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ON A KAEHLER MANIFOLD WHOSE TOTALLY REAL BISECTIONAL CURVATURE IS BOUNDED FROM BELOW

Dedicated to Professor Tsunero Takahashi on his sixtieth birthday

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Abstact The purpose of this paper is to show that a complete $n(\geq 3)$-dimensional Kaehler manifold with positively lower bounded totally real bisectional curvature and constant scalar curvature is globally isometric to a complex projective space $P_n(C)$ with Fubini-Study metric.

0. Introduction

R.L. Bishop and S.I. Goldberg [2] introduced the notion of totally real bisectional curvature $B(X, Y)$ on a Kaehler manifold M. It is determined by a totally real plane $[X, Y]$ and its image $[JX, JY]$ by the complex structure J, where $[X, Y]$ denotes the plane spanned by linearly independent vector fields X, and Y. Moreover the above two planes $[X, Y]$ and $[JX, JY]$ are orthogonal to each other. And it is known that two orthonormal vectors X and Y span a totally real plane if and only if X, Y and JY are orthonormal.

C.S. Houh [7] showed that $(n\geq 3)$-dimensional Kaehler manifold with constant totally real bisectional curvature is congruent to a complex space form of constant

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holomorphic sectional curvature $H(X) = c$, where $H(X)$ is determined by the holomorphic plane $[X, JX]$.

On the other hand, S.I. Goldberg and S. Kobayashi [5] introduced the notion of holomorphic bisectional curvature $H(X, Y)$, which is determined by two holomorphic planes $[X, JX]$ and $[Y, JY]$, and asserted that a complex projective space $P_n(C)$ is the only compact Kaehler manifold with positive holomorphic bisectional curvature $H(X, Y)$ and constant scalar curvature. If we compare the notion of $B(X, Y)$ with $H(X, Y)$ and $H(X)$, it can be easily seen that the positiveness of $B(X, Y)$ is weaker than the positiveness of $H(X, Y)$, because $H(X, Y) > 0$ implies that both of $B(X, Y)$ and $H(X)$ are positive but neither $B(X, Y) > 0$ nor $H(X) > 0$ implies $H(X, Y) > 0$.

In section 1 we introduce a local formula for Kaehler manifolds, which will be used to prove our main result. And in section 2 let us find a relation between the totally real bisectional curvature and the sectional curvature of a Kaehler manifold M. Also the further relation between the totally real bisectional curvature and the holomorphic sectional curvature of M will be treated. Moreover in this section we calculate the totally real bisectional curvature of the complex quadric Q_n immersed in a complex projective space $P_{n+1}(c)$ with the constant holomorphic sectional curvature c. In section 3 we will prove that a complete Kaehler manifold M with positively lower bounded totally real bisectional curvature $B(X, Y) \geq b > 0$ and constant scalar curvature is congruent to a complex projective space $P_n(C)$. Before to obtain this result we should verify that a Kaehler manifold M with $B(X, Y) \geq b > 0$ is Einstein. Moreover we also show that the positive constant b in the above estimation is best possible, because we can find that there is a complete Kaehler manifold with non-negative totally real bisectional curvature $B(X, Y) \geq 0$ but not Einstein.

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1. Local formulas.

This section is concerned with local formula for Kaehler manifolds. Let M be a complex n-dimensional connected Kaehler manifold. Then we can choose a local unitary frame field $\{E_A\} = \{E_1, \ldots, E_n\}$ on a neighborhood of M. With respect to this frame field, let $\{\omega_A\}$ be its local dual frame fields. Then the Kaehlerian metric tensor g of M is given by $g = 2\Sigma_A \omega_A \otimes \bar{\omega}_A$. The canonical forms ω_A and the connection forms ω_{AB} of M satisfy the following equations:

$$d\omega_A + \Sigma \omega_{AB} \wedge \omega_B = 0, \quad \omega_{AB} + \bar{\omega}_{BA} = 0, \tag{1.1}$$

$$\begin{aligned} d\omega_{AB} + \Sigma \omega_{AC} \wedge \omega_{CB} &= \Omega_{AB}, \\ \Omega_{AB} &= \Sigma R_{\bar{A}BC\bar{D}} \omega_C \wedge \bar{\omega}_D, \end{aligned} \tag{1.2}$$

where Ω_{AB} (resp. $R_{\bar{A}BC\bar{D}}$) denotes the Riemannian curvature form (resp. the components of the Riemannian curvature tensor R) on M.

The second equation of (1.1) means the skew-hermitian symmetry of Ω_{AB}, which is equivalent to the symmetric conditions

$$R_{\bar{A}BC\bar{D}} = \bar{R}_{\bar{B}AD\bar{C}}.$$

The Bianchi identities $\Sigma_B \Omega_{AB} \wedge \omega_B = 0$ obtained by the exterior derivative of (1.1) and (1.2) give the further symmetric relations

$$(1.3) \qquad R_{\bar{A}BC\bar{D}} = R_{\bar{A}CB\bar{D}} = R_{\bar{D}BC\bar{A}} = R_{\bar{D}CB\bar{A}}.$$

Now, with respect to the frame chosen above, the Ricci-tensor S of M can be expressed as follows;

$$S = \Sigma(S_{C\bar{D}}\omega_C \otimes \bar{\omega}_D + S_{\bar{C}D}\bar{\omega}_C \otimes \omega_D),$$

where $S_{C\bar{D}} = \Sigma_B R_{\bar{B}BC\bar{D}} = S_{\bar{D}C} = \bar{S}_{\bar{C}D}$. The scalar curvature r is also given by

$$r = 2\Sigma_D S_{D\bar{D}}.$$

The Kaehlerian manifold M is said to be *Einstein* if the Ricci tensor S is given by

$$S_{C\bar{D}} = \lambda\delta_{CD}, \quad \lambda = \frac{r}{2n},$$

for a constant λ, where λ is called the Ricci curvature of the Einstein manifold.

The component $R_{\bar{A}BC\bar{D}E}$ and $R_{\bar{A}BC\bar{D}\bar{E}}$ of the covariant derivative of the Riemannian curvature tensor R(resp. $S_{A\bar{B}C}$ and $S_{A\bar{B}\bar{C}}$ of the Ricci tensor S) are defined by

$$\Sigma_E(R_{\bar{A}BC\bar{D}E}\omega_E + R_{\bar{A}BC\bar{D}\bar{E}}\bar{\omega}_E) = dR_{\bar{A}BC\bar{D}} - \Sigma(R_{\bar{E}BC\bar{D}}\bar{\omega}_{EA}$$

$$+ R_{\bar{A}EC\bar{D}}\omega_{EB} + R_{\bar{A}BE\bar{D}}\omega_{EC} + R_{\bar{A}BC\bar{E}}\bar{\omega}_{ED}),$$

$$\Sigma_C(S_{A\bar{B}C}\omega_C + S_{A\bar{B}\bar{C}}\bar{\omega}_C) = dS_{A\bar{B}} - \Sigma(S_{C\bar{B}}\omega_{CA} + S_{A\bar{C}}\bar{\omega}_{CB}).$$

The second Bianchi formula is given by

$$R_{\bar{A}BC\bar{D}E} = R_{\bar{A}BE\bar{D}C}, \tag{1.4}$$

and hence we have

$$S_{A\bar{B}C} = S_{C\bar{B}A} = \Sigma_D R_{\bar{B}AC\bar{D}D}, \quad r_A = 2\Sigma_C S_{B\bar{C}C}, \tag{1.5}$$

where $dr = \Sigma_C(r_C\omega_C + \bar{r}_C\bar{\omega}_C)$. The components $S_{A\bar{B}CD}$ and $S_{A\bar{B}C\bar{D}}$ of the covariant derivative of $S_{A\bar{B}C}$ are expressed by

$$\begin{aligned}\Sigma_D(S_{A\bar{B}CD}\omega_D + S_{A\bar{B}C\bar{D}}\bar{\omega}_D) = dS_{A\bar{B}C} - \Sigma_D(&S_{D\bar{B}C}\omega_{DA} \\ &+ S_{A\bar{D}C}\bar{\omega}_{DB} + S_{A\bar{B}D}\omega_{DC}).\end{aligned} \tag{1.6}$$

By the exterior differentiation of the definition of $S_{A\bar{B}C}$ and by taking account of (1.6) the Ricci formula for the Ricci tensor S is given as follows:

$$S_{A\bar{B}C\bar{D}} - S_{A\bar{B}\bar{D}C} = \Sigma_E(R_{\bar{D}CA\bar{E}}S_{E\bar{B}} - R_{\bar{D}CE\bar{B}}S_{A\bar{E}}). \tag{1.7}$$

The sectional curvature of the holomorphic plane P spanned by u and Ju is called the *holomorphic sectional curvature*, which is denoted by $H(P) = H(u)$. A Kaehler manifold M is said to be of *constant holomorphic sectional curvature* if its holomorphic sectional curvature $H(P)$ is constant for all P and for all points of M. Then M is called a complex space form, which is denoted by $M_n(c)$, provided that it is of constant holomorphic sectional curvature c, of complex dimension n. The standard models of complex space forms are the following three kinds: the complex Euclidean space C^n, the complex projective space $P_n(C)$ or the complex hyperbolic space $H_n(C)$, according as $c = 0, c > 0$ or $c < 0$.

Now, the Riemannian curvature tensor $R_{\bar{A}BC\bar{D}}$ of $M_n(c)$ is given by

$$R_{\bar{A}BC\bar{D}} = \frac{c}{2}(\delta_{AB}\delta_{CD} + \delta_{AC}\delta_{BD}). \tag{1.8}$$

First of all, let us introduce a fundamental property for the generalized maximal principal due to H.Omori [10] and S.T.Yau [12].

Theorem 1.1. *Let M be an n-dimensional Riemannian manifold whose Ricci curvature is bounded from below on M. Let F be a C^2-function bounded from below on M, then for any $\epsilon > 0$, there exists a point p such that*

$$|\nabla F(p)| < \epsilon, \quad \Delta F(p) > -\epsilon \quad \textit{and} \quad \inf F + \epsilon > F(p).$$

2. Totally real bisectional curvature.

Let (M, g) be an n-dimensional Kaehlerian manifold with almost complex structure J. In this section, we consider a Kaehlerian manifold with totally real bisectional curvature, which is determined by a totally real plane $[u, v]$ and its image $[Ju, Jv]$ by the complex structure J. That is, the totally real bisectional curvature is defined by

$$B(u, v) = g(R(u, Ju)Jv, v), \tag{2.1}$$

where $[u, v]$ means the totally real plane section such that $g(u, u) = g(v, v) = 1$, $g(u, v) = 0$ and $g(u, Jv) = 0$. Then for a Kaehlerian manifold, using the first Bianchi-identity to (2.1), we get

$$\begin{aligned} B(u, v) &= g(R(u, Jv)Jv, u) + g(R(u, v)v, u) \\ &= K(u, v) + K(u, Jv), \end{aligned} \tag{2.2}$$

where $K(u, v)$ means the sectional curvature of the plane spanned by u and v.

Now if we put $u' = \frac{u+v}{\sqrt{2}}$ and $v' = \frac{J(u-v)}{\sqrt{2}}$, then it is easily seen that $g(u', u') = g(v', v') = 1$, and $g(u', Jv') = 0$. Thus $B(u', v') = g(R(u', Ju')Jv', v')$ implies that

$$4B(u', v') - 2B(u, v) = H(u) + H(v) - 4K(u, Jv), \tag{2.3}$$

where $H(u) = K(u, Ju)$, and $H(v) = K(v, Jv)$ means the holomorphic sectional curvatures of the plane $[u, Ju]$ and $[v, Jv]$ respectively.

If we put $u'' = \frac{u+Jv}{\sqrt{2}}$, and $v'' = \frac{Ju+v}{\sqrt{2}}$, then we get $g(u'', u'') = g(v'', v'') = 1$ and $g(u'', v'') = 0$. Using the similar method as in (2.3), we get

$$4B(u'', v'') - 2B(u, v) = H(u) + H(v) - 4K(u, v). \tag{2.4}$$

Summing up (2.3) and (2.4), we obtain

$$2B(u', v') + 2B(u'', v'') = H(u) + H(v). \tag{2.5}$$

Now we calculate the totally real bisectional curvatures of some manifolds.

Example 2.1 Let $M_n(c)$ be a complex space form of constant holomorphic sectional curvature c and $[u, v]$ be a totally real plane section. Then

$$\begin{aligned} B(u, v) &= g(R(u, Ju)Jv, v) \\ &= \frac{c}{4}\{g(u, v)g(Ju, Jv) - g(u, Jv)g(Ju, v) + g(Ju, v)g(-u, Jv) \\ &\quad - g(Ju, Jv)g(-u, v) - 2g(Ju, Jv)g(-u, v)\} \\ &= \frac{c}{2}. \end{aligned}$$

Thus $M_n(c)$ is a space of complex space form of constant totally real bisectional curvature $\frac{c}{2}$.

As a Kaehler manifold which is not of constant totally real bisectional curvature, we introduce the following example.

Example 2.2 Let Q_n be a complex quadric in $P_{n+1}(c)$ and $[u, v]$ a totally real plane section. Since Q_n is represented as a Hermitian symmetric space of compact type, its sectional curvature is non-negative(cf [8]). Thus by (2.2) we know that the totally real bisectional curvature $B(u, v)$ of Q_n is non-negative. Now let us estimate the upper bounds of $B(u, v)$ of Q_n. For the action of $G = SO(n+2)$ on Q_n, the isotropy group H turns out to be $SO(2)\times SO(n)$, where $SO(n)$ denotes the group of special orthogonal $n\times n$-matrices.

The canonical decomposition of the Lie algebra of the group G is

$$\mathcal{G} = \mathcal{H} + \mathcal{M},$$

where $\mathcal{G} = \mathcal{O}(n+2)$, $\mathcal{H} = \mathcal{O}(2) + \mathcal{O}(n)$, $\mathcal{M} = \{\begin{pmatrix} 0 & 0 & -{}^t\xi \\ 0 & 0 & -{}^t\eta \\ \xi & \eta & 0 \end{pmatrix} | \xi, \eta \in R^n\}$, and $\mathcal{O}(n)$ denotes the Lie algebra of the special orthogonal group $SO(n)$.

Identifying $(\xi, \eta) \in R^n + R^n$ with the above matrix in $\mathcal{M}$, we define an inner product g on $\mathcal{M}\times \mathcal{M}$ by

$$g((\xi, \eta), (\xi', \eta')) = \frac{2}{c}\{< \xi, \xi' > + < \eta, \eta' >\},$$

where $< \xi, \xi' >$ is the standard inner product in R^n. We also define a complex structure J on $\mathcal{M}$ by

$$J(\xi, \eta) = (-\eta, \xi).$$

The curvature tensor R at the origin is given by the following

$$R((\xi, \eta), (\xi', \eta')) = ad\begin{pmatrix} 0 & -\lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & B \end{pmatrix}, \quad B \in O(n),$$

where $\lambda =< \xi', \eta > - < \xi, \eta' >$, $B = \frac{c}{4}\{\xi \wedge \xi' + \eta \wedge \eta'\}$, and $(\xi \wedge \xi')\eta = \frac{4}{c}\{< \xi', \eta > \xi - < \xi, \eta > \xi'\}$. Thus for unit elements $u = (\xi, \eta)$, $v = (\xi', \eta')$ in $\mathcal{M}$, the holomorphic bisectional curvature is given by

$$(2.6) \quad \begin{aligned} H(u, v) = g(R(u, Ju)Jv, v) &= \frac{2}{c}\{< -B\eta', \xi' > + < B\xi', \eta' >\} + \frac{c}{2}g(v, v) \\ &= \frac{8}{c}\{< \xi, \xi' >< \eta, \eta' > - < \xi, \eta' >< \xi', \eta >\} + \frac{c}{2}. \end{aligned}$$

And the holomorphic sectional curvature $H(u)$ is given by

$$H(u) = g(R(u, Ju)Ju, u) = \frac{8}{c}(|\xi|^2|\eta|^2 - < \xi, \eta >^2) + \frac{c}{2} \geq \frac{c}{2}.$$

In fact, since the complex quadric Q_n is a Hermitian symmetric space of compact type with rank 2, by K. Ogiue and R. Takagi [9] the holomorphic sectional curvature $H(u)$ of Q_n is holomorphically pinched as $\frac{c}{2} \leq H(u) \leq c$.

Now we consider the totally real bisectional curvature of the complex quadric Q_n. Let $[u, v]$ be a totally real plane section such that $u = (\xi, \eta), v = (\xi', \eta')$, and

$Jv = (-\eta', \xi')$. Then u, v, Ju and Jv constitute orthonormal unit elements in $\mathcal{M}$. That is

$$g(u, v) = \frac{2}{c}\{< \xi, \xi' > + < \eta, \eta' >\} = 0,$$

$$g(u, Jv) = \frac{2}{c}\{< \xi, -\eta' > + < \eta, \xi' >\} = 0.$$

From these together with (2.6) the totally real bisectional curvature is given by

$$B(u, v) = -\frac{8}{c}\{< \xi, \xi' >^2 + < \xi, \eta' >^2\} + \frac{c}{2}.$$

From this, using the elementary method of Lagrange multiplier rule, it can be easily seen that the totally real bisectional curvature $B(u, v)$ is bounded as

$$-\frac{3}{2}c \leq B(u, v) \leq \frac{1}{2}c,$$

where the upper equality holds if and only if ξ is orthogonal to ξ' and η' in R^n. Accordingly, it follows that

$$0 \leq B(u, v) \leq \frac{1}{2}c$$

for any totally real plane $[u, v]$ of M, because we have already known that the totally real bisectional curvature of the complex quadric Q_n is non- negative.

3. Complete Kaehler manifolds with positive totally real bisectional curvature.

Let M be an n-dimensional Kaehler manifold with the complex structure J. We can choose a local field of orthonormal frames $u_1, ..., u_n, u_{1^*} = Ju_1, ..., u_{n^*} = Ju_n$

on a neighborhood on M. With respect to this frame field, let $\theta_1, ..., \theta_n, \theta_{1^*}, ..., \theta_{n^*}$ be the field of dual frames.

Let us denote by $\theta = (\theta_{AB}, \theta_{A^*B}, \theta_{AB^*}, \theta_{A^*B^*})$, $A, B = 1, ..., n$ the connection form of M. Then we have

$$\theta_{AB} = \theta_{A^*B^*}, \theta_{AB^*} = -\theta_{A^*B}, \theta_{AB} = -\theta_{BA}, \text{ and } \quad \theta_{AB^*} = \theta_{BA^*}. \tag{3.1}$$

Now we set $e_A = \frac{1}{\sqrt{2}}(u_A - iu_A)$, $e_{\bar{A}} = \frac{1}{\sqrt{2}}(u_A + iu_A)$. Then $\{e_A, e_{\bar{A}}\}$ constitute a local field of unitary frames. And let us denote by $\omega_A = \theta_A + i\theta_{A^*}$ and $\bar{\omega}_A = \theta_A - i\theta_{A^*}$ its dual frame fields respectively. Then the components of Kaehler metric $g = 2\Sigma_A \omega_A \otimes \bar{\omega}_A$ and the metric components of the Riemannian curvature tensor are given by the following respectively

$$g_{B\bar{C}} = g_{BC} + ig_{BC^*}, \tag{3.2}$$

$$R_{\bar{A}BC\bar{D}} = -\{K_{ABCD} + K_{A^*BC^*D} + i(-K_{ABC^*D} + K_{A^*BCD})\}, \tag{3.3}$$

where $R_{\bar{A}BC\bar{D}} = g_{\bar{A}E}R^E{}_{BC\bar{D}}$. Thus for the case of $A = B$, $C = D$, $B \neq C$ in (3.3), the totally real bisectional curvature is given by

$$R_{\bar{B}BC\bar{C}} = -K_{B^*BC^*C} = K_{BB^*C^*C} = B(u_B, u_C). \tag{3.4}$$

For the case of $A = B = C = D$ in (3.3), the holomorphic sectional curvature is given by

$$R_{\bar{B}BB\bar{B}} = g(R(u_B, Ju_B)Ju_B, u_B) = H(u_B). \tag{3.5}$$

Remark 3.1 From (1.8) and (3.4) we know that for any totally real plane section $[u, v]$ the totally real bisectional curvature $B(u, v)$ of a complex space form $M_n(c)$ is $\frac{c}{2}$ which is the same value as in Example 2.1.

On the other hand, S.I. Goldberg and S. Kobayashi [5] showed that a Kaehler manifold with positive holomorphic bisectional curvature and constant scalar curvature is Einstein. It is well known that the Ricci 2-form is harmonic if and only if the scalar curvature is constant. In order to prove that the second Betti number of a compact connected Kaehler manifold M with positive holomorphic bisectional curvature $H(X, Y) > 0$ is one they have used the fact that $H(X) > 0$. Thus the Ricci 2-form is propotional to the Kaehler 2-form, so that M becomes to an Einstein manifold. But the condition $B(X, Y) > 0$ is weaker than the condition of $H(X, Y) > 0$ we can not use $H(X) > 0$ to obtain the above result. From this point of view by means of Theorem 1.1 we can obtain the following.

Theorem 3.1 *Let M be a complete n-dimensional Kaehler manifold with constant scalar curvature. Assume that the totally real bisectional curvature is lower bounded for some positive constant b. Then M is Einstein.*

Proof. Since $(S_{B\bar{C}})$ is a Hermitian matrix, it can be diagonalizable. Thus $S_{B\bar{C}} = \lambda_B \delta_{BC}$, where λ_B is a real valued function. From this it follows that $r = 2\Sigma_B S_{B\bar{B}} = 2\Sigma_B \lambda_B$. Now we put $S_2 = \Sigma_{B,\bar{C}} S_{B\bar{C}} S_{C\bar{B}}$. Then it yields easily that

$$S_2 - \frac{r^2}{4n} = \Sigma\lambda_B^2 - \frac{(\Sigma\lambda_B)^2}{n} = \frac{1}{2n}\Sigma_{B,C}(\lambda_B - \lambda_C)^2. \tag{3.6}$$

Since we have assumed that the scalar curvature r of M is constant, from (1.5) it follows $\Sigma_B S_{B\bar{B}C} = \Sigma_B S_{C\bar{B}B} = 0$. Together with this fact using (1.5) and the Ricci formula (1.7) we have that

$$\Delta S_{B\bar{C}} = \Sigma_D S_{B\bar{C}D\bar{D}} = \Sigma_D S_{D\bar{C}B\bar{D}}$$

$$= \Sigma_{E,D}(R_{\bar{D}BD\bar{E}} S_{E\bar{C}} - R_{\bar{D}BE\bar{C}} S_{D\bar{E}}),$$

from which, if we use the first Bianchi-identity (1.3) to the final term, we have

$$\Delta S_{B\bar{C}} = \Sigma_E(S_{B\bar{E}} S_{E\bar{C}} - \Sigma_D R_{\bar{D}EB\bar{C}} S_{D\bar{E}})$$

$$= \lambda_B S_{B\bar{C}} - \Sigma_A \lambda_A R_{\bar{A}AB\bar{C}}.$$

Thus we get

$$\frac{1}{2}\Delta S_2 = \frac{1}{2}|\nabla S|^2 + \Sigma_{B,C} S_{\bar{C}B}(\lambda_B S_{B\bar{C}} - \Sigma_A \lambda_A R_{\bar{A}AB\bar{C}}), \tag{3.7}$$

where $|\nabla S|^2 = 2\Sigma S_{A\bar{B}C} \bar{S}_{A\bar{B}C}$. Since the second term of the right hand side is reduced to

$$\Sigma_{A,B}(\lambda_B^2 R_{\bar{A}AB\bar{B}} - \lambda_A \lambda_B R_{\bar{A}AB\bar{B}}) = \frac{1}{2}\Sigma_{A,B}(\lambda_A - \lambda_B)^2 R_{\bar{A}AB\bar{B}},$$

we get the following inequality by (3.7)

$$\Delta S_2 \geq \Sigma(\lambda_A - \lambda_B)^2 R_{\bar{A}AB\bar{B}}, \tag{3.8}$$

where the above equality holds if and only if the Ricci tensor S is parallel on M.

Now let us consider a non-negative function $f = S_2 - \frac{r^2}{4n}$. Then from (3.6),(3.8) and the assumption it follows that

$$\Delta f \geq 2nbf, \tag{3.9}$$

where the above equality holds if and only if the Ricci tensor S is parallel on M. In order to prove this theorem, we need the following lemma.

Lemma 3.2 *Under the same assumption as stated in Theorem 3.1 the Ricci-curvature is bounded from below.*

Proof. From the assumption and (2.5) it follows that

$$H(u) + H(v) \geq 4b.$$

Using (3.5) to the above equation for $u = u_A$, $v = u_B$, $A \neq B$, then we can rewrite the above inequality as the following

$$R_{\bar{A}A A\bar{A}} + R_{\bar{B}BB\bar{B}} \geq 4b.$$

If we put $R_A = R_{\bar{A}AA\bar{A}}$, then

$$R_A + R_B \geq 4b \qquad (A \neq B). \tag{3.10}$$

Thus $\Sigma_{A<B}(R_A + R_B) \geq 2n(n-1)b$ implies that

$$\Sigma_A R_A \geq 2nb, \tag{3.11}$$

where the equality holds if and only if $R_A = 2b$ for any A.

On the other hand, from the fact that

$$r = 2\Sigma_A S_{A\bar{A}} = 2\Sigma_{A,B} R_{\bar{A}AB\bar{B}} = 2(\Sigma_A R_A + \Sigma_{A\neq B} R_{\bar{A}AB\bar{B}})$$
$$\geq 2\Sigma_A R_A + 2n(n-1)b$$

it follows

$$\Sigma_A R_A \leq \frac{r}{2} - n(n-1)b, \tag{3.12}$$

where the equality holds if and only if $R_{\bar{A}AB\bar{B}} = b$ for any A, B ($A\neq B$). In this case due to C.S.Houh [7] M is congruent to $M_n(2b)$. From (3.11) and (3.12) we know that $r \geq 2n(n+1)b$. Thus from the assumption the scalar curvature r is positive constant. Also (3.10) gives $\Sigma^n_{B=2}(R_1 + R_B) \geq 4(n-1)b$, so that

$$(n-2)R_1 + \Sigma_B R_B \geq 4(n-1)b. \tag{3.13}$$

From this and (3.12) it follows

$$(n-2)R_1 \geq 4(n-1)b - \Sigma_B R_B \geq 4(n-1)b - \{\frac{r}{2} - n(n-1)b\}.$$

Thus if we use the similar method to the other index, we can assert the following

$$(n-2)R_B \geq (n-1)(n+4)b - \frac{r}{2}$$

for any index B, so that R_B is bounded from below for $n \geq 3$. Moreover the above equality holds for some index B if and only if M is congruent to $M_n(2b)$. Accordingly the Ricci-curvature is given by

$$\lambda_A = S_{A\bar{A}} = \Sigma_B R_{\bar{A}AB\bar{B}} = R_A + \Sigma_{A\neq B} R_{\bar{A}AB\bar{B}}$$
$$\geq R_A + (n-1)b.$$

Thus the Ricci-curvature is also bounded from below. Now Lemma 3.2 is proved.

Now we will complete the proof of Theorem 3.1. For a constant $a > 0$, we consider a smooth positive function $F = (f + a)^{-\frac{1}{2}}$. Thus, from Lemma 3.2 we can apply Theorem 1.1(H.Omori [10] and S.T.Yau [12]) to the function $F = (f + a)^{-\frac{1}{2}}$ for the given f. Given any positive number $\epsilon > 0$, there exists a point p such that

(3.15) $$|\nabla F|(p) < \epsilon, \quad \Delta F(p) > -\epsilon, \quad F(p) < inf\ F + \epsilon.$$

On the other hand, the Laplacian of the function F can be calculated by

$$\Delta F = \Sigma_k\{(f + a)^{-\frac{1}{2}}\}_{k\bar{k}} = \frac{3}{4}F^5\Sigma_k f_{\bar{k}} f_k - \frac{1}{2}F^3\Delta f,$$

where f_k and $f_{\bar{k}}$ denote $\frac{\partial f}{\partial z_k}$ and $\frac{\partial f}{\partial \bar{z}_k}$ respectively. From this and (3.15), together with the fact that

$$|\nabla F| = |grad\ F|^2 = 2\Sigma_k F_{\bar{k}} F_k = \frac{1}{2}F^6\Sigma_k f_{\bar{k}} f_k$$

it follows that

(3.16) $$\epsilon(3\epsilon + 2F(p)) > F(p)^4\Delta f(p) \geq 0.$$

Thus for a convergent sequence $\{\epsilon_m\}$ such that $\epsilon_m > 0$ and $\epsilon_m \to 0$ as $m \to \infty$, there is a point sequence $\{p_m\}$ so that the sequence $\{F(p_m)\}$ satisfies (3.15) and converges to F_0, by taking a subsequence, if necessary, because the sequence $\{F(p_m)\}$ is bounded. From the definition of the infimum and (3.15) we have $F_0 = infF$ and hence $f(p_m) \to f_0 = supf$. It follows from (3.16) that we have

$$\epsilon_m\{3\epsilon_m + 2F(p_m)\} > F(p_m)^4\Delta f(p_m)$$

and the left hand side converges to 0 because the function F is bounded. Thus we get

$$F(p_m)^4 \Delta f(p_m) \to 0 \quad (m \to \infty).$$

As is already seen, the Ricci-curvature is bounded from below i.e., so is any λ_B. Since $r = 2\Sigma_B \lambda_B$ is constant, λ_B is bounded from above. Hence $F = (f + a)^{-\frac{1}{2}}$ is bounded from below by a positive constant. From (3.17) it follows that $\Delta f(p_m) \to 0$ as $m \to \infty$. Since $b > 0$, by (3.9) we have that

$$\Delta f(p_m) \geq \frac{n}{2} b f(p_m) \geq 0.$$

Thus we have $f(p_m) \to 0 = inf\ f$. Since $f(p_m) \to sup\ f$, we have $sup\ f = inf\ f = 0$. Hence $f = 0$ on M. That is, M is Einstein. This completes the above proof of Theorem 3.1.

Remark 3.2 The positive constant $b > 0$ in Theorem 3.1 is best possible. Because there is a complete Kaehler manifold with non-negative totally real bisectional curvature $B(u, v) \geq 0$ but not Einstein as follows: Consider a product manifold $M = P_{n^1}(c_1) \times P_{n^2}(c_2)$. Then from (3.8) we know that its totally real bisectional curvature is given by

$$R_{\bar{A}AB\bar{B}} = \begin{cases} R_{\bar{a}ab\bar{b}} = \frac{c_1}{2} & (A = a, B = b), \\ 0 & (A = a, B = s), \\ R_{\bar{r}rs\bar{s}} = \frac{c_2}{2} & (A = r, B = s), \end{cases}$$

where indices $A, B (A \neq B), \ldots; 1, \ldots, n_1, n_1 + 1, \ldots, n_2$, and $a, b, ..; 1, \ldots, n_1, r, s, ..; n_1 + 1, \ldots, n_2$.

And its Ricci-tensor is given by the following

$$S_{A\bar{B}} = \Sigma_C R_{\bar{B}AC\bar{C}} = \Sigma_a R_{\bar{B}Aa\bar{a}} + \Sigma_r R_{\bar{B}Ar\bar{r}}$$

$$= \begin{cases} \frac{n_1+1}{2} c_1 \delta_{bc} & (B = c, A = b), \\ 0 & (B = s, A = b), \\ \frac{n_2+1}{2} c_2 \delta_{ts} & (B = s, A = t). \end{cases}$$

Thus for the case where $(n_1+1)c_1 \neq (n_2+1)c_2$, $M = P_{n^1}(c_1) \times P_{n^2}(c_2)$ is not Einstein.

Since a complete Kaehler manifold M with the assumption in Theorem 3.1 is known to be Einstein and its scalar curvature r is positive constant, its Ricci-tensor is positive definite. Thus by using a theorem of Myers we can assert that M is compact [8]. Now let us introduce a theorem of S.I. Goldberg and S. Kobayashi [5], which is slight different from the original one.

Theorem A. *An n-dimensional compact connected Kaehler manifold with an Einstein metric of totally real bisectional curvature is globally isometric to* $P_n(C)$ *with Fubini-Study metric.*

Though the original theorem in [5] are assumed with positive holomorphic bisectional curvature, the above result in Theorem A also holds for the assumption with positive totally real bisectional curvature. Thus combining Theorem A and Theorem 3.1 we can assert the following.

Theorem 3.3 *Let* M *be a complete* $n(\geq 3)$*-dimensional Kaehler manifold with constant scalar curvature. Assume that the totally real bisectional curvature is lower bounded for some positive constant* b. *Then* M *is globally isometric to* $P_n(C)$ *with Fubini-Study metric.*

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