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ON THE CLASS $\mathcal{Y}$ OPERATORS

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Abstract. In [9], one of the authors proved that, for a M-hyponormal operator A^* and for a dominant operator B, $CA = BC$ implies $CA^* = B^*C$. In the case where A^* and B are normal, this result are well known as the Putnam-Fuglede theorem. In this paper, we will generalize this result to the cases where A^* or both A^* and B belong to some class $\mathcal{Y}$ which includes the class of M-hyponormal operators. And also we prove that every compact operators in the class $\mathcal{Y}$ are normal.

We denote the set of all bounded linear operators on a Hilbert space $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. The following results are well known.

Fuglede's theorem. ([2]) If $T \in \mathcal{B}(\mathcal{H})$ is normal and if $TS = ST$ for some $S \in \mathcal{B}(\mathcal{H})$, then $T^*S = ST^*$.

Putnam's corollary. ([3]) If A^* and B are normal operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively and if C is a bounded linear operator from $\mathcal{H}$ to $\mathcal{K}$ such that $CA = BC$, then $CA^* = B^*C$.

Lemma 1. ([1]) For $A,\ B \in \mathcal{B}(\mathcal{H})$, the following assertions are equivalent.
(i) $A\mathcal{H} \subseteq B\mathcal{H}$.
(ii) $AA^* \leq \lambda^2 BB^*$ for some $\lambda \geq 0$.
(iii) There exists a $C \in \mathcal{B}(\mathcal{H})$ such that $A = BC$.
In particular, there exists a $C \in \mathcal{B}(\mathcal{H})$ uniquely such that

$$(a)\ \ \|C\|^2 = \inf\{\mu\ ;\ AA^* \leq \mu BB^*\}$$
$$(b)\ \ \mathcal{N}_A = \mathcal{N}_C \qquad \text{and} \qquad (c)\ \ C\mathcal{H} \subseteq [B^*\mathcal{H}]^{\sim}.$$

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According to [7] and [8], a bounded linear operator T on a Hilbert space $\mathcal{H}$ is dominant if

$$(T - zI)\mathcal{H} \subseteq (T - zI)^*\mathcal{H} \quad \text{for all } z \in \sigma(T),$$

where $\sigma(T)$ denotes the spectrum of T. This condition is equivalent, by Lemma 1, to the existence of a positive constant M_z for each $z \in \mathbb{C}$ such that

$$(T - zI)(T - zI)^* \leq {M_z}^2(T - zI)^*(T - zI).$$

If there is a constant M such that $M_z \leq M$ for all $z \in \mathbb{C}$, then T is called M-hyponormal, and if $M = 1$, T is hyponormal.

Easily we see the following inclusion relations :

$$\{\text{Hyponormal}\} \subseteq \{M\text{-hyponormal}\} \subseteq \{\text{Dominant}\}.$$

The following results are well known, but, for convenience' sake we state here them as lemmas without proof.

Lemma 2. ([9]) The restriction $T|_{\mathcal{M}}$ of the dominant (respectively, M-hyponormal) operator T to its invariant subspace $\mathcal{M}$ is dominant (respectively, M-hyponormal).

Lemma 3. ([5]) Let $(T - zI)(T - zI)^* \geq D \geq O$ for all $z \in \mathbb{C}$, then, for each $x \in D^{\frac{1}{2}}\mathcal{H}$, there exists a bounded function $f(z) : \mathbb{C} \to \mathcal{H}$ such that $(T - zI)f(z) \equiv x$.

Lemma 4. ([4]) Let T on $\mathcal{H}$ be a normal operator with the spectral decomposition $T = \int \lambda dE_\lambda$ and let $E(\delta)$ be the associated projection measure defined on the Borel set δ of the plane $\mathbb{C}$, then

$$\bigcap_{\omega \notin \delta}(T - \omega I)\mathcal{H} \subseteq E(\delta)\mathcal{H} \subseteq \bigcap_{\omega \notin \bar{\delta}}(T - \omega I)\mathcal{H} \qquad \text{for any Borel set } \delta.$$

As a special case, we have

$$E(\delta)\mathcal{H} = \bigcap_{\omega \notin \delta}(T - \omega I)\mathcal{H} \qquad \text{for any closed set } \delta \subseteq \mathbb{C}.$$

Lemma 5. ([7]) Let $T \in \mathcal{B}(\mathcal{H})$ be dominant. Let $\delta \subseteq \mathbb{C}$ be closed. If there exists a bounded function $f(z) : \mathbb{C} \setminus \delta \to \mathcal{H}$ such that $(T - zI)f(z) \equiv x$ for some non-zero $x \in \mathcal{H}$, then $f(z)$ is analytic on $\mathbb{C} \setminus \delta$.

Lemma 6. ([7]) Let T be dominant and let $\mathcal{M}$ be an invariant subspace of T for which $T|_{\mathcal{M}}$ is normal. Then $\mathcal{M}$ reduces T.

In [9], one of the authors generalized the Putnam's corollary as follows.

Proposition. If $A^* \in \mathcal{B}(\mathcal{H})$ is M-hyponormal and $B \in \mathcal{B}(\mathcal{K})$ is dominant and if C is a bounded linear transformation from $\mathcal{H}$ to $\mathcal{K}$ such that $CA = BC$, then $CA^* = B^*C$.

Moreover, $[\text{range}(C)]^{\sim}$ and $[\text{kernel}(C)]^{\perp}$ are reducing subspaces of B and A respectively and the restrictions $B|_{[\text{range}(C)]^{\sim}}$ and $A|_{[\text{kernel}(C)]^{\perp}}$ are normal.

In Proposition, let $A = B = C$, then $AA^* = A^*A$ and we have the following.

Corollary 1. If A is co-M-hyponormal (i.e., A^* is M-hyponormal) and dominant, then A is normal.

Definition. For a bounded linear operator T on a Hilbert space $\mathcal{H}$, we say that T^* belongs to the class $\mathcal{Y}_\alpha$ for some $\alpha \geq 1$ if there is a positive number K_α such that

$$|TT^* - T^*T|^\alpha \leq K_\alpha{}^2 (T - zI)(T - zI)^* \quad \text{for all } z \in \mathbb{C}.$$

Let $\mathcal{Y} = \bigcup_{\alpha \geq 1} \mathcal{Y}_\alpha$.

Lemma 7. For each α, β such as $1 \leq \alpha < \beta$, we have $\mathcal{Y}_\alpha \subseteq \mathcal{Y}_\beta$.

Proof.

$$\begin{aligned} |TT^* - T^*T|^\beta &= |TT^* - T^*T|^{\frac{\alpha}{2}} |TT^* - T^*T|^{\beta-\alpha} |TT^* - T^*T|^{\frac{\alpha}{2}} \\ &\leq \|TT^* - T^*T\|^{\beta-\alpha} |TT^* - T^*T|^\alpha \\ &\leq \{2\|T\|^2\}^{\beta-\alpha} K_\alpha{}^2 (T - zI)(T - zI)^* \\ &= K_\beta{}^2 (T - zI)(T - zI)^*, \quad \text{where } K_\beta{}^2 = \{2\|T\|^2\}^{\beta-\alpha} K_\alpha{}^2. \end{aligned}$$

Since, for each $z \in \mathbb{C}$,

$$TT^* - T^*T = (T - zI)(T - zI)^* - (T - zI)^*(T - zI),$$

we have $TT^* - T^*T \le (T - zI)(T - zI)^*$. And if T^* is hyponormal, then

$$|TT^* - T^*T| = TT^* - T^*T \le (T - zI)(T - zI)^*$$

and $T^* \in \mathcal{Y}_1$. Therefore we have the following.

Lemma 8. If T^* is hyponormal, then $T^* \in \mathcal{Y}_1$.

Conversely if $T^* \in \mathcal{Y}_1$, then we have the following.

Lemma 9. If $T^* \in \mathcal{Y}_1$, then T^* is M-hyponormal.

Proof. Let $TT^* - T^*T = V|TT^* - T^*T|$ be the polar decomposition of $TT^* - T^*T$, then V is a Hermitian partial isometry and commutes with $|TT^* - T^*T|$ because $TT^* - T^*T$ is Hermitian. Hence, for any $x \in \mathcal{H}$ such as $\|x\| = 1$, we have

$$\begin{aligned}|\langle (TT^* - T^*T)x,\ x\rangle| &= |\langle |TT^* - T^*T|^{\frac{1}{2}}x,\ |TT^* - T^*T|^{\frac{1}{2}}V^*x\rangle| \\ &\le \|\ |TT^* - T^*T|^{\frac{1}{2}}x\|\ \|\ |TT^* - T^*T|^{\frac{1}{2}}V^*x\| \\ &= \|\ |TT^* - T^*T|^{\frac{1}{2}}x\|^2 = \langle |TT^* - T^*T|x,\ x\rangle\end{aligned}$$

and $\pm(TT^* - T^*T) \le |TT^* - T^*T|$. And, if $T^* \in \mathcal{Y}_1$, then, for each $z \in \mathbb{C}$,

$$\begin{aligned}(T - zI)^*(T - zI) &= (T - zI)(T - zI)^* - (TT^* - T^*T) \\ &\le (T - zI)(T - zI)^* + |TT^* - T^*T| \\ &\le (T - zI)(T - zI)^* + {K_1}^2(T - zI)(T - zI)^* \\ &= M^2(T - zI)(T - zI)^*, \quad \text{where } M^2 = 1 + {K_1}^2.\end{aligned}$$

In [6], Radjabalipour proved the following.

Lemma 10. If T^* is M-hyponormal, then $T^* \in \mathcal{Y}_2$.

Proof. If T^* is M-hyponormal, then, by Lemma 1, there exists a family of operators C_z such that $\|C_z\| \le M$ and $(T-zI)^* = (T-zI)C_z$. Thus, for all $z \in \mathbb{C}$,

$$\begin{aligned} TT^* - T^*T &= (T-zI)(T-zI)^* - (T-zI)^*(T-zI) \\ &= (T-zI)[(T-zI)^* - C_z(T-zI)] \end{aligned}$$

and

$$|TT^* - T^*T|^2 = (T-zI)[(T-zI)^* - C_z(T-zI)][(T-zI)^* - C_z(T-zI)]^*(T-zI)^*.$$

And, for $|z| > \|T\| + 1$, we have

$$TT^* - T^*T = (T-zI)(T-zI)^{-1}(TT^* - T^*T)$$

and

$$|TT^* - T^*T|^2 = (T-zI)(T-zI)^{-1}(TT^* - T^*T)^2(T-zI)^{-1*}(T-zI)^*.$$

Choose K_2 is larger than $2\|T\|^2$ and $(1+M)(2\|T\|+1)$, then $T^* \in \mathcal{Y}_2$ because

$$\begin{aligned} \sup_{|z|\le\|T\|+1} \|(T-zI)^* - C_z(T-zI)\| &\le \sup_{|z|\le\|T\|+1} \|(T-zI)\|(1+\|C_z\|) \\ &\le (2\|T\|+1)(1+M) \le K_2 \end{aligned}$$

and

$$\begin{aligned} \sup_{|z|>\|T\|+1} \|(T-zI)^{-1}(TT^* - T^*T)\| &\le \sup_{|z|>\|T\|+1} \|(T-zI)^{-1}\| \; \|TT^* - T^*T\| \\ &\le \|TT^* - T^*T\| \le 2\|T\|^2 \le K_2. \end{aligned}$$

Firstly, we can generalize Proposition as follows.

Theorem 1. If $A^* \in \mathcal{B}(\mathcal{H})$ is in the class $\mathcal{Y}$ and $B \in \mathcal{B}(\mathcal{K})$ is dominant and if C is a bounded linear transformation from $\mathcal{H}$ to $\mathcal{K}$ such that $CA = BC$, then $CA^* = B^*C$.

Moreover, $[\text{range}(C)]^\sim$ and $[\text{kernel}(C)]^\perp$ are reducing subspaces of B and A respectively and the restrictions $B|_{[\text{range}(C)]^\sim}$ and $A|_{[\text{kernel}(C)]^\perp}$ are normal.

Proof. By Lemma 7, there is an integer $n > 1$ such that $A^* \in \mathcal{Y}_{2^n}$. Then there exists $K_{2^n} > 0$ such that

$$|AA^* - A^*A|^{2^n} \leq K_{2^n}{}^2(A - zI)(A - zI)^* \quad \text{for all } z \in \mathbb{C}.$$

And then, by Lemma 3, for each $x \in |AA^* - A^*A|^{2^{n-1}}\mathcal{H}$, there exists a bounded function $f(z) : \mathbb{C} \to \mathcal{H}$ such that $(A - zI)f(z) \equiv x$. But then

$$Cx \equiv C(A - zI)f(z) = (B - zI)Cf(z) \quad \text{for all } z \in \mathbb{C}.$$

If $Cx \neq o$, then, by Lemma 5, $Cf(z)$ is a bounded entire function and hence it is constant by Liouville's theorem. And hence $Cx = o$ because

$$Cf(z) = \lim_{z \to \infty}(B - zI)^{-1}Cx = o.$$

This contradiction implies that $C|AA^* - A^*A|^{2^{n-1}}\mathcal{H} = \{o\}$ and hence

$$C|AA^* - A^*A|^2\mathcal{H} = \{o\}. \tag{1}$$

From the equation $CA = BC$ we know that $[\text{range}(C)]^\sim$ and $[\text{kernel}(C)]^\perp$ are invariant subspaces of B and A^* respectively. By the decompositions

$$\mathcal{H} = [\text{kernel}(C)]^\perp \oplus [\text{kernel}(C)] \quad \text{and} \quad \mathcal{K} = [\text{range}(C)]^\sim \oplus [\text{range}(C)]^\perp,$$

we have

$$A = \begin{pmatrix} A_1 & O \\ T & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & S \\ O & B_2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} C_1 & O \\ O & O \end{pmatrix}.$$

Hence C_1 is injective, with dense range, $C_1A_1 = B_1C_1$ and $B_1 = B|_{[\text{range}(C)]^\sim}$ is dominant by Lemma 2. Since

$$\begin{aligned} AA^* - A^*A &= \begin{pmatrix} A_1 & O \\ T & A_2 \end{pmatrix}\begin{pmatrix} A_1{}^* & T^* \\ O & A_2{}^* \end{pmatrix} - \begin{pmatrix} A_1{}^* & T^* \\ O & A_2{}^* \end{pmatrix}\begin{pmatrix} A_1 & O \\ T & A_2 \end{pmatrix} \\ &= \begin{pmatrix} A_1A_1{}^* - A_1{}^*A_1 - T^*T & A_1T^* - T^*A_2 \\ (A_1T^* - T^*A_2)^* & TT^* + A_2A_2{}^* - A_2{}^*A_2 \end{pmatrix} \\ &= \begin{pmatrix} A_1A_1{}^* - A_1{}^*A_1 - T^*T & E_1 \\ E_1{}^* & F_1 \end{pmatrix}, \end{aligned}$$

we have

$$|AA^* - A^*A|^2 = \begin{pmatrix} (A_1A_1{}^* - A_1{}^*A_1 - T^*T)^2 + E_1E_1{}^* & E_2 \\ E_2{}^* & F_2 \end{pmatrix}$$

and since, by (1), $C|AA^* - A^*A|^2[\text{kernel}(C)]^\perp = \{o\}$, we have

$$C_1[(A_1A_1{}^* - A_1{}^*A_1 - T^*T)^2 + E_1E_1{}^*] = O$$

and $(A_1A_1{}^* - A_1{}^*A_1 - T^*T)^2 + E_1E_1{}^* = O$ because C_1 is injective and hence

$$A_1A_1{}^* - A_1{}^*A_1 - T^*T = O. \tag{2}$$

This implies that $A_1{}^*$ is hyponormal. And then, by Proposition, A_1 and B_1 are normal and $T = O$ by (2). Therefore $[\text{kernel}(C)]^\perp$ reduces A. And also $[\text{range}(C)]^\sim$ reduces B by Lemma 6. Hence we have

$$A = \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & O \\ O & B_2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} C_1 & O \\ O & O \end{pmatrix},$$

where A_1 and B_1 are normal and C_1 is injective and with dense range. Since $CA = BC$, $C_1A_1 = B_1C_1$ and $C_1A_1{}^* = B_1{}^*C_1$ by Putnam's corollary. Therefore $CA^* = B^*C$.

Corollary 2. If A^* is in the class $\mathcal{Y}$ and if A is dominant, then A is normal.

Proof. In Theorem 1, let $A = B = C$, then $AA^* = A^*A$.

There is an example of compact operator which is dominant, co-dominant (i.e., its adjoint operator is dominant) and not normal. The following example is due to [7].

Example. Let $\{f_n\}_{-\infty}^{\infty}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. Define $Tf_n = 2^{-|n|}f_{n+1}$. Then $T^*f_{n+1} = 2^{-|n|}f_n$ and clearly T is non-normal.

Since $T^*Tf_n = 2^{-|n|}T^*f_{n+1} = 2^{-2|n|}f_n$, T^*T is a compact operator and hence both T and T^* are also compact operators.

By the definition of T, $\sigma_p(T) = \emptyset$ and $\sigma(T) = \{0\}$ by the compactness of T, where $\sigma_p(T)$ denotes the point spectrum of T.

For

$$x = \sum_{n=-\infty}^{\infty} \alpha_n f_n \text{ and } y = \sum_{n=-\infty}^{\infty} \beta_n f_n,$$

we have

$$Tx = \sum_{n=-\infty}^{\infty} \alpha_n 2^{-|n|} f_{n+1} \text{ and } T^*y = \sum_{n=-\infty}^{\infty} \beta_{n+1} 2^{-|n|} f_n$$

and

$$Tx = T^*y \Leftrightarrow \alpha_n 2^{-|n|} = \beta_{n+2} 2^{-|n+1|} \text{ for all } n = 0, \pm 1, \pm 2, \cdots .$$

And since

$$\sum_{n=-\infty}^{\infty} |\alpha_n|^2 < +\infty \Leftrightarrow \sum_{n=-\infty}^{\infty} |\beta_n|^2 < +\infty$$

because

$$\frac{\beta_{n+2}}{\alpha_n} = 2^{|n+1|-|n|} = \begin{cases} 2, & (n = 0, 1, 2, \cdots) \\ \frac{1}{2}, & (n = -1, -2, \cdots) \end{cases},$$

$T\mathcal{H} = T^*\mathcal{H}$ and both T and T^* are dominant because $\sigma(T) = \{0\}$.

And, by Corollary 1, both T and T^* are not class $\mathcal{Y}$.

And hence it is natural to consider the generalized Putnam's corollary in the cases where both A^* and B are in the class $\mathcal{Y}$. For our purpose we need the following lemmas.

Lemma 11. If $T \in \mathcal{Y}$, then $Tx = \lambda x$ implies $T^*x = \overline{\lambda} x$.

Proof. If $T \in \mathcal{Y}$, then $T \in \mathcal{Y}_\alpha$ for some $\alpha \geq 1$ and there exists a positive number K_α such that

$$|TT^* - T^*T|^\alpha \leq K_\alpha{}^2 (T - zI)^*(T - zI) \text{ for all } z \in \mathbb{C}.$$

And $Tx = \lambda x$ implies $|TT^* - T^*T|^{\frac{\alpha}{2}} x = o$ and $(TT^* - T^*T)x = o$ and hence $\|(T - \lambda I)^* x\| = \|(T - \lambda I)x\| = 0$.

Lemma 12. If $T \in \mathcal{Y}$ and if $\mathcal{M}$ is an invariant subspace of T for which $T|_\mathcal{M}$ is normal, then $\mathcal{M}$ reduces T.

Proof. Let

$$T = \begin{pmatrix} N & A \\ O & B \end{pmatrix} \text{ on } \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp.$$

Since $T \in \mathcal{Y}$, by Lemma 7, there is an integer $n > 1$ such that $T \in \mathcal{Y}_{2^n}$. Then there exists $K_{2^n} > 0$ such that

$$|TT^* - T^*T|^{2^n} \leq {K_{2^n}}^2 (T - zI)^*(T - zI) \text{ for all } z \in \mathbb{C}.$$

Since

$$\begin{aligned} TT^* - T^*T &= \begin{pmatrix} N & A \\ O & B \end{pmatrix} \begin{pmatrix} N^* & O \\ A^* & B^* \end{pmatrix} - \begin{pmatrix} N^* & O \\ A^* & B^* \end{pmatrix} \begin{pmatrix} N & A \\ O & B \end{pmatrix} \\ &= \begin{pmatrix} NN^* + AA^* - N^*N & AB^* - N^*A \\ (AB^* - N^*A)^* & BB^* - A^*A - B^*B \end{pmatrix} \\ &= \begin{pmatrix} AA^* & E_1 \\ {E_1}^* & F_1 \end{pmatrix}, \end{aligned}$$

we have

$$|TT^* - T^*T|^2 = \begin{pmatrix} (AA^*)^2 + E_1 {E_1}^* & E_2 \\ {E_2}^* & F_2 \end{pmatrix}$$

and, by repeating this, we have

$$\begin{aligned} &|TT^* - T^*T|^{2^n} \\ &= \begin{pmatrix} (\cdots(((AA^*)^2 + E_1{E_1}^*)^2 + E_2{E_2}^*)^2 \cdots)^2 + E_{n-1}{E_{n-1}}^* & E_n \\ {E_n}^* & F_n \end{pmatrix}. \end{aligned}$$

And since

$$\begin{aligned} (T - zI)^*(T - zI) &= \begin{pmatrix} (N - zI)^* & O \\ A^* & (B - zI)^* \end{pmatrix} \begin{pmatrix} (N - zI) & A \\ O & (B - zI) \end{pmatrix} \\ &= \begin{pmatrix} (N - zI)^*(N - zI) & (N - zI)^*A \\ ((N - zI)^*A)^* & A^*A + (B - zI)^*(B - zI) \end{pmatrix}, \end{aligned}$$

we have

$$\begin{aligned} D &= (\cdots(((AA^*)^2 + E_1{E_1}^*)^2 + E_2{E_2}^*)^2 \cdots)^2 + E_{n-1}{E_{n-1}}^* \\ &\leq {K_{2^n}}^2 (N - zI)^*(N - zI) \text{ for all } z \in \mathbb{C}. \end{aligned}$$

And, by Lemma 1, $D^{\frac{1}{2}}\mathcal{M} \subseteq (N - zI)^*\mathcal{M}$ for all $z \in \mathbb{C}$ and, by Lemma 4,

$$D^{\frac{1}{2}}\mathcal{M} \subseteq \bigcap_{z \in \mathbb{C}} (N^* - \bar{z}I)\mathcal{M} \subseteq E(\emptyset)\mathcal{M} = \{o\},$$

where $E(\cdot)$ denotes the spectral measure of N and hence $D = O$. And then $AA^* = O$ and $A = O$. Therefore $\mathcal{M}$ reduces T.

Lemma 13. If $T \in \mathcal{Y}_\alpha$ for some $\alpha \geq 1$ and if, for a closed set $\delta \subseteq \mathbb{C}$, there exist a bounded function $f(z) : \mathbb{C} \setminus \delta \to \mathcal{H}$ and a non-zero $x \in \mathcal{H}$ such that $(T - zI)f(z) \equiv x$, then $g(z) = (I - E(\{0\}))f(z)$ is analytic on $\mathbb{C} \setminus \delta$ where $E(\cdot)$ denotes the spectral measure of $|TT^* - T^*T|^{\frac{\alpha}{2}}$. Moreover, if $0 \notin \sigma_p(TT^* - T^*T)$, then $f(z)$ is analytic on $\mathbb{C} \setminus \delta$.

Proof. We may assume that T is completely non-normal (i.e., T has no normal part) because the assertions of Lemma 13 for normal operator are clear by Lemma 5. And then, by Lemmas 11 and 12, we may assume that $\sigma_p(T) = \emptyset$ and hence f is weakly continuous. In fact, if weak $\lim f(\lambda_n) = u \neq f(z)$ for $\lambda_n \to z \in \mathbb{C} \setminus \delta$, then, for any $h \in \mathcal{H}$,

$$\begin{aligned}
\langle (T - zI)f(z),\ h\rangle &= \langle x,\ h\rangle = \langle (T - \lambda_n I)f(\lambda_n),\ h\rangle \\
&= \langle f(\lambda_n),\ (T - \lambda_n I)^* h\rangle \\
&\to \langle f(\lambda_n),\ (T - zI)^* h\rangle\ (n \to \infty) \quad \text{because } f \text{ is bounded} \\
&\to \langle u,\ (T - zI)^* h\rangle\ (n \to \infty) \\
&= \langle (T - zI)u,\ h\rangle
\end{aligned}$$

and $f(z) - u$ would be a non-zero eigenvector of $T - zI$.

Let Δ be a triangle in $\mathbb{C} \setminus \delta$. Define $u = \int_{\partial\Delta} f(\lambda)d\lambda$, where the integral exists in the weak topology. Since, for any $\varepsilon > 0$, $E([\varepsilon,\ \infty))u \in |TT^* - T^*T|^{\frac{\alpha}{2}}\mathcal{H}$ and since $|TT^* - T^*T|^{\frac{\alpha}{2}}\mathcal{H} \subseteq (T - zI)^*\mathcal{H}$ for all $z \in \mathbb{C}$ by Lemma 1, $E([\varepsilon,\ \infty))u = (T - zI)^* v(z)$ for all $z \in \text{Int}\Delta$ and some $v(z) \in \mathcal{H}$.

We now show that $\langle f(\lambda),\ E([\varepsilon,\ \infty))u\rangle$ is analytic in $\text{Int}\Delta$.

Fix $z \in \text{Int}\Delta$. Then

$$\begin{aligned}
\lim_{\lambda \to z}\left\langle \frac{f(\lambda) - f(z)}{\lambda - z},\ E([\varepsilon,\ \infty))u\right\rangle &= \lim_{\lambda \to z}\left\langle \frac{f(\lambda) - f(z)}{\lambda - z},\ (T - zI)^* v(z)\right\rangle \\
&= \lim_{\lambda \to z}\left\langle \frac{(T - zI)f(\lambda) - (T - zI)f(z)}{\lambda - z},\ v(z)\right\rangle \\
&= \lim_{\lambda \to z}\left\langle \frac{(T - \lambda I)f(\lambda) + (\lambda - z)f(\lambda) - x}{\lambda - z},\ v(z)\right\rangle \\
&= \lim_{\lambda \to z}\left\langle \frac{x + (\lambda - z)f(\lambda) - x}{\lambda - z},\ v(z)\right\rangle \\
&= \lim_{\lambda \to z}\langle f(\lambda),\ v(z)\rangle = \langle f(z),\ v(z)\rangle.
\end{aligned}$$

The function $\langle f(\lambda),\ u\rangle$ is analytic in IntΔ and continuous on the boundary. Thus

$$\|E([\varepsilon,\ \infty))u\|^2 = \langle u,\ E([\varepsilon,\ \infty))u\rangle = \int_{\partial\Delta} \langle f(\lambda),\ E([\varepsilon,\ \infty))u\rangle d\lambda = 0$$

by Cauchy's integral theorem and hence $E([\varepsilon,\ \infty))u = o$. Since $\varepsilon > 0$ is arbitrary, $u = E(\{0\})u \in E(\{0\})\mathcal{H}$.

Let $g(z) = (I - E(\{0\}))f(z)$ and let $h(z) = E(\{0\})f(z)$. Then g and h are bounded and wearly continuous on $\mathbb{C} \setminus \delta$ and

$$u = \int_{\partial\Delta} f(\lambda)d\lambda = \int_{\partial\Delta} g(\lambda)d\lambda + \int_{\partial\Delta} h(\lambda)d\lambda$$

and hence

$$\int_{\partial\Delta} g(\lambda)d\lambda = u - \int_{\partial\Delta} h(\lambda)d\lambda \in (I - E(\{0\}))\mathcal{H} \cap E(\{0\})\mathcal{H} = \{0\}.$$

Therefore $\int_{\partial\Delta} g(\lambda)d\lambda = o$ for any triangle in $\mathbb{C} \setminus \delta$ and g is analytic on $\mathbb{C} \setminus \delta$ by Morera's theorem.

If $0 \notin \sigma_p(TT^* - T^*T)$, then $E(\{0\}) = O$ and $f(z) = g(z)$ is analytic on $\mathbb{C} \setminus \delta$.

Lemma 14. If $T \in \mathcal{Y}_\alpha$ for some $\alpha \geq 1$ and if there exist a bounded function $f(z) : \mathbb{C} \to \mathcal{H}$ and an $x \in \mathcal{H}$ such that $(T - zI)f(z) \equiv x$, then $x = o$.

Proof. In this proof, we use the same notations as in the proof of Lemma 13. By Lemma 13, $g(z)$ is a bounded entire function and, by Liouville's theorem, it is a constant. And $g(z) = \lim_{\lambda\to\infty} (I - E(\{0\}))f(\lambda) = o$ and $f(z) = h(z) \in E(\{0\})\mathcal{H}$ for all $z \in \mathbb{C}$.

Since, for any positive integer n, $(T - zI)T^n f(z) = T^n(T - zI)f(z) \equiv T^n x$, by the same reasons as above, $T^n f(z) \in E(\{0\})\mathcal{H}$ for all $z \in \mathbb{C}$. And then

$$\begin{aligned} o &= (TT^* - T^*T)T^n f(z) \\ &= \{(T - zI)T^* - T^*(T - zI)\}T^n f(z) = (T - zI)T^*T^n f(z) - T^*T^n x \end{aligned}$$

and $(T - zI)T^*T^n f(z) \equiv T^*T^n x$ and hence we have also $T^*T^n f(z) \in E(\{0\})\mathcal{H}$ for all $z \in \mathbb{C}$. And then

$$\begin{aligned} o &= (TT^* - T^*T)T^*T^n f(z) \\ &= \{(T - zI)T^* - T^*(T - zI)\}T^*T^n f(z) = (T - zI)T^{*2}T^n f(z) - T^{*2}T^n x \end{aligned}$$

and $(T - zI)T^{*2}T^n f(z) \equiv T^{*2}T^n x$ and hence $T^{*2}T^n f(z) \in E(\{0\})\mathcal{H}$ for all $z \in \mathbb{C}$. By repeating this argument, for any non-negative integers m, we have $(T - zI)T^{*m}T^n f(z) \equiv T^{*m}T^n x$ and $T^{*m}T^n f(z) \in E(\{0\})\mathcal{H}$ for all $z \in \mathbb{C}$.

Particularly, $(T-zI)T^{*m} f(z) \equiv T^{*m}x$ and $T^{*m} f(z) \in E(\{0\})\mathcal{H}$ for all $z \in \mathbb{C}$ and $(T - zI)T^nT^{*m} f(z) = T^n(T - zI)T^{*m} f(z) \equiv T^nT^{*m}x$ and hence $T^nT^{*m} f(z) \in E(\{0\})\mathcal{H}$ for all $z \in \mathbb{C}$.

Therefore we have, for any non-negative integers m, n and for all $z \in \mathbb{C}$,

$$\begin{aligned} T^nT^{*m} f(z) &= T^{n-1}TT^*T^{*m-1} f(z) = T^{n-1}T^*TT^{*m-1} f(z) \\ &= \cdots = T^{n-1}T^{*m}T f(z) \\ &= \cdots = T^{*m}T^n f(z). \end{aligned}$$

Let

$$\mathcal{N} = \vee\{T^nT^{*m} f(z) \;:\; m,\ n = 0, 1, 2, \cdots,\ z \in \mathbb{C}\}.$$

Then $\mathcal{N}$ reduces T and the restriction $T|_{\mathcal{N}}$ is normal. And let $F(\cdot)$ be the spectral measure of $T|_{\mathcal{N}}$. Then

$$x \equiv (T - zI)f(z) = (T|_{\mathcal{N}} - zI)f(z) \in \bigcap_{z\in\mathbb{C}} (T|_{\mathcal{N}} - zI)\mathcal{N} = F(\emptyset)\mathcal{N} = \{o\}$$

by Lemma 4 and $x = o$.

Now we can generalize Proposition as follows.

Theorem 2. If both $A^* \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ are in the class $\mathcal{Y}$ and if C is a bounded linear transformation from $\mathcal{H}$ to $\mathcal{K}$ such that $CA = BC$, then $CA^* = B^*C$.

Moreover, $[\text{range}(C)]^{\sim}$ and $[\text{kernel}(C)]^{\perp}$ are reducing subspaces of B and A respectively and the restrictions $B|_{[\text{range}(C)]^{\sim}}$ and $A|_{[\text{kernel}(C)]^{\perp}}$ are normal.

Proof. By Lemma 7, there is an integer $n > 1$ such that $A^* \in \mathcal{Y}_{2^n}$. Then there exists $K_{2^n} > 0$ such that

$$|AA^* - A^*A|^{2^n} \leq K_{2^n}{}^2(A - zI)(A - zI)^* \quad \text{for all } z \in \mathbb{C}.$$

And then, by Lemma 3, for each $x \in |AA^* - A^*A|^{2^{n-1}}\mathcal{H}$, there exists a bounded function $f(z) : \mathbb{C} \to \mathcal{H}$ such that $(A - zI)f(z) \equiv x$. But then

$$Cx \equiv C(A - zI)f(z) = (B - zI)Cf(z) \quad \text{for all } z \in \mathbb{C}.$$

Since $B \in \mathcal{Y}$, $B \in \mathcal{Y}_\alpha$ for some $\alpha \geq 1$ by Lemma 7 and hence $Cx = o$ by Lemma 14. This implies that $C|AA^* - A^*A|^{2^{n-1}}\mathcal{H} = \{o\}$ and hence

$$C|AA^* - A^*A|^2\mathcal{H} = \{o\}. \tag{1}$$

From the equation $CA = BC$ we know that $[\text{range}(C)]^\sim$ and $[\text{kernel}(C)]^\perp$ are invariant subspaces of B and A^* respectively. By the decompositions

$$\mathcal{H} = [\text{kernel}(C)]^\perp \oplus [\text{kernel}(C)\} \quad \text{and} \quad \mathcal{K} = [\text{range}(C)]^\sim \oplus [\text{range}(C)]^\perp,$$

we have

$$A = \begin{pmatrix} A_1 & O \\ T & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & S \\ O & B_2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} C_1 & O \\ O & O \end{pmatrix}.$$

Hence C_1 is injective, with dense range and $C_1A_1 = B_1C_1$. Since

$$\begin{aligned} AA^* - A^*A &= \begin{pmatrix} A_1 & O \\ T & A_2 \end{pmatrix}\begin{pmatrix} A_1{}^* & T^* \\ O & A_2{}^* \end{pmatrix} - \begin{pmatrix} A_1{}^* & T^* \\ O & A_2{}^* \end{pmatrix}\begin{pmatrix} A_1 & O \\ T & A_2 \end{pmatrix} \\ &= \begin{pmatrix} A_1A_1{}^* - A_1{}^*A_1 - T^*T & A_1T^* - T^*A_2 \\ (A_1T^* - T^*A_2)^* & TT^* + A_2A_2{}^* - A_2{}^*A_2 \end{pmatrix} \\ &= \begin{pmatrix} A_1A_1{}^* - A_1{}^*A_1 - T^*T & E_1 \\ E_1{}^* & F_1 \end{pmatrix}, \end{aligned}$$

we have

$$|AA^* - A^*A|^2 = \begin{pmatrix} (A_1A_1{}^* - A_1{}^*A_1 - T^*T)^2 + E_1E_1{}^* & E_2 \\ E_2{}^* & F_2 \end{pmatrix}$$

and since, by (1), $C|AA^* - A^*A|^2[\text{kernel}(C)]^\perp = \{o\}$, we have

$$C_1[(A_1A_1{}^* - A_1{}^*A_1 - T^*T)^2 + E_1E_1{}^*] = O$$

and $(A_1A_1{}^* - A_1{}^*A_1 - T^*T)^2 + E_1E_1{}^* = O$ because C_1 is injective and hence

$$A_1A_1{}^* - A_1{}^*A_1 - T^*T = O. \tag{2}$$

This implies that $A_1{}^*$ is hyponormal.

And since $B \in \mathcal{Y}$, by Lemma 7, there is an integer $m > 1$ such that $T \in \mathcal{Y}_{2^m}$. Then there exists $K_{2^m} > 0$ such that

$$|BB^* - B^*B|^{2^m} \leq K_{2^m}{}^2(B - zI)^*(B - zI) \quad \text{for all } z \in \mathbb{C}.$$

Since

$$BB^* - B^*B = \begin{pmatrix} B_1 & S \\ O & B_2 \end{pmatrix}\begin{pmatrix} B_1{}^* & O \\ S^* & B_2{}^* \end{pmatrix} - \begin{pmatrix} B_1{}^* & O \\ S^* & B_2{}^* \end{pmatrix}\begin{pmatrix} B_1 & S \\ O & B_2 \end{pmatrix}$$
$$= \begin{pmatrix} B_1B_1{}^* + SS^* - B_1{}^*B_1 & SB_2{}^* - B_1{}^*S \\ (SB_2{}^* - B_1{}^*S)^* & B_2B_2{}^* - S^*S - B_2{}^*B_2 \end{pmatrix}$$
$$= \begin{pmatrix} B_1B_1{}^* + SS^* - B_1{}^*B_1 & G_1 \\ G_1{}^* & H_1 \end{pmatrix},$$

we have

$$|BB^* - B^*B|^2 = \begin{pmatrix} (B_1B_1{}^* + SS^* - B_1{}^*B_1)^2 + G_1G_1{}^* & G_2 \\ G_2{}^* & H_2 \end{pmatrix}$$

and, by repeating this, we have

$$|BB^* - B^*B|^{2^m}$$
$$= \begin{pmatrix} (\cdots(((B_1B_1{}^* + SS^* - B_1{}^*B_1)^2 + G_1G_1{}^*)^2 + G_2G_2{}^*)^2 \cdots)^2 + G_{m-1}G_{m-1}{}^* & G_m \\ G_m{}^* & H_m \end{pmatrix}.$$

And since

$$(B - zI)^*(B - zI) = \begin{pmatrix} (B_1 - zI)^* & O \\ S^* & (B_2 - zI)^* \end{pmatrix}\begin{pmatrix} (B_1 - zI) & S \\ O & (B_2 - zI) \end{pmatrix}$$
$$= \begin{pmatrix} (B_1 - zI)^*(B_1 - zI) & (B_1 - zI)^*S \\ ((B_1 - zI)^*S)^* & S^*S + (B_2 - zI)^*(B_2 - zI) \end{pmatrix},$$

we have

$$D = (\cdots(((B_1B_1{}^* + SS^* - B_1{}^*B_1)^2 + G_1G_1{}^*)^2 + G_2G_2{}^*)^2 \cdots)^2 + G_{m-1}G_{m-1}{}^*$$
$$\leq K_{2^m}{}^2(B_1 - zI)^*(B_1 - zI) \quad \text{for all } z \in \mathbb{C}.$$

And for each $x \in D^{\frac{1}{2}}[\mathrm{range}(C)]^\sim$, by Lemma 3, there exists a bounded function $f(z) : \mathbb{C} \to [\mathrm{range}(C)]^\sim$ such that $(B_1{}^* - zI)f(z) \equiv x$. Since $C_1A_1 = B_1C_1$, $C_1{}^*B_1{}^* = A_1{}^*C_1{}^*$ and $C_1{}^*x \equiv C_1{}^*(B_1{}^* - zI)f(z) = (A_1{}^* - zI)C_1{}^*f(z)$ and hence $C_1{}^*x = o$ by Lemma 14 because $A_1{}^*$ is hyponormal and $A_1{}^* \in \mathcal{Y}_1$ by Lemma 8. And then $x = o$. This implies $D = O$ and $B_1B_1{}^* + SS^* - B_1{}^*B_1 = O$ and hence B_1 is also hyponormal. Therefore, by Proposition, A_1 and B_1 are normal and $C_1A_1{}^* = B_1{}^*C_1$. Since the normality of A_1 and B_1 implies that $T = O$ and $S = O$ and hence we have

$$A = \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & O \\ O & B_2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} C_1 & O \\ O & O \end{pmatrix}.$$

Therefore $CA^* = B^*C$.

Corollary 3. If both A^* and A are in the class $\mathcal{Y}$, then A is normal.

Proof. In Theorem 2, let $A = B = C$, then $AA^* = A^*A$.

Concerning the normality of operators in the class $\mathcal{Y}$, we have the following.

Theorem 3. If $T \in \mathcal{B}(\mathcal{H})$ is in the class $\mathcal{Y}$ and if $\sigma(T) = \{0\}$, then $T = O$.

Proof. If $T \in \mathcal{Y}$, then $T \in \mathcal{Y}_\alpha$ for some $\alpha \geq 1$ and there exists a positive number K_α such that

$$|TT^* - T^*T|^\alpha \leq K_\alpha{}^2(T - zI)^*(T - zI) \quad \text{for all } z \in \mathbb{C}.$$

And for each $x \in |TT^* - T^*T|^{\frac{\alpha}{2}}\mathcal{H}$, by Lemma 3, there exists a bounded function $f(z) : \mathbb{C} \to \mathcal{H}$ such that $(T^* - zI)f(z) \equiv x$ and $f(z) = (T^* - zI)^{-1}x$ is analytic on $\mathbb{C} \setminus \{0\}$ by the assumption. Hence $z = 0$ is a removable singular point of $f(z)$ by Riemann's theorem and, by Liouville's theorem, $f(z)$ is a constant and hence $f(z) = \lim_{z \to \infty} (T^* - zI)^{-1}x = o$ and $x = o$. Since $x \in |TT^* - T^*T|^{\frac{\alpha}{2}}\mathcal{H}$ is arbitrary, $TT^* = T^*T$ and $T = O$ because $\sigma(T) = \{0\}$.

Corollary 4. If $T \in \mathcal{Y}$ is compact, then T is normal.

Proof. Since T is compact, each non-zero $\lambda \in \sigma(T)$ is a point spectrum of T.

Let $\mathcal{M}_\lambda = \{x \in \mathcal{H} \ : \ Tx = \lambda x\}$. Then $\mathcal{M}_\lambda$ is a reducing subspace of T and $\mathcal{M}_\lambda \perp \mathcal{M}_\mu$ for $\lambda \neq \mu$ by Lemma 11. And let $\mathcal{M} = \bigoplus_{\lambda \in \sigma_p(T)} \mathcal{M}_\lambda$. Then $T = T|_\mathcal{M} \oplus T|_{\mathcal{M}^\perp}$ where $T|_\mathcal{M}$ is normal and $\sigma(T|_{\mathcal{M}^\perp}) = \{0\}$. Since $T|_{\mathcal{M}^\perp} \in \mathcal{Y}$, $T|_{\mathcal{M}^\perp} = O$ by Theorem 3.

As a special case, we have the following.

Corollary 5. Every compact M-hyponormal operators are normal.

Proof. It is clear by Lemma 10 and by Corollary 4.

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