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ON THE SPACE SATISFYING CONDITION (T^{**})

SANG-EON HAN

Department of Mathematics
College of Natural Sciences, Honam University
Kwangju 506-090, Korea

ABSTRACT. In this paper,we define the locally nilpotent space as the extensive concept of the nilpotent space and the condition (T^*) and (T^{**}). We study the conditions that locally nilpotent space has a fixed point free deformation with relation to the condition (T^{**}).

1. Introduction

There are many results on the nilpotent space with respect to the homotopy equivalence,localization,completion and Euler characteristic [3,7,8,9,].

In this paper, we define the locally nilpotent spaces as the extensive concept of the nilpotent space. There is an effort applying the fixed point free deformation property to the space satisfying condition (T^{**}).

We make some results of the locally nilpotent spaces with relation to the condition (T^*) and (T^{**}). Furthermore, we study the homotopy equivalent conditions of the locally nilpotent spaces and spaces satisfying condition (T^{**}).

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The problems of the fixed point free deformation of the spaces satisfying condition (T^{**}) by use of the Euler charateristic number will be studied. We work in the category of the connected CW-complexes with base point and denoted as the T.

2. Some properties of the conditions (T^*) and (T^{**})

In this section, we define the locally nilpotent space and condition (T^{**}) and study their properties respectively. We recall that locally nilpotent group is the group whose all finitely generated subgroups are nilpotent groups [10].

We know that a space $X(\in T)$ is said to be a nilpotent space if

(1) $\pi_1(X)$ is a nilpotent group,

(2) the action $\pi_1(X) \times \pi_n(X) \to \pi_n(X)$ is nilpotent for all $n \geq 2$ [1].

And we denote the category of nilpotent spaces and continuous maps as T_N.

Now we extend the concept of the nilpotent spaces like following;

Definition 2.1. A space $X(\in T)$ is said to be a locally nilpotent space if

(1) $\pi_1(X)$ is a locally nilpotent group,

(2) the action $\pi_1(X) \times \pi_n(X) \to \pi_n(X)$ is nilpotent for all ≥ 2.

And we denote the category of locally nilpotent spaces and continuous maps as T_{LN}.

We know that the category T_N is a full subcategory of T_{LN}.

Generally, for a group G and a fixed $g \in G$, we denote by $[g, G]$ the subgroup of G generated by all commutators $[g, a]$,where $a \in G$. Since $[g, a]^b = [g, b]^{-1}[g, ab]$ for each $a, b \in G$ (where $a^b = b^{-1}ab$), we know that [g,G] is a normal subgroup of G.

Definition 2.2. We say that a space $X(\in T)$ satisfies condition (T^*) if for all $g, t \in \pi_1(X)$

either $g[g, \pi_1(X)] = t[t, \pi_1(X)]$

or $g[g, \pi_1(X)] \bigcap t[t, \pi_1(X)] = \phi$.

Lemma 2.3 [4]. *Let G be an arbitrary group. If $b \in a[a, G]$ $(a, b \in G)$ then $b[b, G] \subset a[a, G]$.*

The following lemma is followed by the topological reformation of the locally nilpotent group [4].

Lemma 2.4. *For $X \in T_{LN}$, then X satisfies the condition (T^*).*

Proof. Since $\pi_1(X)$ is a locally nilpotent group, suppose $c \in a[a, \pi_1(X)] \cap b[b, \pi_1(X)]$ for some $a, b, c \in \pi_1(X)$. We only show that $a[a, \pi_1(X)] = b[b, \pi_1(X)]$. By Lemma 2.3,

$$c[c, \pi_1(X)] \subset a[a, \pi_1(X)] \cap b[b, \pi_1(X)] \cdots\cdots\cdots\cdots\cdots (*)$$

Clearly, $c = h^{-1}a$ for some $h = \prod_{i=1}^{m} [a, g_i]^{\epsilon_i} \in [a, \pi_1(X)]$ $(g_i \in \pi_1(X), \epsilon_i = \pm 1)$. Let $G_1 =< a, g_1, \cdots, g_m >$. Since $a = hc$, $h \equiv \prod_{i=1}^{m} [h, g_i]^{\epsilon_i}$ modulo $[c, G_1]$, that is, $h = \prod_{i=1}^{m} [h, g_i\ ]^{\epsilon_i}$ in $\frac{G_1}{[c, G_1]}$. However, since the groups G_1 is nilpotent it follows that $h = 1$ in $\frac{G_1}{[c, G_1]}$ and $h \in [c, G_1]$. Therefore, $a = hc \in c[c, \pi_1(X)]$ and by Lemma 2.3, $a[a, \pi_1(X)] \subset c[c, \pi_1(X)]$. It follows from $(*)$that $a[a, \pi_1(X)] = c[c, \pi_1(X)]$.

Similary, $b[b, \pi_1(X)] = c[c, \pi_1(X)]$ and consequently, $a[a, \pi_1(X)] = b[b, \pi_1(X)]$.

Lemma 2.5. *For $X(\in T)$, the following conditions are equivalent.*

(1) *X satisfies the condition (T^*).*

(2) *For each* $a, b \in \pi_1(X)$, $a[a, \pi_1(X)] \subset b[b, \pi_1(X)]$

$\Rightarrow a[a, \pi_1(X)] = b[b, \pi_1(X)]$.

(3) *For each* $a \in \pi_1(X)$, $h \in [a, \pi_1(X)] \Rightarrow [ah, \pi_1(X)] = [a, \pi_1(X)]$.

Proof. By use of the Lemma 2.3 and Dokuchaev's result of [4 ,Lemma 3.1], our proof is completed.

Theorem 2.6. *For* $X \in T_{LN}$ *, if* $b \in [a, \pi_1(X)]$ *then* $a[a, \pi_1(X)] = b[b, \pi_1(X)]$, *for a,b* $\in \pi_1(X)$.

Proof. By Lemma 2.3, we know $b[b, \pi_1(X)] \subset a[a, \pi_1(X)]$ and by Lemma 2.4, X satisfies the condition (T^*). Thus our proof is completed by Lemma 2.5.

Now we define effective concept with respect to the locally nilpotent space.

Definition 2.7. For $X \in T$, we say that X satisfies the condition (T^{**}) if for all $g(\neq 1) \in \pi_1(X)$, then $g \notin [g, \pi_1(X)]$.

Since the $[g, \pi_1(X)]$ is a normal subgroup of $\pi_1(X)$, condition (T^{**}) is a homotopy invariant property.Furthermore the condition (T^{**}) is more powerful than the condition (T^*) in the point of view of homotopy invariant property in general.

In fibration $F \to E \to B$,any path $\alpha : I \to B$ and singular q-complex $g : \Delta^q \to p^{-1}(\alpha(0))$ determine a map $G : \Delta^q \times I \to E$ over $\alpha o pr_2 : \Delta^q \times I \to I \to B$ and extending $G_0 = g : \Delta^q \times \{ 0 \} \to E$. If α is a loop ,then $G_1 : \Delta^q \times \{ 1 \} \to E$ is a q-simplex in $p^{-1}(\alpha(1) = p^{-1}(\alpha(0))$. Now do elements of $\pi_1(B)$ operate on $H_*(F)$.

Definition 2.8. A fibration $F \to E \to B$ is said to be quasi-nilpotent if the action of $\pi_1(B)$ on $H_*(F)$ is nilpotent, $* \geq 0$.

Theorem 2.9. *For $X \in T_{LN}$, X satisfies the condition (T^{**}).*

Proof. Assume that $g \in [g, \pi_1(X)]$ for some $g(\neq 1) \in \pi_1(X)$. Then $g^{-1} \in [g, \pi_1(X)]$ and $1 \in g[g, \pi_1(X)]$. Thus $g[g, \pi_1(X)] \bigcap 1[1, \pi_1(X)] \neq \phi$. Since X satisfies the condition (T^*) by Lemma 2.4, $g[g, \pi_1(X)] = 1$. Since $g(\neq 1) \in g[g, \pi_1(X)]$, we have a contradiction.

3. An applications to the fixed point free deformation of the space satisfying condition (T^{**})

In this section we make results with relation to the fixed point free deformation of the locally nilpotent spaces and spaces satisfying condition (T^{**}) by use of the Euler characteristic number. Furthermore,we study about the condition that the spaces satisfying the condition (T^{**}) are homotopy equivalent.

We know the following ; if $\pi_1(X)$ is a nilpotent group then there exist finite upper central series of $\pi_1(X)$ by the virtue of center of $\pi_1(X)$ [7].

Lemma 3.1. *For X satisfying condition (T^{**}) with*

(1) *$\pi_1(X)$ finite ,*

(2) *the action $\pi_1(X) \times \pi_n(X) \to \pi_n(X)$ is nilpotent for all $n \geq 2$,*

then $X \in T_N$.

Proof. We only prove that $\pi_1(X)$ is a nilpotent group under the above hypothesis. So assume that $\pi_1(X)$ is not nilpotent,then we don't have finite upper central series of $\pi_1(X)$. If $Z_n(\pi_1(X))$ denote the $n-th$ center of $\pi_1(X)$, we can find an integer n such that $Z_{n+1}(\pi_1(X)) = Z_n(\pi_1(X)) \subsetneq \pi_1(X)$. It follows that if $x \notin Z_n(\pi_1(X))$, then $[x, \pi_1(X)] \not\subseteq Z_n(\pi_1(X))$. Choose any $x_1 \notin Z_n(\pi_1(X))$, we know $[x_1, \pi_1(X)] \not\subseteq Z_n(\pi_1(X))$ by above. If $x_1 \in [x_1, \pi_1(X)]$ then we have shown that the condition (T^{**}) does not hold, as required, so assume $x_1 \notin [x_1, \pi_1(X)]$. Then choose $x_2 \in [x_1, \pi_1(X)]$, $x_2 \notin Z_n(\pi_1(X))$. Since $[x_1, \pi_1(X)]$ is a normal subgroup of $\pi_1(X), [x_2, \pi_1(X)] \subseteq [x_1, \pi_1(X)]$. If $x_2 \in [x_2, \pi_1(X)]$, we are done.

Otherwise, we have $[x_2, \pi_1(X)] \subsetneq [x_1, \pi_1(X)]$ but also we noted $[x_2, \pi_1(X)] \nsubseteq Z_n(\pi_1(X))$. So pick $x_3 \in [x_2, \pi_1(X)]$, $x_3 \notin Z_n(\pi_1(X))$ and continue. Since $\pi_1(X)$ is finite, this process must stop. After all we have α for which $x_\alpha(\neq 1) \in [x_\alpha, \pi_1(X)]$. This is a contradiction to the fact that X satisfies the condition (T^{**}).Thus we know that $\pi_1(X)$ is nilpotent group. Thus our proof is completed.

We recall that map $f : X \to X$ is called a fixed point free deformation if f has no fixed point and is homotopic to 1_X [2].

Lemma 3.2 [3]. *If X is a polyhedron and $\chi(X) = 0$,then X adimits a fixed point free deformation.*

Lemma 3.3 [5]. *For finite X if $\pi_1(X)$ contains a torsion free normal abelian subgroup $A \neq 1$ which acts nilpotently on $H_*(\tilde{X})$ then $\chi(X) = 0$.*

Theorem 3.4. *For finite X satisfying condition (T^{**}), if*

(1) *the action $\pi_1(X) \times \pi_n(X) \to \pi_n(X)$ is nilpotent for all $n \geq 2$*

(2) *$\pi_1(X)$ is finite*

then X adimits a fixed point free deformation.

Proof. When $\pi_1(X)$ is finite, since X satisfies the condition (T^{**}) and by Lemma 3.1, $\pi_1(X)$ is nilpotent group. Since $X \in T_N$, $\chi(\tilde{X}) = \chi(X)$ and another general property $\chi(\tilde{X}) = |\pi_1(X)|\chi(X)$ [9] where $|\quad|$ means the order of $\pi_1(X)$ and $\tilde{X}$ means the univeral covering space of X. If $\pi_1(X) \neq 1$,

$\chi(X) = 0$. Thus X admits a fixed point free deformation by Lemma 3.2.

We recall that a group G satisfies the maximal condition if it has no infinite strictly increasing chain of subgroups [10].

Theorem 3.5. *For finite $X(\in T_{LN})$,if*

either $\pi_1(X)$ is infinite with the maximal condition on normal subgroups of $\pi_1(X)$

or $\pi_1(X)(\neq 1)$ is finite

then X admits a fixed point free deformation .

Proof. (case 1) When $\pi_1(X)$ is finite,we know that X satisfies the condition (T^{**}) by the Theorem 2.9. By the similar method of the Theorem 3.3 we get $\chi(X) = 0$.

(case 2) When $\pi_1(X)$ is infinite and $\pi_1(X)$ has maximal condition on normal subgroups then $\pi_1(X)$ is a finitely generated nilpotent group. Thus $\pi_1(X)$ has the center as the infinite normal abelian subgroup which acts nilpotently on $H_*(\tilde{X})$ then by Lemma 3.3 we have $\chi(X) = 0$.

At any cases, X admits a fixed point free deformation by Lemma 3.2.

Theorem 3.6. *For finite X satisfying condition (T^{**}), suppose that*

(1) *the map $f : \tilde{X} \to X$ is a universal covering map with the condition that $\pi_1(X)$ $(\neq 1)$ is finite*

(2) *the action $\pi_1(X) \times \pi_n(X) \to \pi_n(X)$ is nilpotent for all $n \geq 2$*

then $\tilde{X}$ also admits a fixed point free deformation.

Proof. Since $\chi(\tilde{X}) = |\pi_1(X)|\chi(X)$ where $|\quad|$ means the order of $\pi_1(X)$ and $\tilde{X}$ means the univeral covering space of X,and $X \in T_N$ by Lemma 3.1, $\chi(X) = 0$

and $\chi(\tilde{X}) = 0$. Thus our proof is completed.

In fibration $F_f \to E \xrightarrow{f} B$, if reduced homology group $\tilde{H}_*(F_f) = 0, \quad * \geq 0$ we call that f is an acyclic map, where F_f is a homotopy fiber of f.

Theorem 3.7. *For finite X satisfying condition (T^{**}), if*

(1) *the action $\pi_1(X) \times \pi_n(X) \to \pi_n(X)$ is nilpotent for all $n \geq 2$*

(2) *$f : X \to Y$ is an acyclic map with $\pi_1(X)$ finite*

then Y also admits a fixed point free deformation.

Proof. By Lemma 3.1 , $\pi_1(X)$ is a nilpotent group. Thus $X \in T_N$. From the fact that $f : X \to Y$ is an acyclic map and the classical homotopy exact sequence of fibration : $F_f \to X \xrightarrow{f} Y$, we know that $\pi_1(f)$ is an epimorphism. $\pi_1(F_f)$ is a perfect group and the homomorphic image of a perfect group is also a perfect group. Thus $\pi_1(X) \cong \frac{\pi_1(Y)}{P\pi_1(X)}$ where $P\pi_1(X)$ means a perfect normal subgroup of $\pi_1(X)$. Since $X \in T_N$, we have $\chi(X) = 0$ under the above hypothesis and $P\pi_1(X)$ is trivial. Thus $\pi_1(f)$ is an isomorphism. By use of the Hurewicz Theorem inductively, $\pi_i(F_f) = 0$. Thus f is a weak homotopy equivalence. By the Whitehead Theorem [6], f is a homotopy equivalence. Therefore, our proof is completed .

Theorem 3.8. *For finite X satisfying the condition (T^{**}),if*

(1) *$f : X \to Y$ is quasi-nilpotent homology equivalence with $\pi_1(X)$ finite*

(2) *the action $\pi_1(X) \times \pi_n(X) \to \pi_n(X)$ is nilpotent for all $n \geq 2$,*

then Y admits a fixed point free deformation.

Proof. By Lemma 3.1, X is a nilpotent space and so the homotopy fiber F_f of f is also nilpotent space. From the fact that f is quasi-nilpotent,we know that Y

is a nilpotent space, and f is a homotopy equivalence.We get $\chi(Y) = 0$. Thus our proof is completed by Lemma 3.2.

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