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NEW CRITERIA FOR MEROMORPHIC UNIVALENT FUNCTIONS OF ORDER α

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ABSTRACT. Let $M_n(\alpha)$ be the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k$$

which are regular in the punctured disc $U^* = \{z : 0 < |z| < 1\}$ and satisfying

$$\operatorname{Re}\left\{\frac{D^{n+1} f(z)}{D^n f(z)} - 2\right\} < -\frac{n+\alpha}{n+1}, \qquad |z| < 1,$$

$n \in N_o = \{0, 1, \ldots\}$, and $0 \le \alpha < 1$, where

$$D^n f(z) = \frac{1}{z}\left(\frac{z^{n+1} f(z)}{n!}\right)^{(n)}.$$

It is proved that $M_{n+1}(\alpha) \subset M_n(\alpha)$. Since $M_o(\alpha)$ is the class of meromorphically starlike functions of order α, $0 \le \alpha < 1$, all functions in $M_n(\alpha)$ are univalent. Further we consider the integrals of functions in $M_n(\alpha)$.

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1. Introduction.

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k \tag{1.1}$$

which are regular in the punctured disc $U^* = \{z : 0 < |z| < 1\}$. The Hadamard product of two functions $f,\ g \in \Sigma$ will be denoted by $(f*g)(z)$. Let

$$D^n f(z) = \frac{1}{z(1-z)^{n+1}} * f(z), \qquad n \geq 0 \tag{1.2}$$

$$= \frac{1}{z}\left(\frac{z^{n+1} f(z)}{n!}\right)^{(n)} \tag{1.3}$$

$$= \frac{1}{z} + (n+1)a_o + \frac{(n+1)(n+2)a_1 z}{2!} + \ldots . \tag{1.4}$$

In this paper along with other things we shall show that a function $f(z) \in \Sigma$, which satisfies one of the conditions

$$\mathrm{Re}\left\{\frac{D^{n+1} f(z)}{D^n f(z)} - 2\right\} < -\frac{n+\alpha}{n+1}, \quad |z| < 1, \tag{1.5}$$

$n \in N_o = N \cup \{0\}$, and $0 \leq \alpha < 1$, is univalent in $0 < |z| < 1$. More preceisely it is proved that for the classes $M_n(\alpha)$ of functions in Σ satisfying (1.5),

$$M_{n+1}(\alpha) \subset M_n(\alpha) \tag{1.6}$$

holds. Since $M_o(\alpha)$ eqqual $\Sigma^*(\alpha)$ (the class of meromorphically starlike functions of order α, $0 \leq \alpha < 1$) the univalence of members in $M_n(\alpha)$ is a consequence of (1.6).

Further for $c > 0$, let

$$F(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)\, dt, \tag{1.7}$$

it is shown that $F(z) \in M_n(\alpha)$ whenever $f(z) \in M_n(\alpha)$. Also it is shown that if $f(z) \in M_n(\alpha)$ then

$$F(z) = \frac{n+1}{z^{n+2}} \int_0^z t^{n+1} f(t)\, dt \tag{1.8}$$

belongs to $M_{n+1}(\alpha)$ for $F(z) \neq 0$ in U^*. Some known results of Bajpai [1], Goel and Sohi [3] and Ganigi and Uralegaddi [2] are extended. In [5] Ruscheweyh obtained the new criteria for univalent functions.

2. Properties of the class $M_n(\alpha)$.

In proving our main results (Theorem 1 and Theorem 2 below), we shall need the following lemma due to I. S. Jack [4].

Lemma. Let $w(z)$ be non-constant and regular in $U = \{z: |z| < 1\}$, $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at z_o, we have $z_o w'(z_o) = k\, w(z_o)$, where k is a real number and $k \geq 1$.

Theorem 1. $M_{n+1}(\alpha) \subset M_n(\alpha)$ for each integer $n \in N_o$.

Proof. Let $f(z) \in M_{n+1}(\alpha)$. Then

$$\mathrm{Re}\left\{\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 2\right\} < -\frac{n+\alpha+1}{n+2}. \tag{2.1}$$

We have to show that (2.1) implies the inequality

$$\mathrm{Re}\left\{\frac{D^{n+1}f(z)}{D^{n}f(z)} - 2\right\} < -\frac{n+\alpha}{n+1}. \tag{2.2}$$

Define a regular function $w(z)$ in $U = \{z: |z| < 1\}$ by

$$\frac{D^{n+1}f(z)}{D^{n}f(z)} - 2 = -\left\{\frac{n+\alpha}{n+1} + \frac{1-\alpha}{n+1} \cdot \frac{1-w(z)}{1+w(z)}\right\}. \tag{2.3}$$

Clearly $w(0) = 0$. Equation (2.3) may be written as

$$\frac{D^{n+1}f(z)}{D^{n}f(z)} = \frac{(n+1)+(n+3-2\alpha)w(z)}{(n+1)(1+w(z))}. \tag{2.4}$$

Differentiating (2.4) logarithmically and using the identity

$$z(D^n f(z))' = (n+1)D^{n+1}f(z) - (n+2)D^n f(z). \tag{2.5}$$

We obtain

$$\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 2 + \frac{n+\alpha+1}{n+2} = \frac{n+1+(n+3-2\alpha)w(z)}{(n+2)(1+w(z))}$$

$$- 1 + \frac{\alpha}{n+2} + \frac{2(1-\alpha)\ z\ w'(z)}{(n+2)(1+w(z))[n+1+(n+3-2\alpha)w(z)]}.$$

That is

$$\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 2 + \frac{n+\alpha+1}{n+2} = \frac{1-\alpha}{n+2}\left\{-\frac{1-w(z)}{1+w(z)}\right.$$

$$\left. + \frac{2zw'(z)}{(1+w(z))[n+1+(n+3-2\alpha)w(z)]}\right\}. \tag{2.6}$$

We claim that $|w(z)| < 1$. For otherwise (by Jack's lemma) there exists z_o in U such that

$$z_o w'(z_o) = k\ w(z_o), \tag{2.7}$$

where $|w(z_o)| = 1$ and $k \geq 1$. From (2.6) and (2.7), we obtain

$$\frac{D^{n+2}f(z_o)}{D^{n+1}f(z_o)} - 2 + \frac{n+\alpha+1}{n+2} = \frac{1-\alpha}{n+2}\left\{-\frac{1-w(z_o)}{1+w(z_o)}\right.$$

$$\left. + \frac{2kw(z_o)}{(1+w(z_o))[n+1+(n+3-2\alpha)w(z_o)]}\right\}. \tag{2.8}$$

Thus

$$\operatorname{Re}\left\{\frac{D^{n+2}f(z_o)}{D^{n+1}f(z_o)} - 2 + \frac{n+\alpha+1}{n+2}\right\} \geq \frac{1-\alpha}{2(n+2)(n+2-\alpha)} > 0$$

which contradicts (2.1). Hence $|w(z)| < 1$ in U and from (2.3) it follows that $f(z) \in M_n(\alpha)$.

<u>Theorem 2</u>. Let $f(z) \in \Sigma$ and for a given $n \in N_o$ and $c > 0$ satisfy the condition

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^{n}f(z)} - 2\right\} < \frac{(1-\alpha)-2(n+\alpha)(c+1-\alpha)}{2(n+1)(c+1-\alpha)}. \tag{2.9}$$

Then

$$F(z) = \frac{c}{z^{c+1}}\int_o^z t^c\, f(t)\, dt \tag{2.10}$$

belongs to $M_n(\alpha)$ for $F(z) \neq 0$ in $0 < |z| < 1$.

<u>Proof</u>. Using the identities

$$z(D^nF(z))' = cD^nf(z) - (c+1)D^nF(z). \tag{2.11}$$

and

$$z(D^nF(z))' = (n+1)D^{n+1}F(z) - (n+2)D^nF(z) \tag{2.12}$$

the condition (2.9) may be written as

$$\operatorname{Re}\left\{\frac{(n+2)\dfrac{D^{n+2}F(z)}{D^{n+1}F(z)} - (n+2-c)}{(n+1) - (n+1-c)\dfrac{D^{n}F(z)}{D^{n+1}F(z)}} - 2\right\} < \frac{(1-\alpha)-2(n+\alpha)(c+1-\alpha)}{2(n+1)(c+1-\alpha)}. \tag{2.13}$$

We have to prove that (2.13) implies the inequality

$$\operatorname{Re}\left\{\frac{D^{n+1}F(z)}{D^{n}F(z)} - 2\right\} < -\frac{n+\alpha}{n+1}.$$

Define a regular function $w(z)$ in U by

$$\frac{D^{n+1}F(z)}{D^{n}F(z)} - 2 = -\left\{\frac{n+\alpha}{n+1} + \frac{1-\alpha}{n+1}\cdot\frac{1-w(z)}{1+w(z)}\right\}. \tag{2.14}$$

Clearly $w(0) = 0$. The equation (2.14) may be written as

$$\frac{D^{n+1}F(z)}{D^{n}F(z)} = \frac{(n+1)+(n+3-2\alpha)w(z)}{(n+1)(1+w(z))}. \tag{2.15}$$

Differentiating (2.15) logarithmically and simplifying we obtain

$$\frac{(n+2)\dfrac{D^{n+2}F(z)}{D^{n+1}F(z)} -(n+2-c)}{(n+1) - (n+1-c)\dfrac{D^{n}F(z)}{D^{n+1}F(z)}} - 2 = -\left\{\frac{n}{n+1} + \frac{1}{n+1}\left[(1-\alpha)\frac{1-w(z)}{1+w(z)} + \alpha\right]\right\}$$

$$+\frac{2(1-\alpha)z\,w'(z)}{(n+1)(1+w(z))[c+(c+2-2\alpha)w(z)]}. \qquad (2.16)$$

We claim that $|w(z)| < 1$. For otherwise by Jack's lemma there exists a z_o, $|z_o| <1$ such that

$$z_o w'(z_o) = k\, w(z_o), \qquad (2.17)$$

where $|w(z_o)| = 1$ and $k \geq 1$.

Combining (2.16) and (2.17), we obtain

$$\frac{(n+2)\dfrac{D^{n+2}F(z_o)}{D^{n+1}F(z_o)} -(n+2-c)}{(n+1)-(n+1\ -c)\dfrac{D^{n}F(z_o)}{D^{n+1}F(z_o)}} - 2 = -\left\{\frac{n}{n+1} + \frac{1}{n+1}\left[(1-\alpha)\frac{1-w(z_o)}{1+w(z_o)} + \alpha\right]\right\}$$

$$+\frac{2(1-\alpha)z_o w'(z_o)}{(n+1)(1+w(z_o))[c+(c+2-2a)w(z_o)]}.$$

Thus

$$\mathrm{Re}\left\{\frac{(n+2)\dfrac{D^{n+2}F(z_o)}{D^{n+1}F(z_o)} -(n+2-c)}{(n+1)-(n+1\ -c)\dfrac{D^{n}F(z_o)}{D^{n+1}F(z_o)}} - 2\right\} > \frac{(1-\alpha)-2(n+\alpha)(c+1-\alpha)}{2(n+1)(c+1-\alpha)}.$$

which contradicts (2.9). Hence $|w(z)| < 1$ and from (2.14) it follows that $F(z) \in M_n(\alpha)$.

Putting $n = 0$ and $c = 1$ in the statement of Theorem 2, we obtain the following result.

Corollary 1. If $f(z) \in \Sigma$ and satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \frac{(2-\alpha)(1-2\alpha) - 1}{2(2-\alpha)}$$

then

$$F(z) = \frac{1}{z^2} \int_0^z t\ f(t)\ dt$$

belongs to $\Sigma^*(\alpha)$ for $F(z) \neq 0$ in $0 < |z| < 1$.

Putting $\alpha = 0$ in Corollaey 1 we obtain the following result of Goel and Sohi [3].

Corollary 2. If $f(z) \in \Sigma$ and satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \frac{1}{4}$$

then

$$F(z) = \frac{1}{z^2} \int_0^z t\ f(t)\ dt$$

belongs to Σ^* for $F(z) \neq 0$ in $0 < |z| < 1$.

Remark 1. Corollary 2 extends a result of Bajpai [1].

Theorem 3. If $f(z) \in M_n(\alpha)$, then

$$F(z) = \frac{n+1}{z^{n+2}} \int_0^z t^{n+1} f(t)\, dt \in M_{n+1}(\alpha)$$

for $F(z) \neq 0$ in $0 < |z| < 1$.

Proof. We have

$$cD^n f(z) = (n+1)D^{n+1}F(z) - (n+1-c)D^n F(z)$$

and

$$cD^{n+1} f(z) = (n+2)D^{n+2}F(z) - (n+2-c)D^{n+1}F(z).$$

Taking $c = n+1$ in the above relations we obtain

$$\frac{(n+2)D^{n+2}F(z) - D^{n+1}F(z)}{(n+1)D^{n+1}F(z)} = \frac{D^{n+1}f(z)}{D^n f(z)}.$$

Thus

$$\mathrm{Re}\left\{\frac{n+2}{n+1} \cdot \frac{D^{n+2}F(z)}{D^{n+1}F(z)} - \frac{1}{n+1} - 2\right\} = \mathrm{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)} - 2\right\} < -\frac{n+\alpha}{n+1}$$

from which it follows that

$$\mathrm{Re}\left\{\frac{D^{n+2}F(z)}{D^{n+1}F(z)} - 2\right\} < -\left(\frac{n+\alpha+1}{n+2}\right).$$

This completes the proof of Theorem 3.

Remark 2. Taking $\alpha = 0$ in the above theorems, we get the results obtained by Ganigi and Uralegaddi [2].

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