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ENVELOPE OF HYPERHOLOMORPHY AND HYPERHOLOMORPHIC CONVEXITY

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Nôno investigated the hyperholomorphy of functions of quaternion variables in [13], [14], [15] and [16]. In the present paper, making use of his results [13] and [16] on series expansion and integral representation, for a Riemann's domain (Ω, φ) over $\mathbf{C}^2 \times \mathbf{C}^2$, we define the envelope $(\tilde{\Omega}, \tilde{\varphi})$ of hyperholomorphy of the domain (Ω, φ) and prove that the domain $(\tilde{\Omega}, \tilde{\varphi})$ is hyperholomorphically convex.

1. Hyperholomorphic function on a domain in $\mathbf{C}^2 \times \mathbf{C}^2$

The field $\mathcal{H}$ of quaternions

(1) $$z = x_1 + ix_2 + jx_3 + kx_4, \qquad x_1, x_2, x_3, x_4 \in \mathbf{R}$$

is a four dimensional non-commutative $\mathbf{R}$-field generated by four base elements 1, i, j and k with the following non-commutative multiplication rule:

(2) $$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

R. Fueter[7]-[8] and his school established the theory of quaternionic functions, called *regular functions*, of a quaternionic variable. F. Brackx[1]-[2] also developped the theory of quaternionic functions, called *monogenic functions*, of a quaternionic variable in the view point of (1).

By the second view point, an element z of the field $\mathcal{H}$ of quaternions is regarded as

(3) $$z = x_1 + P, \qquad P = ix_2 + jx_3 + kx_4, \qquad x_1, x_2, x_3, x_4 \in \mathbf{R}.$$

In the fashion of (3), C. A. Deavours[4] developped the theory of quaternionic regular functions.

As the third view point, we associate two complex numbers

(4) $$z_1 := x_1 + ix_2, \quad z_2 := x_3 + ix_4 \in \mathbf{C}$$

to (1), regarding as

(5) $$z = z_1 + z_2 j \in \mathcal{H}.$$

Thus, we identify $\mathcal{H}$ with $\mathbf{C}^2 \cong \mathbf{R}^4$ and speak of the topology of $\mathcal{H}$. We define the non-commutative multiplication of two quaternions $z = z_1 + z_2 j, w = w_1 + w_2 j \in \mathcal{H}$ by

(6) $$zw := (z_1 w_1 - z_2 \overline{w_2}) + (z_1 w_2 + z_2 \overline{w_1}) j \in \mathcal{H}$$

using the complex conjugate

$$\overline{z_1} := x_1 - ix_2, \quad \overline{z_2} := x_3 - ix_4. \tag{7}$$

On the contrary, the quaternionic conjugate z^* of $z = z_1 + z_2 j \in \mathcal{H}$ is defined by

$$z^* := \overline{z_1} - z_2 j. \tag{8}$$

The absolute value

$$|z| := \sqrt{|z_1|^2 + |z_2|^2} \tag{9}$$

coincides with the usual norm of $z \in \mathbf{C}^2 \cong \mathbf{R}^4$.

In the present paper, we use the following quaternionic differential operators:

$$\frac{d}{dz^*} := \frac{\partial}{\partial \overline{z_1}} + j\frac{\partial}{\partial \overline{z_2}}, \qquad \frac{d}{dw^*} := \frac{\partial}{\partial \overline{w_1}} + j\frac{\partial}{\partial \overline{w_2}}. \tag{10}$$

Let Ω be an open set in $\mathcal{H}^2 \cong \mathbf{C}^4$ and $f(z,w) = f_1(z,w) + f_2(z,w)j$ be a C^1-function on $\Omega \subset \mathcal{H}^2 \cong \mathbf{R}^8$. f is said to be *hyperholomorphic* in Ω if f satisfies the equations

$$\frac{d}{dz^*}f = 0, \qquad f\frac{d}{dw^*} = 0 \tag{11}$$

in Ω, which are equivalent to

$$\frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial \overline{f_2}}{\partial z_2}, \qquad \frac{\partial \overline{f_1}}{\partial z_2} = -\frac{\partial f_2}{\partial \overline{z_1}} \tag{12}$$

and

$$\frac{\partial f_1}{\partial \overline{w_1}} = \frac{\partial f_2}{\partial \overline{w_2}}, \qquad \frac{\partial f_1}{\partial w_2} = -\frac{\partial f_2}{\partial w_1}. \tag{13}$$

by page 3 of Nôno[16].

For any complex valued functions f_1 and f_2 such that f_1 is a holomorphic function of the variables $z_1, z_2, w_1, \overline{w_2}$ and that f_2 is a holomorphic function of the variables $z_1, z_2, \overline{w_1}, w_2$, the quaternion valued function $f := f_1 + f_2 j$ is a hyperholomorphic function of two quaternionic variables $z := z_1 + z_2 j, w := w_1 + w_2 j$.

M. Naser[12] and K. Nôno[13]-[14]-[15] developped the theory of quaternionic hyperholomorphic functions of a quaternionic variable and, moreover, K. Nôno[16] developped that of two quaternionic variables z and w. In this fashion, the hyperholomorphy is a natural extension of the holomorphy of the theory of functions of several complex variables. And problems of the theory of functions of several complex variables can be made clearer. This is a characteristic of the third fashion (5) and is an advantage claimed for this hyperholomorphy. In deed, the notion of hyperholomorphy has aquired citizenship as is shown, e. g., in M. S. Marinov[11].

2. Envelopes of hyperholomorphy of domains over $\mathcal{H}^2$

Let D be a connected Hausdorff space and $\varphi : D \longrightarrow \mathcal{H}^2$ be a local homeomorphism. We induce canonically the quaternionic structure into D using the atlas associated to φ. The pair (D, φ) is called a *domain over* $\mathcal{H}^2$. A point x of D is called *a point over* $\varphi(x)$. When the mapping φ is injective, the domain (D, φ) over $\mathcal{H}^2$ is said to be *schlicht* and is identified with the domain $\varphi(D)$ of $\mathcal{H}^2$. Let $\mathcal{F} := \{f_\iota; \iota \in I\}$ be a family of hyperholomorphic functions on D. A trio (λ, D', φ') is said to be a *hyperholomorphic extension* of the domain (D, φ) over $\mathcal{H}^2$ with respect to the family $\mathcal{F}$ if the pair (D', φ') is a domain over $\mathcal{H}^2$, if the mapping $\lambda : D \longrightarrow D'$ is a local homeomorphism with $\varphi = \varphi' \circ \lambda$ and if, for any $f \in \mathcal{F}$, there exists a hyperholomorphic function f' on D' with $f = f' \circ \lambda$. Let $\mathcal{O}_\mathcal{F}$ be the sheaf of germs of families indexed by the said set I of hyperholomorphic functions over $\mathcal{H}^2$ and $\pi : \mathcal{O}_\mathcal{F} \longrightarrow \mathcal{H}^2$ be the canonical projection. By Theorem 3.5 of Nôno[16], the sheaf $\mathcal{O}_\mathcal{F}$ is a Hausdorff space and $(\mathcal{O}_\mathcal{F}, \pi)$ is also regarded as an open set over $\mathcal{H}^2$.

Now, we define canonically a local homeomorphism $\tilde{\lambda}_\mathcal{F} : D \longrightarrow \mathcal{O}_\mathcal{F}$ as follows:

$$D \ni x \rightsquigarrow \{f_\iota; \iota \in I\}_{\varphi(x)} \in \mathcal{O}_\mathcal{F} \tag{14}$$

associating to each point $x \in D$ the germ $\{f_\iota; \iota \in I\}_{\varphi(x)}$ at $\varphi(x)$ of the family $\{f_\iota; \iota \in I\}$ of hyperholomorphic functions f_ι on D. The image $\tilde{\lambda}_\mathcal{F}(D)$ of the connected space D by the continuous mapping $\tilde{\lambda}_\mathcal{F}$ is connected. Let $\tilde{D}_\mathcal{F}$ be the connected component of the Hausdorff space $\mathcal{O}_\mathcal{F}$ containing the connected set $\tilde{\lambda}_\mathcal{F}(D)$ and $\tilde{\varphi}_\mathcal{F}$ be the restriction of the projection π to $\tilde{D}_\mathcal{F}$. Then the trio $(\tilde{\lambda}_\mathcal{F}, \tilde{D}_\mathcal{F}, \tilde{\varphi}_\mathcal{F})$ is the largest hyperholomorphic extension of (D, φ) with respect to the family $\mathcal{F}$. The trio $(\tilde{\lambda}_\mathcal{F}, \tilde{D}_\mathcal{F}, \tilde{\varphi}_\mathcal{F})$ is called the *envelope of hyperholomorphy* of the domain (D, φ) over $\mathcal{H}^2$ with respect to the family $\mathcal{F}$. In this way, we can prove the following theorem:

Theorem 1. *Let (D, φ) be a domain over $\mathcal{H}^2$ and $\mathcal{F}$ be a family of hyperholomorphic functions on D. Then, there exists uniquely the envelope $(\tilde{\lambda}_\mathcal{F}, \tilde{D}_\mathcal{F}, \tilde{\varphi}_\mathcal{F})$ of hyperholomorphy of a domain (D, φ) over $\mathcal{H}^2$ with respect to the family $\mathcal{F}$.*

The envelope of hyperholomorphy with respect to a family consisting of a single hyperholomorphic function f is called the *domain of hyperholomorphy* of the function f. The envelope of hyperholomorphy with respect to the family of hyperholomorphic functions on D is called simply the *envelope of hyperholomorphy* of the domain D.

Let W be the domain in $\mathbf{C}^2$ and $f(w_1, w_2)$ be the holomorphic function of the complex variables $(w_1, w_2) \in W$ given at page 43 of Gunning-Rossi[9] such that the domain of holomorphy $\tilde{W}$ of the single valued holomorphic function f on the schlicht domain W is not a schlicht domain over $\mathbf{C}^2$ and that the holomorphic extension of the holomorphic function $f(w_1, w_2)$ takes two different values at two points over a point in $\mathbf{C}^2$. Then, for the quaternionic variables $z := z_1 + z_2 j$, $w := w_1 + w_2 j$, the domain of hyperholomorphy of the single valued hyperholomorphic function $h(z, w) := f(z_1, z_2) + f(\overline{w_1}, w_2) j$ on the schlicht domain $D := \{(z_1 + z_2 j, w_1 + w_2 j) \in \mathcal{H}^2; z_1, z_2, \overline{w_1}, w_2 \in W\}$ in $\mathcal{H}^2$ is not schlicht. So, it is not complete to treat domains of hyperholomorphy exclusively in the category of schlicht domains, in order to discuss fully the problem of extension on quaternionic functions. This is the reason why the present paper adds a non trivial result on the theory of quaternionic functions.

Let z be a point in $\mathcal{H}^2$ and r be a positive number. We denote by $B(z,r)$ the euclidean open ball with center z and radius r in $\mathcal{H}^2 \cong \mathbf{C}^4$. Let (D,φ) be a domain over $\mathcal{H}^2$, x be a point of D and r be a positive number. Under the assumption of the existence of an open neighborhood U of the point x in the Hausdorff space D such that φ maps U homeomorphically onto the open ball $B(\varphi(x),r)$ in the euclidean space $\mathbf{R}^4 \cong \mathcal{H}^2$, we speak of the open *ball with center* x *and radius* r in the domain D and denote U by $B(x,r)$.

We define the distance of x from the boundary ∂D of D by

$$\delta(x,\partial D) := \sup\{r : \textit{there} \quad \textit{exists} \quad B(x,r) \subset D\}, \tag{15}$$

i. e., $\delta(x,\partial D)$ is the largest between those r so that $\varphi|_{B(x,r)} : B(x,r) \to B(\varphi(x),r)$ are homeomorphisms.

We differentiate functions on D using the above chart, e. g., for a hyperholomorphic function f on D and a point $x \in D$, we define as follows:

$$\frac{d}{dz^*} f|_{B(x,\delta(x,\partial D))} := (\frac{d}{dz^*} f \circ (\varphi|_{B(x,\delta(x,\partial D))})^{-1}) \circ \varphi|_{B(x,\delta(x,\partial D))} \tag{16}$$

and

$$f\frac{d}{dw^*}|_{B(x,\delta(x,\partial D))} := (f \circ (\varphi|_{B(x,\delta(x,\partial D))})^{-1} \frac{d}{dw^*}) \circ \varphi|_{B(x,\delta(x,\partial D))}. \tag{17}$$

Let K be a compact in D, we use the following notation too:

$$\delta(K,\partial D) := \min\{\delta(x,\partial D); x \in K\}. \tag{18}$$

Moreover, let $\mathcal{F}$ be a family of hyperholomorphic functions on D. The set

$$\tilde{K_{\mathcal{F}}} := \{x \in D; |f(x)| \leq \sup_{y \in K} |f(y)| \qquad \textit{for} \quad \textit{all} \quad f \in \mathcal{F}\} \tag{19}$$

is called the *hyperholomorphic hull of the compact set* K *with respect to the family* $\mathcal{F}$. When $\mathcal{F}$ is the family of all hyperholomorphic functions on D, $\tilde{K_{\mathcal{F}}}$ is called simply the *hyperholomorphic hull of the compact set* K and denoted by $\tilde{K_{\mathcal{F}}}$.

A domain (D,φ) over $\mathcal{H}^2$ is said to be *hyperholomorphically convex with respect to a family* $\mathcal{F}$ of hyperholomorphic functions on D if, for any compact set K in D, the distance $\delta(\tilde{K_{\mathcal{F}}},\partial D)$ from the boundary of the domain D to the hyperholomorphic hull $\tilde{K_{\mathcal{F}}}$ of K with respect to the family $\mathcal{F}$ is positive and, when $\mathcal{F}$ is the family of all hyperholomorphic functions on D, D is said simply to be *hyperholomorphically convex*.

3. Contionuation Theorem of Cartan-Thullen's type

Let (D,φ) be a domain over $\mathcal{H}^2$ and $\mathcal{F}$ be a family of hyperholomorphic functions on D stable under the following differentiations:

$$\mathcal{F} \ni f \hookrightarrow \frac{\partial}{\partial z_j} f \in \mathcal{F}, \quad \mathcal{F} \ni f \hookrightarrow \frac{\partial}{\partial \overline{z_j}} f \in \mathcal{F} \qquad (j = 1,2) \tag{20}$$

and

$$\mathcal{F} \ni f \hookrightarrow f\frac{\partial}{\partial w_j} \in \mathcal{F} \quad \mathcal{F} \ni f \hookrightarrow f\frac{\partial}{\partial \overline{w_j}} \in \mathcal{F} \qquad (j = 1,2) \tag{21}$$

For a quadruplet $p = (p_1, p_2, p_3, p_4)$ of non negative integers $p_j (j \in \{1,2,3,4\}), |p| := p_1 + p_2 + p_3 + p_4$, we use the following differential operators on $z := z_1 + z_2 j \in \mathcal{H}$ in this section:

$$\partial_z^p := \frac{\partial^{|p|}}{\partial z_1^{p_1} \partial \overline{z_1}^{p_2} \partial z_2^{p_3} \partial \overline{z_2}^{p_4}}. \tag{22}$$

Theorem 2. *Let K be a compact set in D, $\tilde{K}_{\mathcal{F}}$ be the hyperholomorphic hull of K with respect to the family $\mathcal{F}$ stable under the differentiation (20) and (21), and x be a point in $\tilde{K}_{\mathcal{F}}$. Then any $f \circ (\varphi|_{B(x,\delta(x,\partial D))})^{-1}$ is hyperholomorphically extended to $B(\varphi(x), (1 - \frac{1}{\sqrt{2}})\delta(K, \partial D))$.*

Proof. Let f be an element of $\mathcal{F}$ and x a point of D. We put $(z_0, w_0) := \varphi(x)$. By Nôno[16], $f \circ \varphi|_{B(x,\delta(x,\partial D))}$ is represented as the series

$$f \circ \varphi|_{B(x,\delta(x,\partial D))}(z,w) = \sum_{m,n=0}^{\infty} \sum_{|p|=m,|q|=n} (z_1 - z_1^0)^{p_1} (\overline{z_1} - \overline{z_1^0})^{p_2} (z_2 - z_2^0)^{p_3} (\overline{z_2} - \overline{z_2^0})^{p_4} \times$$
$$\frac{\partial_z^p f \circ (\varphi|_{B(x,\delta(x,\partial D))})^{-1}(z^0, w^0) \partial_w^q}{p!q!} (w_1 - w_1^0)^{q_1} (\overline{w_1} - \overline{w_1^0})^{q_2} (w_2 - w_2^0)^{q_3} (\overline{w_2} - \overline{w_2^0})^{q_4} \tag{23}$$

which converges normally in the open ball $B(\varphi(x), (1 - \frac{1}{\sqrt{2}})\delta(K, \partial D))$.

4. Hyperholomorphic convexity of envelopes of holomorphy

Theorem 3. *Let (D,φ) be a domain over $\mathcal{H}^2$, $\mathcal{F}$ be a family of hyperholomorphic functions on D which contains the functions z_1, z_2, w_1, w_2 and which is stable under the differentiation (20) and (21), and $(\tilde{D}_{\mathcal{F}}, \tilde{\varphi}_{\mathcal{F}})$ be the envelope of hyperholomorphy of the domain D with respact to the family $\mathcal{F}$. Then $\tilde{D}_{\mathcal{F}}$ is hyperholomorphically convex.*

Proof. Let K be a compct subset of $\tilde{D}_{\mathcal{F}}$. Since $\mathcal{F}$ contains the functions z_1, z_2, w_1, w_2, the set $\varphi(K_{\mathcal{F}})$ is bounded by the theory of functions of several complex variables. Moreover, by Theorem 2, we have

$$(1 - \frac{1}{\sqrt{2}})\delta(K, \partial D) \leq \delta(\tilde{K}_{\mathcal{F}}, \partial D) \leq \delta(K, \partial D) \tag{24}$$

and, making use of this inequality, we can prove the theorem.

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