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ON OPTIMAL NETWORK WITH QUASI-FULL STEINER TOPOLOGY

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In this paper, the notion of quasi-full Steiner tree (QFST) is introduced which is virtually an extension of full Steiner tree (FST). We discuss some properties of QFST, and present a generating algorithm for obtaining a QFST or denying its existence. With this algorithm we can obtain a minimum QFST that is just the Steiner minimal tree (SMT) if it has the quasi-full topology. We also use this algorithm to simplify construction methods for some of the well- known SMTs.

1. Introduction

A Steiner minimal tree (SMT) on a given set X of points called the regular points in the Euclidean plane is the shortest tree interconnecting the points of X. Any intersections of edges which are not in X are called Steiner points(s-points). It is well known[5] that each s--point is of degree three and any two edges in an SMT intersect at an angle with at least 120°. An interconnecting tree satisfying the above two conditions is called a Steiner tree (ST). The problem of finding out an SMT on a set X has been showen to be NP-hard[5]. However, for certain kinds of sets of regular points, say the point sets of the Ladders, the Zig-zag lines and the Bar waves, the related SMT problems have been well-resolved. It has been known[5] that an ST for n given points can have at most n-2 s-points. An ST is called a full Steiner tree (FST) if it has n-2 s-points. It has been also known that an ST can be considered as the union of a certain number of full Steiner tree components[5]. Since it is much easier to construct FST than to construct ST, so one way to construct SMT is to construct its full Steiner tree components. This fact illustrates the significance of studying FST.

In this paper, our discussion is based on the notion of Quasi-full Steiner tree.

Definition 1. *Suppose X is a given set of regular points in Euclidean plane, a Steiner tree on X is called a Quasi-full Steiner tree (QFST) if any angles formed by its two adjacent edges are of* 120°. *A QFST on X is briefly denoted in QFST(X).*

Obviously, the notion of QFST is an extension of that of FST. In this paper we first discuss some properties concerning QFST, and present later a generating algorithm for obtaining a QFST or denying its existence. If a QFST is obtained, by applying this algorithm repeatedly, we improve each interim QFSTs until a minimum QFST is obtained, and this minimal QFST is an SMT itself. Thus, our unified way of constructing SMT with quasi-full topology makes the construction of certain well-known SMTs become very simple.

2. Preliminaries

We introduce some notations as follows.

[u,v] (or uv): line segment between points u and v.

d[u,v]: Euclidean distance between u and v.

(uv): vertex, which is not u and v, of a certain equilateral triangle that contains the edge [u,v], and satisfies that u, v and (uv) are counter-clockwise oriented.

p(u,v): path from u to v.

L(u,v): broken line from u to v.

(I , **I**): partition of a certain set composed of some of the line segments.

x(u): abscissa of point u.

y(u): ordinate of point u.

uv→ $I_0(\mathbf{I}_0)$: translation of [u,v] from $\mathbf{I}_0(I_0)$ to $I_0(\mathbf{I}_0)$.

d[T]: length of tree T.

From definition 1 and the fundamental properties of ST, we have

Lemma 1. *An ST with at least four regular points is a QFST if and only if the ST is consist of at most three groups of parallel edges and each angle between any two adjacent edges belonging to dif-*

ferent groups is of $120°$.

Definition 2. *In an ST, a path* $p(u,v)$ *(both u and v are regular points) is referred as a simple path if there are at most two s-points in* $p(u,v)$ *and when there are exactly two then they locate on the different sides of* $[u,v]$. *The* $[u,v]$ *is called a base of* $p(u,v)$.

Definition 3. *A QFST is called separable if one of three groups of parallel edges is taken off (see to Lemma* 1) *then only isolated simple paths could exist. A separable QFST on X is denoted in SQFST(X).*

Let the set of all bases of an SQFST(X) be M, $|M|=m$, and (I , II) a partition of all bases in M which makes some of the bases members of I and others members of II. The broken line L(c,d) of (I , II) is formed by first translating, one after another, all the bases of I (II) to the lower (upper) side of a fixd point, which we call the basic point, and then connecting them up. $[c,d]$ is called the chord of (I , II) and (cd) is called the characteristic point of (I , II)(or L(c,d)). If we consider each base of L(c,d) as a vector, then $[c,d]$ can be taken as the sum of these vectors. Now, we have

Lemma 2. *The position of the chord of a* (I , II) *on M is determined only by that of the basic point of* (I , II), *and this is independent of the arrangement order of the bases in* I *(or* II). *Furthermore, the chord lengths of all partitions on M are of a constant and all the chords are parallel.*

Since the directions of the chords are uniform, we can take the uniform direction as an ordinate axis, now, if the leftmost regular point is considered as an original point, then we have with us a coordinate system. A regular point with minimum (maxmum) abscissa is called the beginning (terminal) point. A base that contains the beginning (terminal) point is called the beginning (terminal) base. Usually the beginning point is taken as the basic point and the u and v of the base $[u,v]$ is supposed to satisfy $y(u) \geqslant y(v)$.

Lemma 3. *If SQFST(X) exists, then the length of the SQFST(X) that corresponds to* (I , II) *is equal to the distance between the characteristic point of* (I , II) *and the terminal point.*

Proof: We prove this lemma by making induction on the number m of the bases in the given partition (I , II).

For $m=2$, $X=\{a_1, b_1, a_2, b_2\}$, and $M=\{[a_1, b_1], [a_2, b_2]\}$, this lemma can be readily derived from the theorem 6 and its corollary in [2].

Suppose that the conclusion is true for $m=k$, and we prove in the following this is also true for $m=k+1$.

First, we know that if $SQFST(X)=T_1\cup T_2$, and $T_1\cap T_2=s_3$, then

$$d[SQFST(X)]=d[T_1]+d[T_2] \tag{1}$$

We then get on to prove $d[T_1]=d[(ca_1),s_3]$ (see to Fig. 1.)

Extend $[s_2,a_2]$ to a_2' so that $d[a_2,a_2']=d[b_2,s_3]$. Translate $[a_2,b_2]$ to $[b_1,c]$ and form $[b_1,e]/\!/[s_2,s_3]$ so that $d[b_1,e]=d[s_2,s_3]$. Extend $[a_1,s_1]$ to q so that $d[s_1,q]=d[s_2,s_3]$. Now, we have

i) $[c,e]/\!/[b_1,s_1]$ (since $\Delta a_2's_2s_3\equiv\Delta ceb_1$).

ii) $[s_1,s_2]/\!/[q,s_3]$, $[e,q]/\!/[b_1,s_1]$ and $\angle a_1qs_3=\angle a_1s_1s_2=120°$ (since both the quadralaterals $s_1qs_3s_2$ and b_1eqs_1 are parallelograms).

iii) Points c, e and q are collineation points (since $[c,e]/\!/[e,q]$).

iv) Points a_1, (ca_1), c and q are concyclic points (since $\angle a_1qc=120°$, $\angle a_1(ca_1)c=60°$).

v) Points (ca_1), q, s_3 and s_4 are collineation points (since $\angle a_1q(ca_1)=60°$, $\angle a_1qs_3=120°$ and $[q,s_3]/\!/[s_3,s_4]$).

From the construction method by Melzak[2], we have

$$d[(ca_1),s_3]=d[c,q]+d[a_1,q]+d[q,s_3]=d[QFST(a_1,c,s_3)] \tag{2}$$

Since

$$\begin{aligned} d[T_1]&=d[b_1,s_1]+d[s_2,a_2]+d[b_2,s_3]+d[a_1,s_1]+d[s_2,s_3]+d[s_1,s_2]\\ &=d[e,q]+(d[s_2,a_2]+d[a_2,a_2'])+(d[a_1,s_1]+d[s_1,q])+d[q,s_3]\\ &=d[e,q]+d[s_2,a_2']+d[a_1,q]+d[q,s_3]\\ &=(d[e,q]+d[c,e])+d[a_1,q]+d[q,s_3]\\ &=d[c,q]+d[a_1,q]+d[q,s_3]\\ &=d[QFST(a_1,c,s_3)] \end{aligned} \tag{3}$$

then we have $d[T_1]=d[(ca_1),s_3]$.

Lastly, let $M'=M\backslash\{[a_1,b_1]\cup[a_2,b_2]\}\cup\{[a_1,c]\}$, $X'=X\backslash\{b_1,a_2,b_2\}\cup\{c\}$, where X' is the set of regular points cooresponding to M'. By (1), (2) and (3), we have

$$d[QFST(X')]=d[QFST(a_1,c,s_3)]+d[T_2]$$
$$=d[T_1]+d[T_2]=d[SQFST(X)]$$

Since $|M'|=k$, then by the hypothesis of the induction, we know that the conclusion is true for $m=k+1$.

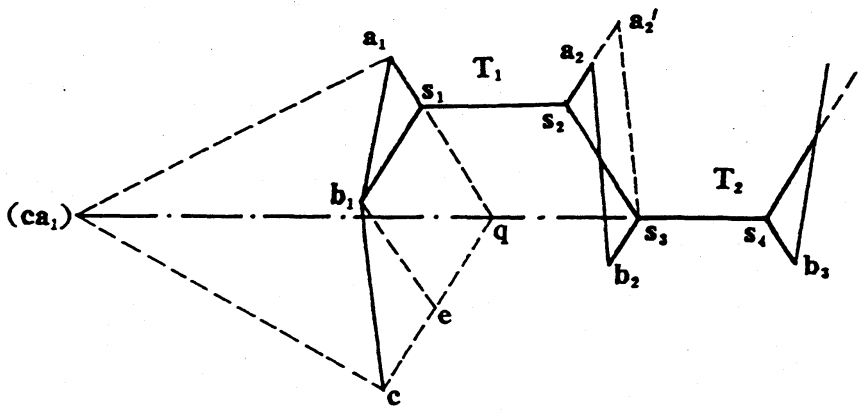

Figure 1. □

3. The Generating Algorithm

3.1. *The Generating Algorithm for QFST on a Given Partition*

Suppose that $(\mathrm{I}_0, \mathbf{I}_0)$ is the given partition on M. With respect to the coordinate system given afore, we arrange the m bases of M orderly from left to right. Let $K(i)$ $(i=1,2,\cdots,m)$ denote the broken line that is formed from the first i bases of M with respect to $(\mathrm{I}_0, \mathbf{I}_0)$.

Algorithm 1.

Step 1: Let $L(c,d)=K(m)$. Draw a straight line R by connecting the characteristic point (cd) of $(\mathrm{I}_0, \mathbf{I}_0)$ and the terminal point (x_0, y_0). Draw two orientation straight lines R_1 and R_2 such that the angle between R and $R_1(R_2)$ is of $60°(120°)$.

Step 2: Let the beginning base be $[u,v]$. Through the characteristic point of $K(1)$ draw a straight line $L(1)//R$. Through u draw $[u,s']//R_2$ so that $[u,s']\cap L(1)=s'$. Connecting s' with v, a simple path $p(u,v)=[u,s']\cup[s',v]$ is obtained. If $u\equiv v$, then $u\equiv s'\equiv v$.

Step 3: Suppose that $1<i<m$ and the ith base is $[u,v]$. Through the characteristic point of $K(i)$ draw a straight line $L(i)//R$.

If $[u,v]\in I_0$:

Through u draw $[u,s]//R_1$ so that $[u,s]\cap L(i-1)=s$. Through v draw $[v,s']//R_1$ so that $[v,s']\cap L(i)=s'$. Connecting the current s with the previous s', an edge $[s',s]$, which we call separating edge in the following, is obtained. Connecting the current s with the current s', a simple path $p(u,v)=[u,s]\cup[s,s']\cup[s',v]$ is obtained.

If $[u,v]\in \mathbf{I}_0$:

Through u draw $[u,s']//R_2$ so that $[u,s']\cap L(i)=s'$. Through v draw $[v,s]//R_2$ so that $[v,s]\cap L(i-1)=s$. Connecting the current s with the previous s', a separating edge $[s',s]$ is obtained. Connecting the current s with the current s', a simple path $p(u,v)=[u,s']\cup[s',s]\cup[s,v]$ is obtained.

Repeat this step until $i=m-1$.

Step 4: Let $[u,v]$ be the terminal base. Through u draw $[u,s]//R_1$ so that $[u,s]\cap L(m-1)=s$. Connecting v with s, a simple path $p(u,v)=[u,s]\cup[s,v]$ is obtained. Connecting the current s with the previous s', a separating edge $[s',s]$ is obtained. If $u\equiv v$, then $u\equiv s\equiv v$. Stop.

Now, the union of all the simple paths and separating edges produced by this algorithm is the very QFST that we want to generate, and this one is specifically denoted in QFST $(I_0, \mathbf{I}_0)$ in the following.

Corollary. *The SQFST* $(I_0, \mathbf{I}_0)$ *exists if and only if its ith base intersects with both* $L(i)$

and $L(i\text{-}1)$, *for* $i=2,3,\cdots,m\text{-}1$, *and the beginning (terminal) base intersects with* $L(1)$ $(L(m\text{-}1))$.

The partition (I_0, II_0) in the above corollary is called a feasible partition.

Definition 4. *A QFST(I, II) is called optimal if its tree length is the shortest among all QFSTs on X. An optimal QFST(X) is denoted in QFST* (X), and the partition (I, II) is called an optimal partition.*

3.2. *The Generating Algorithm for Optimal Partition*

Let P(X) be the set of all partitions on M, (x,y) the characteristic point of (I_0, II_0), and $y(c)<y(d)$ for $L(c,d)$. We have

$$x=-\sqrt{3}/2\Big(\sum_{uv\in II_0}\Delta y[uv]+\sum_{uv\in I_0}\Delta y[uv]\Big) \tag{4}$$

$$y=1/2\Big(\sum_{uv\in II_0}\Delta y[uv]-\sum_{uv\in I_0}\Delta y[uv]\Big) \tag{5}$$

where $\Delta x[uv]=x(u)-x(v)$, $\Delta y[uv]=y(u)\text{-}y(v)$.

Under transformation $uv\rightarrow I_0(II_0)$, the partition (I_0, II_0) becomes (I, II) and Δx and Δy turn out as

$$\Delta x=\sum_{uv\rightarrow II_0}\Delta x[uv]-\sum_{uv\rightarrow I_0}\Delta x[uv] \tag{6}$$

$$\Delta y=\sum_{uv\rightarrow II_0}\Delta y[uv]-\sum_{uv\rightarrow I_0}\Delta y[uv] \tag{7}$$

Define the test number $\lambda(I, II)$ of (I, II) by (8)

$$\lambda(I, II)=2(x\text{-}x_0)\Delta x+2(y\text{-}y_0)\Delta y+\Delta x^2+\Delta y^2 \tag{8}$$

Algorithm 2.

Step 0: $(I_0, II_0):=(M,\varnothing)$

Step 1:

$$\lambda(I', II'):=\min\{\lambda(I, II)\,|\,(I, II)\neq(I_0, II_0),(I, II)\in P(X)\} \tag{9}$$

Step 2: If (I_0, II_0) is feasible, test $\lambda(I', II')\geqslant 0$?

if $\lambda(I', II')\geqslant 0$, trun to step 3, else $(I_0, \mathbf{I}_0) := (I', \mathbf{I}')$, and turn to step 1;

else take $P(X) := P(X) \setminus \{(I_0, \mathbf{I}_0)\}$, test $P(X) = \varnothing$?

if $P(X) = \varnothing$, turn to step 4,

else $(I_0, \mathbf{I}_0) := (I', \mathbf{I}')$, and turn to step 1.

Step 3: Generate $QFST(I_0, \mathbf{I}_0)$ with algorithm 1. Then output

$$QFST^*(X) = QFST(I_0, \mathbf{I}_0),$$

and $$d[QFST^*(X)] = \sqrt{(x-x_0)^2 + (y-y_0)^2}, \text{stop.} \tag{10}$$

Step 4: the SMT(X) is not a QFST, stop.

3.3. *Main Result*

Theorem. *If the SMT(X) is of a separable quasi-full topology, then the $QFST^*(X)$ obtained is an SMT(X) itself. If the $QFST^*(X)$ cannot be generated, then the SMT(X) may not be separably quasi-full.*

Proof: By virtue of the minimum property of SMT, we have

$$d[SMT(X)] \leqslant d[QFST(X)]. \tag{11}$$

From (8) and Lemma 3 we have

$$\lambda(I, \mathbf{I}) = [(x+\Delta x-x_0)^2 + (y+\Delta y-y_0)^2] - [(x-x_0)^2 + (y-y_0)^2]$$

$$= d^2[QFST(I, \mathbf{I})] - d^2[QFST(I_0, \mathbf{I}_0)] \tag{12}$$

Where $(I_0, \mathbf{I}_0)$ is the original feasible partition and $(I, \mathbf{I})$ is a new partition formed through interchanging some of the bases in $(I_0, \mathbf{I}_0)$. $\lambda(I', \mathbf{I}') \geqslant 0$ implies that $\lambda(I, \mathbf{I}) \geqslant 0$ for any feasible $(I, \mathbf{I})$, that is to say no matter how one constructs new feasible partitions from making combinations of the bases, no $QFST(I, \mathbf{I})$ with smaller length can be obtained. Therefore, the $QFST(I_0, \mathbf{I}_0)$ is an optimal separable one which we denote in the following in $QFST^*(X)$. Now, we have

$$d[QFST(I_0, \mathbf{I}_0)] = d[QFST^*(X)] \leqslant d[QFST(I, \mathbf{I})].$$

Since SMT(X) is a separably quasi-full one, so

$$d[QFST^*(X)] \leqslant d[SMT(X)]. \tag{13}$$

By (11) and (13),

$$d[SMT(X)]=d[QFST^{*}(X)]. \qquad (14)$$

Since (I_0, $\mathbf{I}_0$) is feasible, so $QFST^{*}(X)$ exists; Therefore $SMT(X)=QFST^{*}(X)$.

If the $QFST^{*}(X)$ cannot be generated by the generating algorithm, then, of course, the SMT(X) may not be separably quasi-full. □

4. Examples

4.1. *SMT on point set of Ladder*

Example 1. It is known that the SMT on Ladder was first given out by Chung and Graham in [1]. For such SMT, the set of regular points is $L_n=\{a_k,b_k \mid a_k=(2k-2,2), b_k=(2k-2,0), k=1,2,\cdots,n\}$ which is shown in Fig. 2. , where a_k, $b_k(k=1,2,\cdots,n)$ are coordinates of the regular points.

By applying the results gained in [1], we can prove that the $SMT(L_n)$ is separably quasi-full. We now take $[a_k,b_k]$ as the base and b_1 as the basic point.

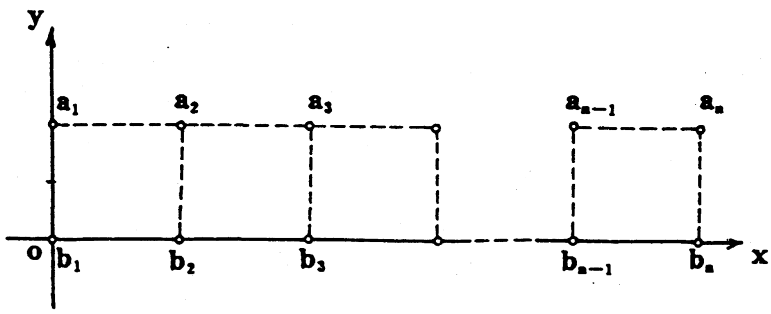

Figure 2.

(1) When n is an odd number, let (I_0, $\mathbf{I}_0$) be such one that $I_0=\{[a_k,b_k] \mid k=2,4,\cdots,n-1\}$ and $\mathbf{I}_0=\{[a_k,b_k] \mid k=1,3,\cdots,n\}$, and let a_n be the terminal point of L_n. Now, the broken line L(c,d) of(I_0, $\mathbf{I}_0$) coincides with its chord [c,d] on the ordinate axis, where c=(0,1-n), d=(0,1+n) (see to Fig. 3.). The $QFST^{*}(L_5)$, as in shown in Fig. 3. , is constructed by appplying the generating algorithm in 3. 1. Obviously, for this (I_0, $\mathbf{I}_0$), the distance between the characteristic point and a_n is the shortest, so the QFST

$(I_0, \mathbf{I}_0)$ is QFST* (L_n) , which is just the one given by Chung and Graham in [1].

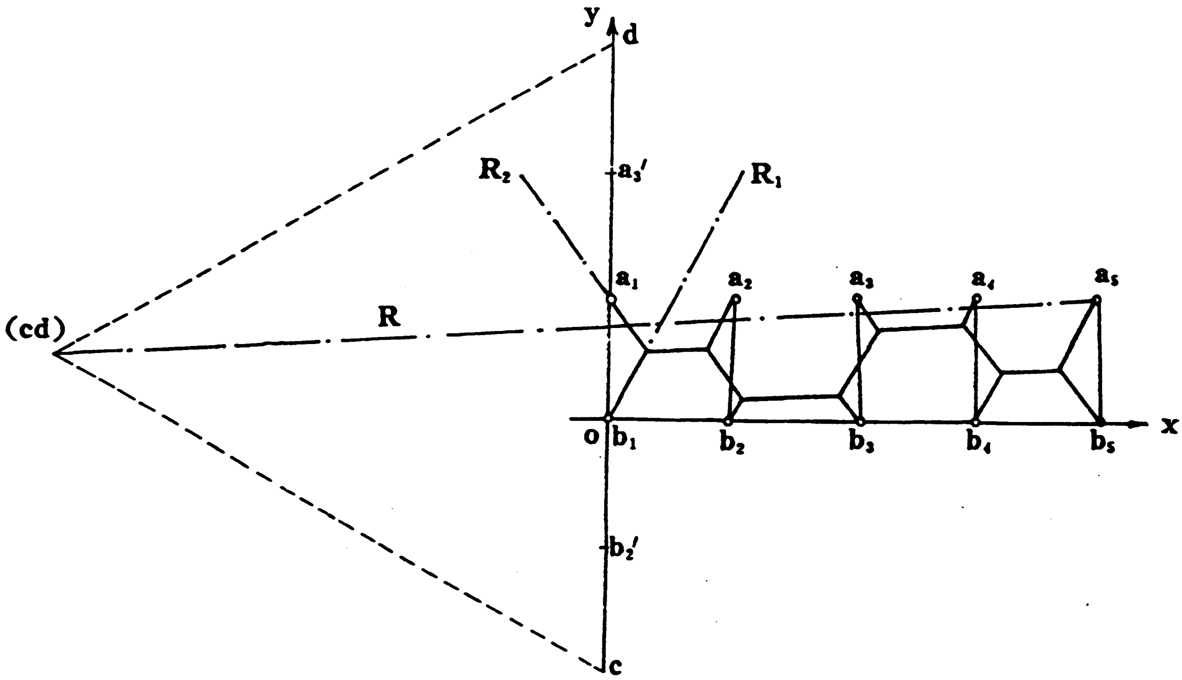

Figure 3.

(2) When n is an even number, let $(I_0, \mathbf{I}_0)$ be such one that $I_0=\{[a_k, b_k] \mid k=2,4,\cdots,n\}$ and $\mathbf{I}_0=\{[a_k, b_k] \mid k=1,3,\cdots,n-1\}$, and let b_n be the terminal point of L_n. The QFST$(I_0, \mathbf{I}_0)$, as is shown in Fig. 4. $(n=4)$, is constructed by Algorithm 1.

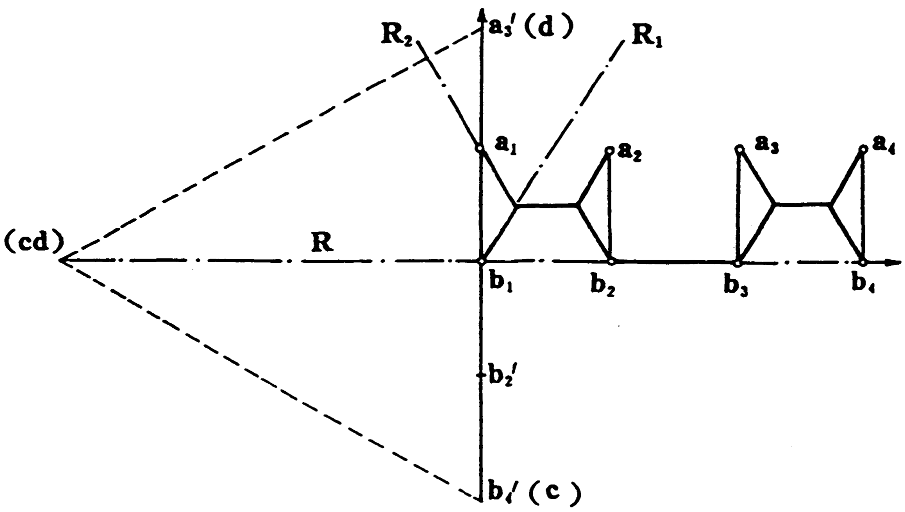

Figure 4.

For this $(I_0, \mathbf{I}_0)$, the basic point b_1 becomes the mid-point of $[c, d]$, and the terminal point b_n locates on the abscissa axis, hence $d[(cd), b_n]$ is the minimum, and we have QFST (I $_0$, **I** $_0$) = QFST* (L_n) = SMT (L_n). This result accords with the one given by Chung and Graham in [1].

4.2 *SMT on point set of Zig-zag line*

Example 2. In [3], Du, Hwang and Weng gave out the SMT on the point set of Zig-zag line on conditions that $Z_n = \{a_1, a_2, \cdots, a_n\}$ is Convex-normal. A Zig-zag line is one as is shown in Fig. 5. (n=7).

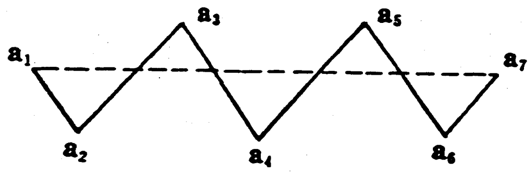

Figure 5.

Du, Hwang and Weng proved[3] that $p(a_k, a_{k+1})$ on the $[a_k, a_{k+1}]$ are simple paths ($k = 1, 3, \cdots, m$; where m=n-2 for odd n, or m=n-1 for even n). Let a_1 be the basic point, a_n the terminal point, I $_0$=M and **I** $_0 = \varnothing$. The QFST(I $_0$, **I** $_0$), as is shown in Fig. 6. (n=7), is constructed by applying Algorithm 1.

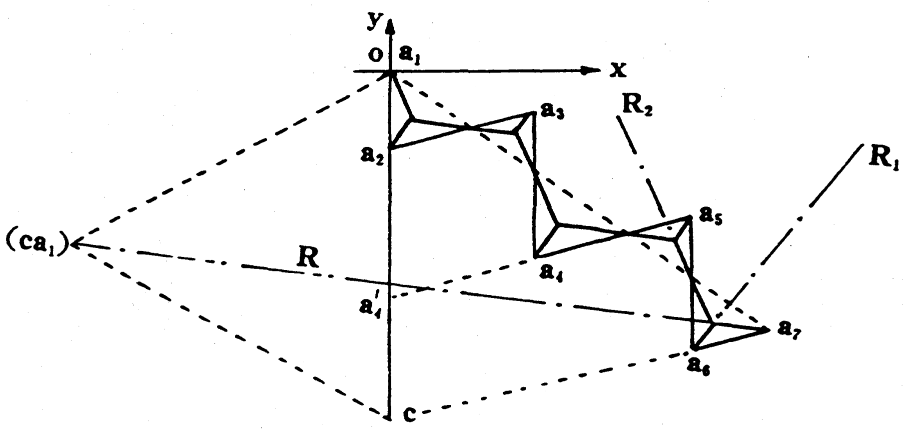

Figure 6.

By the Convex-normal condition set in [3], we have

$$d[a_k, a_{k+2}] \geqslant \max\{d[a_k, a_{k+1}], d[a_{k+1}, a_{k+2}]\}$$

where $\angle a_k = \alpha$ and $60° \leqslant \alpha < 120°$ for every k. Now, we have $\angle a_2 a_1 a_n \leqslant \angle a_2 a_1 a_3 \leqslant \angle a_1 c a_n$ (see to Fig. 6. , n=7). Since $(I_0, \mathbf{I}_0) = (M, \varnothing)$, $d[(ca_1), a_n]$ must be the minimum (see to Fig. 6.) so $QFST(I_0, \mathbf{I}_0) = QFST^*(Z_n) = SMT(Z_n)$. This result is in accord with the one given in [3].

4. 3. *SMT on point set of Bar wave*

Example 3. In [4], Du and Hwang gave out the $SMT(B_n)$, where $B_n = \{a_k, b_k \mid k = 1, 2, \ldots, n\}$ (see to Fig. 7. for n=5), $d[a_k, a_{k+1}] \geqslant \max\{d[a_k, b_k], d[a_{k+1}, b_{k+1}]\}$ for $k = 1, 2, \ldots, n-1$. Du and Hwang proved in [4] that paths on $[a_k, b_k]$ are all simple paths. Let $I_0 = \{[a_k, b_k] \mid k \text{ is even}\}$, $\mathbf{I}_0 = \{[a_k, b_k] \mid k \text{ is odd}\}$, and a_n the terminal point of B_n.

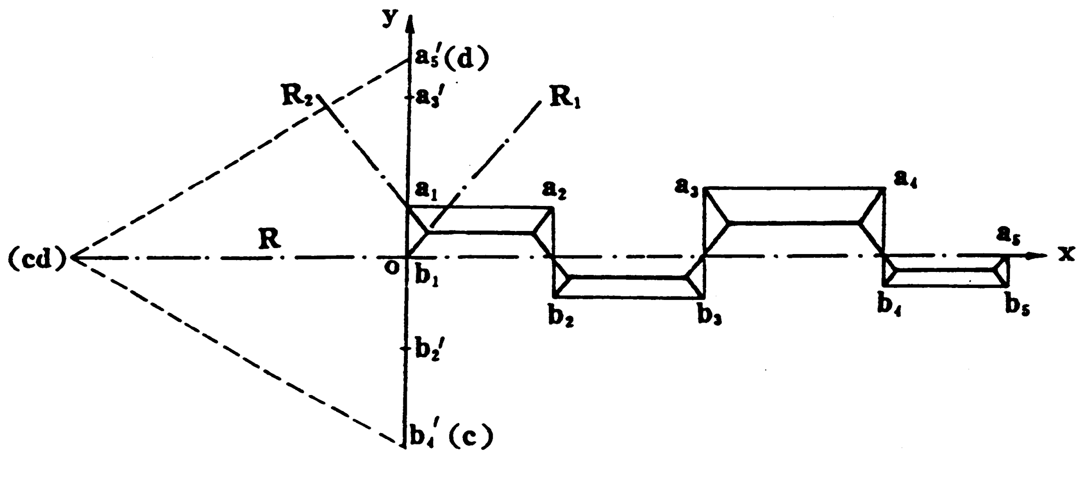

Figure 7.

Obviously, the characteristic point of $(I_0, \mathbf{I}_0)$ is on the abscissa axis, hence the $QFST(B_n)$ constructed by applying Algorithm 1. is an optimal $QFST^*(B_n)$. Therefore, $QFST^*(B_n) = SMT(B_n)$ and this is in accord with the result gained in [4].

Acknowledgements

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References

[1] F. R. K. Chung and R. L. Graham, Steiner trees for Ladders. *Ann. Discrete Math.* **2** (1978), 173-200.

[2] D. Z. Du, F. K. Hwang, G. D. Song and G. Y. Ting, Steiner minimal trees on sets of four points, *Discrete Comput. Geom.* 2(1987), 401-414.

[3] D. Z. Du , F. K. Hwang and J. F. Weng , Steiner minimal trees on Zig-Zag lines, *Trans. Amer. Math. soc.* **278** (1983), 149-156.

[4] D. Z. Du and F. K. Hwang, Steiner minimal trees for Bar waves. *Acta Math. Appl. Sinica* **3** (1987), 246-256.

[5] E. N. Gilbert and H. O. Pollak, Steiner minimal trees, *SIAM J. Appl. Math.* **16** (1968), 1-29.

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