} < \pi/2)$, <u>there exists a lot of noncanonical</u>

metrics of the form (1.5) on $\mathbb{C}P^2$ such that all geodesics which pass through S_∞ and whose Clairaut's first integral d satisfies $d \geq h(\tilde{r})$ are closed.

Proof. We have only to choose η_2 as any smooth function satisfying

$$\eta_2 \equiv 0 \quad \text{on } [0, \sqrt{1 - h(\tilde{r})^2}],$$

$$\eta_2(1) = 0.$$

Then we have h by solving

$$h' = -\sqrt{1 - h^2} \,/\, f_2(\sqrt{1 - h^2}).$$

Since

$$\lim_{r \to r_0} \frac{\sqrt{r_0 - r}}{\sqrt{h^2 - d^2}} = \frac{1}{\sqrt{-2dh'(r_0)}} < \infty,$$

the integral (3.17) is finite. Q.E.D.

Remark. In the above theorem, the restriction for v is only conditions given in (1.6).

§4. General geodesics.

We shall consider geodesics with $c \neq 0$ and $d \neq 0$. At first, we prove

Lemma 4.1. Let γ be a geodesic with $c \neq 0$ and $d \neq 0$. Let P be the projection $\mathbb{C}P^2 \setminus \{p_0\} \longrightarrow S_\infty$ introduced in the proof of Prop. 3.3. Then $P(\gamma)$

is a small circle with perimeter $d\pi / \sqrt{c^2 + d^2}$.

Proof. Change the parameter s for t defined by

$$t = \int^s \frac{d}{h(r(s))^2}\, ds.$$

Then the equations (2.4) and (2.5) of the geodesic γ reduce to

$$(4.1) \qquad \frac{d^2\varphi}{dt^2} - \sin 4\varphi \left(\frac{d\psi}{dt}\right)^2 + 2\,\frac{c}{d}\sin 2\varphi\,\frac{d\psi}{dt} = 0,$$

$$(4.2) \qquad \frac{d^2\psi}{dt^2} + 4\cot 2\varphi\,\frac{d\varphi}{dt}\,\frac{d\psi}{dt} - 2\,\frac{c}{d}\,\frac{1}{\sin 2\varphi}\,\frac{d\varphi}{dt} = 0.$$

From the definition of d given in Lemma 2.5, we have (3.16). Namely, t is the arclength of $P(\gamma)$ since the induced metric on S_∞ is given by $d\varphi^2 + \sin^2 2\varphi\, d\psi^2$. If we denote $P(\gamma)$ by σ and the Riemannian connection by $\tilde{\nabla}$ on S_∞, then (4.1) and (4.2) are equivalent to

$$(4.3) \qquad \tilde{\nabla}_{\dot{\sigma}}\dot{\sigma} = 2\,\frac{c}{d}\,N,$$

where $N = -\sin 2\varphi\,\frac{d\psi}{dt}\,\frac{\partial}{\partial\varphi} + \frac{1}{\sin 2\varphi}\,\frac{d\varphi}{dt}\,\frac{\partial}{\partial\psi}$. It is easy to see that N is the unit normal vector of σ. Since (4.3) is the equation of small circles with perimeter $d\pi / \sqrt{c^2 + d^2}$, we have the assertion. Q.E.D.

Let v and h be smooth functions on $[0,\pi/2]$ such that they satisfy (1.6) and

$$(4.4) \qquad \begin{gathered} v' > 0 \text{ on } [0,r_1), \quad v'(r_1) = 0, \quad v' < 0 \text{ on } (r_1,\pi/2], \\ h' < 0 \text{ on } (0,\pi/2], \quad v(r_1) = \tfrac{1}{2}. \end{gathered}$$

Define $F : (0,\pi/2) \longrightarrow \mathbb{R}$ by

$$(4.5) \qquad F = 1 - \frac{c^2}{v^2} - \frac{d^2}{h^2}$$

for each $0 < c \leq 1/2$ and $0 \leq d < 1$. Moreover we define functions a,b and ω defined on $(0,r_1]$ by

$$\omega = \sqrt{- \frac{v'}{h'} \frac{h^3}{v^3}},$$

(4.6)

$$a = 1 \Big/ \sqrt{\frac{1}{v^2} + \frac{\omega^2}{h^2}}, \qquad b = \omega \Big/ \sqrt{\frac{1}{v^2} + \frac{\omega^2}{h^2}}.$$

Then we have

(4.7) $$1 - \frac{a^2}{v^2} - \frac{b^2}{h^2} = 0.$$

Lemma 4.2. Let g be a metric of the form (1.5). Let h and v satisfy (1.6) and (4.4). If a geodesic γ with the Clairaut's first integrals c and d lies on a geodesic sphere $S(\pi/2 - u)$ $(0 < u < \pi/2)$, then we have

(1) $F(u) = 0$, $F'(u) = 0$,

(2) $c \neq 0$,

(3) if $d \neq 0$, then $u < r_1$,

(4) $d = 0$ if and only if $u = r_1$ (and hence γ is the equator of S_q for some $q \in S_\infty$).

Proof. Since $r = u$ (constant) along γ, (2.2) reduces to

$$\begin{aligned} 0 &= v(u)v'(u)\left(\frac{d\theta}{ds} + \cos 2\varphi \frac{d\psi}{ds}\right)^2 \\ &\quad + h(u)h'(u)\left\{\left(\frac{d\varphi}{ds}\right)^2 + \sin^2 2\varphi \left(\frac{d\psi}{ds}\right)^2\right\} \\ &= \frac{c^2}{v(u)^3} v'(u) + \frac{d^2}{h(u)^3} h'(u) \\ &= \frac{1}{2} F'(u). \end{aligned}$$

If $c = 0$, then $d = 0$ and hence γ is a geodesic issuing from p_0. This is a contradiction. If $d \neq 0$, then $v'(u) > 0$ i.e., $u < r_1$.

The constant d is equal to zero if and only if $v'(u) = 0$.

Q.E.D.

Let γ be a geodesic with $c \neq 0$, $d \neq 0$. Since $P(\gamma)$ is a small circle with perimeter $d\pi / \sqrt{c^2 + d^2}$ (Lemma 4.1), we may assume that φ is a constant along γ such that $\cos 2\varphi = c / \sqrt{c^2 + d^2} > 0$. Then we get, from (4.1),

$$(4.8) \qquad \frac{d\psi}{dt} = \frac{c}{d} \frac{1}{\cos 2\varphi} = \frac{1}{d} \sqrt{c^2 + d^2}.$$

Since

$$\frac{c}{v^2} = \frac{d\theta}{ds} + \cos 2\varphi \frac{d\psi}{ds}$$
$$= \frac{d\theta}{ds} + \frac{c}{h^2},$$

we have

$$(4.9) \qquad \frac{d\theta}{ds} = c\left(\frac{1}{v^2} - \frac{1}{h^2}\right).$$

We give a necessary and sufficient condition for all geodesics which lie on geodesic spheres centered at p_0 to be closed and have the same length.

<u>Theorem</u> 4.3. <u>Under the same assumption as in Lemma</u> 4.2, <u>every geodesic</u> γ <u>on geodesic spheres</u> $S(\pi/2 - u)$ $(0 < u \leq r_1)$ <u>is closed and of the same length</u> π <u>if and only if</u> h <u>and</u> v <u>satisfy</u>

$$(4.10) \qquad h^4 - h^2 + v^2 \equiv 0$$

<u>on</u> $[0, r_1]$.

Proof. The necessity is proved as following. From (4.8), we have

$$\frac{d\psi}{ds} = \frac{1}{h^2} \sqrt{c^2 + d^2}.$$

If γ is a closed geodesic on $S(\pi/2 - u)$ $(0 < u < r_1)$ and of length π, then we see that

$$(4.11) \qquad \frac{1}{h^2}\sqrt{c^2 + d^2} = m, \qquad c(\frac{1}{v^2} - \frac{1}{h^2}) = n$$

for some integers m and n. By virtue of Lemma 4.2 (1) and (4.6), we have $c = a(u)$ and $d = b(u)$. It follows that the geodesic γ_u, for any $0 < u < r_1$,

$$r = u, \qquad \theta = a(u)(\frac{1}{v(u)^2} - \frac{1}{h(u)^2})s,$$

$$\varphi = \frac{1}{2}\arccos\frac{a(u)}{\sqrt{a(u)^2 + b(u)^2}}, \qquad \psi = \frac{1}{h(u)^2}\sqrt{a(u)^2 + b(u)^2}\, s$$

forms a 1-parameter family of geodesics. By continuity, we know that m and n in (4.11) for γ_u do not depend on u. Since if u tends to zero, then γ_u converges to the equator $(2\varphi = \pi/2, \psi = s)$ on S_∞ and if u tends to r_1, then γ_u converges to the equator of S_q for some q (Lemma 4.2 (4)), we conclude that $m = |n| = 1$, where we note that $a(r_1) = \frac{1}{2}$ and $b(r_1) = 0$. We have shown

$$(4.12) \qquad a(\frac{1}{v^2} - \frac{1}{h^2}) = 1, \qquad \sqrt{a^2 + b^2} = h^2$$

on $(0, r_1]$ (we may assume that $n = 1$). The second equation is equivalent to

$$(4.13) \qquad v'(1 - h^2) - h'(\frac{v^3}{h^3} - vh) = 0.$$

On the other hand, the first equation of (4.12) is equivalent to

$$(4.14) \qquad v'\frac{h^3}{v} + h'(\frac{h^2}{v^2} + \frac{v^2}{h^2} - h^2 - 2) = 0.$$

Since $h' \neq 0$ on $(0, r_1]$, we get, from (4.13) and (4.14),

$$(1 - h^2)(\frac{h^2}{v^2} + \frac{v^2}{h^2} - h^2 - 2) + \frac{h^3}{v}(\frac{v^3}{h^3} - vh) = 0,$$

from which

$$h^4 - h^2 + v^2 = 0$$

on $(0,r_1]$.

Differentiating (4.10), we have

$$2h^3h' - hh' + vv' = 0,$$

which implies (4.13) and (4.14). Therefore (4.12) holds and we have shown the sufficiency. Q.E.D.

Therefore, if we choose v and h in such a way that $dr^2 + v^2d\theta^2$ is a Zoll metric and h is a monotone decreasing function which satisfies (4.10) on $(0,r_1]$ and (1.6), then we have

Corollary 4.4. There is a lot of noncanonical metrics of the form (1.5) on $\mathbb{C}P^2$ such that all geodesics on each S_q ($q \in S_\infty$), S_∞ and geodesic spheres centered at p_0 are closed and of the same length π.

We also have

Corollary 4.5. Let g be a metric of the form (1.5), where h and v satisfy (1.6) and (4.4). If, in addition to the above geodesics, all geodesics which pass through points in S_∞ are closed and of length π, then g coincides with g_{can}. In particular, if g is a C_π-metric, then $g = g_{can}$.

Proof. By virtue of Proposition 3.3 and Lemma 3.5 (where we have assumed that g is a C_π-metric, but the assertions hold under the present assumption), $h(r) = \cos r$ on $[0,\pi/2]$. It follows from (4.10) that $v(r) = \sin r \cos r$ on $[0, r_1]$ where r_1 should be $\pi/4$. Therefore $\eta_1 = 0$ on $[0,1]$ (for the detail, see [3]). Since η_1 is an odd function, $\eta_1 = 0$ on $[-1,1]$, i.e., $v(r) = \sin r \cos r$ on $[0, \pi/2]$. Q.E.D.

Finally, we study geodesics which satisfy $c \neq 0$, $d \neq 0$ and are not contained in any geodesic sphere centered at p_0.

<u>Lemma</u> 4.6. <u>Under the same assumption as in Lemma</u> 4.2, <u>we see that if</u> h <u>satisfies</u> (4.10), <u>then there exists a unique zero of</u> F' <u>and</u> $F'' < 0$ <u>at the zero</u>.

Proof. Since

$$\lim_{r \to 0+0} F(r) = \lim_{r \to \pi/2-0} F(r) = -\infty,$$

there is a point where F attains the maximum. Let $\tilde{r}$ be arbitrarily fixed zero of F'. Since

$$F'(\tilde{r}) = 2\left(\frac{c^2}{v(\tilde{r})^3} v'(\tilde{r}) + \frac{d^2}{h(\tilde{r})^3} h'(\tilde{r})\right) = 0,$$

we know that $\tilde{r} < r_1$. It suffices to prove that $F''(\tilde{r}) < 0$. By a straightforward computation,

$$(4.15) \qquad \frac{1}{c^2} F''(\tilde{r}) + 4 \frac{h'(\tilde{r})}{h(\tilde{r})^3} \omega(\tilde{r})\omega'(\tilde{r}) = 0.$$

Let us assume that $F''(\tilde{r}) = 0$. Then $\omega'(\tilde{r}) = 0$. Since

$$\omega' = \frac{1}{a^2}(ab' - a'b),$$

we have $ab' = a'b$ at $r = \tilde{r}$. Moreover (4.10) implies (4.12), from which we get

$$aa' + bb' - 2(a^2 + b^2)\frac{h'}{h} = 0.$$

Substituting $ab' = a'b$ into the above equation, we have a'

$= 2ah'/h$ at $r = \tilde{r}$. From the first equation of (4.12), we have

$$(4.16) \qquad a'\left(\frac{1}{v^2} - \frac{1}{h^2}\right) = 2a\left(\frac{v'}{v^3} - \frac{h'}{h^3}\right)$$

and hence $h'/h = v'/v$ at $r = \tilde{r}$. However $h' < 0$ and $v' > 0$ on $(0,r_1)$. We have shown that $F''(\tilde{r}) \neq 0$.

Next, assume that $F''(\tilde{r}) > 0$. Then (4.15) implies $\omega'(\tilde{r}) > 0$. Differentiating the defining equation of a, we see that $a'(\tilde{r}) < 0$. Therefore $v(\tilde{r}) > h(\tilde{r})$ since the right hand side of (4.16) is positive on $(0,r_1)$. Taking account of (4.4), we know that $v > h$ on $[\tilde{r},r_1]$. By the assumption $F'(\tilde{r}) = 0$ and $F''(\tilde{r}) > 0$, $F'(r) > 0$ if $r \in (\tilde{r},\tilde{r}+\varepsilon)$, where ε (> 0) is sufficiently small. It follows from Rolle's theorem that there is another zero $r^* \in (\tilde{r},r_1)$ of F' such that $F''(r^*) \leq 0$. However, if we apply the argument of the preceding paragraph to r^*, then we have $F''(r^*) < 0$ and hence $\omega'(r^*) < 0$ because of (4.15). So $a'(r^*) > 0$. This inequality and (4.16) imply that $h > v$ at $r^* \in (\tilde{r},r_1)$. We have a contradiction. Q.E.D.

Let γ be a geodesic whose Clairaut's first integrals do not vanish. Furthermore we assume that γ is not contained in any geodesic sphere centered at p_0. Since $F(r(s)) = (dr/ds)^2 \geq 0$ along γ, we know that $w_1 \leq r \leq w_2$ along γ, where w_1 and w_2 are the zeroes of F determined by the Clairaut's first integrals c, d of γ (by virtue of Lemma 4.6, F has only two zeroes).

<u>Theorem</u> 4.7. <u>Let</u> g <u>be a metric of the form</u> (1.5), <u>where</u> h <u>and</u> v <u>satisfy</u> (1.6), (4.4) <u>and</u> (4.10) <u>on</u> $[0,r_1]$. <u>Let</u> γ <u>be a</u>

geodesic with $c \neq 0$, $d \neq 0$. Suppose that γ is not contained in any geodesic sphere centered at p_0. The geodesic γ is a closed curve with length L if h and v satisfy

(4.17) $$2 \int_{w_1}^{w_2} \frac{d \cdot dr}{h^2\sqrt{F}} = \frac{d\pi}{\sqrt{c^2 + d^2}},$$

(4.18) $$2 \int_{w_1}^{w_2} \frac{c\ dr}{v^2\sqrt{F}} = (1 + \frac{c}{\sqrt{c^2 + d^2}})\pi,$$

(4.19) $$2 \int_{w_1}^{w_2} \frac{dr}{\sqrt{F}} = L < \infty.$$

Proof. We derive from (2.2)

$$\frac{d^2r}{ds^2} = \frac{v'}{v^3} c^2 + \frac{h'}{h^3} d^2$$
$$= \frac{1}{2} F'(r(s)).$$

Note that $F'(w_1) > 0$ and $F'(w_2) < 0$. Therefore, by the similar argument to the first paragraph in the proof of Theorem 3.4, we see that γ is tangent to the geodesic spheres $S(\pi/2 - w_1)$ and $S(\pi/2 - w_2)$ alternately and the length between consecutive two points where γ is tangent to $S(\pi/2 - w_i)$ is equal to L because of (4.19). However, such consecutive two points q_1 and q_2 on $S(\pi/2 - w_1)$ must coincide. This can be proved as following. The projection P gives a one to one correspondence between the segment of γ from q_1 to q_2 and the small circle $P(\gamma)$. By (4.17), we know that q_1 and q_2 are contained in the same S_q for some $q \in S_\infty$. From (4.17) and (4.18), we have

$$2 \int_{w_1}^{w_2} (\frac{1}{v^2} - \frac{1}{h^2}) \frac{c}{\sqrt{F}}\ dr = \pi.$$

Therefore (4.9) implies that $q_1 = q_2$ since ψ and θ change from 0 to π (cf. (1.3)).

It is easy to prove that $\dot{\gamma}(q_1) = \dot{\gamma}(q_2)$. Q.E.D.

Remark. In the proof of Lemma 4.6, we have $\tilde{r} < r_1$ and hence $w_1 < r_1$. Equation (4.10) implies that $v(r_1) = h^2(r_1) = 1/2$. Since $F(r) \geq 0$ along γ, we have

$$(h^2)^2 - (1 + d^2)h^2 + c^2 + d^2 \leq 0$$

on $(0, r_1]$, where the equality holds if and only if $r = w_1$. Therefore

$$1 - d^2 \geq 2c \tag{4.20}$$

and if $r_1 \geq w_2$, then

$$\frac{1}{2} \leq \frac{1}{2}\{d^2 + 1 - \sqrt{(1 - d^2)^2 - 4c^2}\}$$

or

$$1 - d^2 \leq \frac{1}{2}(1 + 4c^2). \tag{4.21}$$

If $r_1 \geq w_2$, the conditions (4.17) ~ (4.19) can be rewrited in terms of only h. Equations (4.20), (4.21) give a restriction for the Clairaut's first integrals c, d of geodesics with $r_1 \geq w_2$.

References.

[1] Anderson, M.T. : Metrics of positive Ricci curvature with large diameter, Manuscripta Math. 68 (1990), 405-415.

[2] Besse, A. : Einstein manifolds, Springer-Verlag 1987.

[3] Besse, A. : Manifolds all of whose Geodesics are Closed, Springer-Verlag 1978.

[4] Cheeger, J. : Some examples of manifolds of nonnegative

curvature, J. Differential Geometry 8 (1973), 623-628.

[5] Guillemin, V. : The Radon transform on Zoll surfaces, Advances in Math. 22 (1976), 85-119.

[6] Kazdan, J.L. and F.W. Warner : Curvature functions for open 2-manifolds, Ann. of Math. 99 (1974), 203-219.

[7] Kiyohara, K. : Compact Liouville surfaces, J. Math. Soc. Japan 43 (1991), 555-591.

[8] Sha, J-P. and D-G. Yang : Examples of manifolds of positive Ricci curvature, J. Differential Geometry 29 (1989), 95-103.

[9] Sha, J-P. and D-G. Yang : Positive Ricci curvature on compact simply connected 4-manifolds, preprint.

[10] Zoll, O. : Uber Flächen mit Scharen geschlossener geodätischer Linien, Math. Ann. 57 (1903), 108-133.

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