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CLASSES OF OPERATORS DETERMINED BY THE HEINZ–KATO–FURUTA INEQUALITY AND THE HÖLDER–MCCARTHY INEQUALITY

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ABSTRACT. The class $H(p)$ of operators determined by the Heinz-Kato-Furuta inequality is characterized as the p-hyponormal operators introduced by Aluthge, in the preceding note [6]. From the viewpoint of this, we discuss relations among several classes of operators around p-hyponormal and paranormal operators, in which the Hölder-McCarthy inequality works as well as the Heinz-Kato-Furuta inequality. In addition, we consider some conditions that the Aluthge transform $T \to |T|^{1/2}U|T|^{1/2}$ preserves the norm, where $T = U|T|$ is the polar decomposition of T.

1. Introduction. First of all, we state the following extension of the Heinz-Kato inequality due to Furuta [9] :

The Heinz-Kato-Furuta inequality. *Let A and B be positive operators on a Hilbert space H. If T sastisfies*

$$T^*T \le A^2 \text{ and } TT^* \le B^2, \tag{1}$$

then the inequality

$$|(T|T|^{p+q-1}x, y)| \le \|A^px\|\|B^qy\| \tag{2}$$

holds for all $x, y \in H$ and $0 \le p, q \le 1$ with $p + q \ge 1$, where $|T|$ is the square root of T^*T.

An operator T is said to be *hyponormal* if $T^*T \ge TT^*$. For a given operator T, if we take $A = B = |T|$, then the assumption (1) is just the hyponormality of T. Based on this and the work of Watanabe [16], we introduced in [6] the class $H(p)$ of operators satisfying

$$|(U|T|^{2p}x, y)| \le \||T|^px\|\||T|^py\| \tag{3}$$

for $x, y \in H$, where $T = U|T|$ is the polar decomposition of T. And we showed that the class $H(p)$ determined by the Heinz-Kato-Furuta inequality is characterized by the p-hyponormal operators in the sense of Aluthge, i.e., $(TT^*)^p \le (T^*T)^p$ for $0 < p < 1$.

Now Ando [2] proved Berberian's conjecture that every hyponormal operator is normaloid, i.e., $\|T\| = r(T)$, the spectral radius of T. It induced an intermediate class between the hyponormal operators and the normaloid operators ; an operator T is called paranormal if

$$\|T^2x\|\|x\| \ge \|Tx\|^2$$

for all vectors x, see [3,7,11,13]. Related to p-hyponormal operators, Ando pointed out in [3; Theorem 2] that every p-hyponormal operator is paranormal, though Aluthge [1] showed that every p-hyponoraml operator is normaloid under an additional assumption.

On the other hand, McCarthy [15 ; Lemma 2.1] proposed the following inequalities as an operator variant of the Hölder inequality.

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The Hölder-McCarthy inequality. *Let A be a positive operator on a Hilbert space H. Then for all $x \in H$*

$$(4) \qquad (Ax,x)^r \le \|x\|^{2(r-1)}(A^r x,x) \text{ if } 1 \le r.$$

$$(5) \qquad (Ax,x)^r \ge \|x\|^{2(r-1)}(A^r x,x) \text{ if } 0 \le r \le 1.$$

Let us take $r = 2$ and $A = T^*T$. Then (4) implies

$$\|Tx\|^2 \le \|x\|\|T^*Tx\|.$$

Therefore, if T is hyponormal, then we have

$$\|Tx\|^2 \le \|x\|\|T^*Tx\| \le \|x\|\|T^2x\|,$$

that is, T is paranormal. Recalling that the class $H(p)$ is defined by the inequality which follows from the Heinz-Kato-Furuta inequality under the hyponormality, the paranormality of operators is, in this sense, determined by the Hölder-McCarthy inequality.

In this note, from viewpoint of this, we consider some relations among several classes of operators around the hyponormal and paranormal operators. In particular, we introduce the p-paranormality and generalize Ando's result that every p-hyponormal operator is paranormal. Moreover, we discuss the Aluthge transform $T \to \tilde{T} = |T|^{1/2}U|T|^{1/2}$, where $T = U|T|$ is the polar decomposition. As a matter of fact, we give some conditions equivalent to $\|T\| = \|\tilde{T}\|$. Consequently we have a simple proof of a weaker version of Ando's result.

2. The Hölder-McCarthy inequality. The Hölder-McCarthy inequality (4) implies that

$$(6) \qquad \|Tx\|^{2r} \le ((T^*T)^r x,x)\|x\|^{2(r-1)}$$

for arbitrary operator T and $r \ge 1$. On the other hand, an operator T is *k-paranormal* for a positive integer k if

$$(7) \qquad \|Tx\|^k \le \|T^k x\|\|x\|^{k-1}$$

for all $x \in H$, see [7,11]. To compare with (6) and (7) reminds us of perinormal operators introduced by Furuta and Haketa [10]. They called an operator T *perinormal* if

$$(8) \qquad (T^*T)^n \le T^{*n}T^n$$

for all positive integers n. For a fixed positive integer k, we here call an operator T *k-perinormal* if T satisfies (8) for $n = k$. As in the case of p-hyponormality for $0 < p \le 1$, an operator T is *k-hyponormal* if $(T^*T)^k \ge (TT^*)^k$ for a positive integer k, see [5].

Theorem 1. *Let T be an operator and k a positive integer. If T is k-perinormal, then T is k-paranormal, and if T is k-hyponormal, then T is m-perinormal for $m = 2, 3, \cdots, k+1$.*

Proof. The first half is a simple consequence of (6). The second one is proved by induction. For $k = 1$, since T is hyponormal, we have

$$T^{*2}T^2 - (T^*T)^2 = T^*(T^*T - TT^*)T \ge 0.$$

Next suppose that the statement is true for k and T is (k+1)-hyponormal. Then we have

$$T^{*k+1}T^{k+1} - (T^*T)^{k+1} = T^*(T^{*k}T^k - (TT^*)^k)T \ge T^*(T^{*k}T^k - (T^*T)^k)T \ge 0.$$

Next we turn our attention to the case $0 < r \le 1$ in the Hölder-McCarthy inequality. Thus we state the following simple lemma [4 ; Lemma 1], which implicitly plays an important role.

Lemma 2. *Let $T = U|T|$ be the polar decomposition of T and $p > 0$. Then T is p-hyponormal if and only if $S = U|T|^p$ is hyponormal.*

Based on this, we here define the p-paranormality of operators as follows : An operator T on H is *p-paranormal* if T satisfies

$$(9) \qquad \||T|^p U|T|^p x\| \|x\| \geq \||T|^p x\|^2 \text{ for } x \in H \text{ and } p > 0,$$

where $T = U|T|$ is the polar decomposition of T. It is clear that the 1-paranormality is the paranormality and moreover we have the following.

Lemma 3. *Let $T = U|T|$ be the polar decomposition of T and $p > 0$. Then T is p-paranormal if and only if $S = U|T|^p$ is paranormal. Consequently every p-hyponormal is p-paranormal.*

A generalization of Ando's result is given as follows :

Theorem 4. *Every p-paranormal operator is paranormal.*

Proof. First of all, we note that the Hölder inequality by McCarthy (5) has the following form ;

$$(5') \qquad \|A^p y\| \leq \|Ay\|^p \|y\|^{1-p}.$$

for all $y \in H$. Putting $A = |T|$ and $y = U|T|^p x$ in (5), we have

$$\||T|^p U|T|^p x\| \leq \||T|U|T|^p x\|^p \||T|^p x\|^{1-p}.$$

Since the left hand side of the above inequality is greater than $\||T|^p x\|^2/\|x\|$ by the p-paranormality, it follows that

$$(10) \qquad \||T|^p x\|^{1+p} \leq \||T|U|T|^p x\|^p \|x\|.$$

Hence, if we replace x by $|T|^{1-p}x$ in (10), then

$$\|Tx\|^{p+1} \leq \||T|^{1-p}x\| \|T^2 x\|^p.$$

Applying (5') again, it follows that

$$\||T|^{1-p}x\| \leq \|Tx\|^{1-p}\|x\|^p.$$

Therefore it implies that

$$\begin{aligned}\|Tx\|^{p+1} &\leq \||T|^{1-p}x\| \|T^2 x\|^p \\ &\leq \|Tx\|^{1-p}\|x\|^p \|T^2 x\|^p,\end{aligned}$$

so that

$$\|Tx\|^2 \leq \|x\| \|T^2 x\|.$$

This completes the proof.

Though Lemma 2 is implicitly used in the definition of the p-paranormality, we just apply it to the following result due to Ando appeared in a privately circulated note.

Theorem A. *If T is hyponormal and T^* is paranormal, then T is normal.*

Applying Lemma 2, Theorem A is generalized as follows :

Theorem 5. *If T is p-hyponormal and T^* is p-paranormal for some $0 < p \leq 1$, then T is normal.*

Proof. Let S be as in Lemma 2. Then S is hyponormal and S^* is paranormal. Hence Theorem A implies that S is normal. As in the proof of [4 ; Theorem 1], it follows that T is normal.

3. The Aluthge transform. Aluthge introduced the transform

$$T \to \tilde{T} = |T|^{1/2}U|T|^{1/2},$$

where $T = U|T|$ is the polar decomposition of T. First of all, we point out the following fact :

Lemma 6. *An operator T is normaliod if and only if $\tilde{T}$ is normaloid and $\|\tilde{T}\| = \|T\|$.*

Proof. We only note the following inequality;

$$r(T) = r(\tilde{T}) \leq \|\tilde{T}\| \leq \|T\|,$$

where $r(T)$ is the spectral radius of T.

Thus we discuss some conditions on T equivalent to $\|\tilde{T}\| = \|T\|$. For this, we need the following lemma.

Lemma 7. *Let A be a positive operator on H with norm 1 and $\{x_n\}$ a sequence of unit vectors in H. Then the following statements are mutually equivalent :*

(1) $(1-A)x_n \to 0$.

(2) $(1-A^c)x_n \to 0$ *for some* $c > 0$.

(3) $(1-A^c)x_n \to 0$ *for any* $c > 0$.

Proof. It follows from the elementary fact that for any $c > 0$

$$m_c(1-A) \leq 1 - A^c \leq M_c(1-A),$$

where $m_c = \min\{1, c\}$ and $M_c = \max\{1, c\}$.

Theorem 8. *The Aluthge transform preserves the norm of T if and only if there exist $a, b > 0$ and a sequence $\{x_n\}$ of unit vectors such that*

$$(\|T\|^{2a} - (T^*T)^a)x_n \to 0 \text{ and } (\|T\|^{2b} - (TT^*)^b)x_n \to 0.$$

Proof. We may assume that $\|T\| = 1$. Suppose that $\|\tilde{T}\| = \|T\| = 1$. Then we have

$$\||T|^{1/2}|T^*|^{1/2}x_n\| = \||T|^{1/2}U|T|^{1/2}U^*x_n\| \to 1$$

for some sequence $\{x_n\}$ of unit vectors. Since

$$1 \geq \||T^*|^{1/2}x_n\| = \||T|^{1/2}\|\||T^*|^{1/2}x_n\| \geq \||T|^{1/2}|T^*|^{1/2}x_n\| \to 1,$$

it follows that

$$(|T^*|x_n, x_n) - (x_n, x_n) \to 0$$

and so

$$\|(1-|T^*|)^{1/2}x_n\|^2 = ((1-|T^*|)x_n, x_n) \to 0$$

Hence we have

$$\begin{aligned}\|(1-TT^*)x_n\| &= \|(1+|T^*|^{1/2})(1-|T^*|^{1/2})x_n\| \\ &\le \|1+|T^*|^{1/2}\|\|(1-|T^*|^{1/2})x_n\| \to 0.\end{aligned}$$

On the other hand, since

$$\begin{aligned}\||T|^{1/2}x_n\| &= \||T|^{1/2}|T^*|^{1/2}x_n + |T|^{1/2}(1-|T^*|^{1/2})x_n\| \\ &\ge \||T|^{1/2}|T^*|^{1/2}x_n\| - \|(1-|T^*|^{1/2})x_n\| \to 1,\end{aligned}$$

we have $\||T|^{1/2}x_n\| \to 1$ and so

$$((1-|T|)x_n, x_n) = 1 - \||T|^{1/2}x_n\|^2 \to 0.$$

Hence it implies that $(1-T^*T)x_n \to 0$, as seen in the above.

Next we prove the converse. Since $\|T\| = 1$ is assumed, it follows from Lemma 7 that

$$(1-(T^*T)^{1/4})x_n \to 0 \text{ and } (1-(TT^*)^{1/4})x_n \to 0.$$

That is,

$$(1-|T|^{1/2})x_n \to 0 \text{ and } (1-|T^*|^{1/2})x_n \to 0.$$

Hence we have

$$\||T|^{1/2}|T^*|^{1/2}x_n\| \ge \||T|^{1/2}x_n\| - \||T|^{1/2}(|T^*|^{1/2}x_n - x_n)\| \to 1,$$

which implies that $\|\tilde{T}\| \ge 1$ and so $\|\tilde{T}\| = 1$.

We have the following corollary, as in the proof of Theorem 8.

Corollary 9. *Suppose that $\|T\| = 1$. Then the following statements are equivalent :*

(1) $\|\tilde{T}\| = \|T\|(= 1)$.
(2) *There exists a sequence $\{x_n\}$ of unit vectors such that $\||T|^{1/2}|T^*|^{1/2}x_n\| \to 1$.*
(3) *There exists a sequence $\{x_n\}$ of unit vectors such that*

$$(1-T^*T)x_n \to 0 \text{ and } (1-TT^*)x_n \to 0.$$

Corollary 10. *Suppose that $\|T\| = 1$. If either $|T|^\alpha \ge |T^*|^\beta$ or $|T|^\alpha \le |T^*|^\beta$ for some $\alpha, \beta > 0$, then $\|\tilde{T}\| = \|T\|$.*

Proof. We may assume that $(T^*T)^a \le (TT^*)^b$ for some $a, b > 0$. Since $\|T\| = 1$, there exists a sequence $\{x_n\}$ of unit vectors such that $\|(T^*T)^{a/2}x_n\| \to 1$. Therefore we have

$$0 \le (x_n, x_n) - ((TT^*)^b x_n, x_n) \le (x_n, x_n) - ((T^*T)^a x_n, x_n) \to 0.$$

It follows that

$$(1-(T^*T)^a)x_n \to 0 \text{ and } (1-(TT^*)^b)x_n \to 0,$$

which implies that $\|\tilde{T}\| = \|T\|$ by Theorem 8.

Though Corollary 10 implies that $\|\tilde{T}\| = \|T\|$ for a p-hyponormal operator T, we pose another proof of it by the use of Hansen's inequality that

$$(11) \qquad (X^*AX)^p \geq X^*A^pX$$

for $0 < p \leq 1$, $A \geq 0$ and contractions X, [12] and also [14]. Actually, we assume that $\|T\| = 1$. Since $U^*|T|^{2p}U \geq |T|^{2p}$ by the p-hyponormality of T, we have

$$\begin{aligned}(\tilde{T}^*\tilde{T})^{2p} &= (|T|^{1/2}U^*|T|U|T|^{1/2})^{2p} \\ &\geq |T|^{1/2}U^*|T|^{2p}U|T|^{1/2} \text{ by (11)} \\ &\geq |T|^{1/2}|T|^{2p}|T|^{1/2} \\ &= |T|^{2p+1}.\end{aligned}$$

Hence it follows that $\|\tilde{T}^*\tilde{T}\| \geq 1$ and so $1 = \|T\| \geq \|\tilde{T}\| \geq 1$.

4. Concluding remarks. The Aluthge transform makes p-hyponormal operators grow up in the following sense [1; Theorem 1]:

Theorem B. *If T is a p-hyponormal operator for some $0 < p \leq 1/2$, then $\tilde{T}$ is $(p + 1/2)$-hyponormal.*

Aluthge's proof of Theorem B is a typical application of the Furuta inequality [8]. As a consequence, if T is p-hyponormal, then $\tilde{\tilde{T}}$ is hyponormal and so normaloid, i.e., $r(\tilde{\tilde{T}}) = \|\tilde{\tilde{T}}\|$. Hence we have

$$r(T) = r(\tilde{T}) = r(\tilde{\tilde{T}}) = \|\tilde{\tilde{T}}\| = \|\tilde{T}\| = \|T\|$$

by Corollary 10, cf. Lemma 6, so that T is normaloid.

Remark 1. Though the Aluthge transform preserves the spectral radius obviously, it does not preserve the operator norm in general: Let

$$T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $T = TP$ is the polar decomposition of T and so $\tilde{T} = PTP = 0$.

Remark 2. Finally we consider the class of operators satisfying $\|\tilde{T}x\| \geq \|Tx\|$ for all $x \in H$. Thus we have

$$\begin{aligned}\tilde{T}^*\tilde{T} - T^*T &= |T|^{1/2}U^*(|T| - U|T|U^*)U|T|^{1/2} \\ &= |T|^{1/2}U^*(|T| - |T^*|)U|T|^{1/2}.\end{aligned}$$

Since $\overline{\text{ran}}U|T|^{1/2} = \overline{\text{ran}}T$, an operator T satisfies $\|\tilde{T}x\| \geq \|Tx\|$ for all $x \in H$ if and only if

$$T^*(|T| - |T^*|)T \geq 0.$$

This means that T belongs to this class if and only if T is quasi-1/2-hyponormal, provided that we define the quasi-p-hyponormality of T (for $p > 0$) by

$$T^*(|T|^p - |T^*|^p)T \geq 0.$$

References

1. A.Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integral Equations Operator Theory **13** (1990), 307-315.
2. T.Ando, *On hyponormal operators*, Proc. Amer. Math. Soc. **14** (1963), 290-291.
3. T.Ando, *Operators with a norm condition*, Acta. Sci. Math. Szeged **33** (1972), 169-178.
4. M.Fujii, C.Himeji and A.Matsumoto, *Theorems of Ando and Saito for p-hyponormal operators*, Math. Japon., to appear.
5. M.Fujii and Y.Nakatsu, *On subclasses of hyponormal operators*, Proc. Japan Acad. **51** (1975), 243-246.
6. M.Fujii and H.Watanabe, *The Heinz-Kato-Furuta inequarity and hyponormal operators*, in preparation.
7. T.Furuta, *On the class of paranormal operators*, Proc. Japan Acad. **43** (1967), 594-598.
8. T.Furuta, $A \geq B \geq 0$ *assures* $(B^rA^pB^r)^{1/q} \geq B^{(p+2r)/q}$ *for* $r \geq 0, p \geq 0, q \geq 1$ *with* $(1+2r)q \geq p+2r$, Proc. Amer. Math. Soc. **101** (1987), 85-88.
9. T.Furuta, *An extension of the Heinz-Kato theorem*, Proc. Amer. Math. Soc., to appear.
10. T.Furuta and J.Haketa, *Applications of norm inequalities equivalent to Löwner-Heinz theorem*, Nihonkai J. Math. **1** (1990), 11-17.
11. T.Furuta, M.Horie and R.Nakamoto, *A remark on a class of operators*, Proc. Japan Acad. **43** (1967), 607-609.
12. F.Hansen, *An operator inequality*, Math. Ann. **246** (1980), 325-338.
13. W.Istratescu, T.Saito and T.Yoshino, *On a class of operators*, Tohoku J. Math. **18** (1966), 410-413.
14. D.Kainuma and M.Nakamura, *Around Jensen's inequality*, Math. Japon. **25** (1980), 585-588.
15. C.A.McCarthy, c_p, Israel J. Math. **5** (1967), 249-271.
16. H.Watanabe, *Operators characterized by certain Cauchy-Schwarz type inequalities*, preprint.

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