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Generic submanifolds of an odd-dimensional sphere

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Dedicated to professor Younki Chae on his 60th birthday

Introduction

Submanifolds of a Sasakian manifold were investigated from two different points of view, namely, one is the case where submanifolds are tangent to the structure vector, and the other is the case where those are normal to the structure vector [3], [6], [7] etc. But, without these considerations generic submanifolds of a Sasakian manifold are defined as follows : Let M be a submanifold of a Sasakian manifold $\tilde{M}$ with almost contact metric structure (ϕ, G, V). If each normal space is mapped into the tangent space under the action of ϕ, M is called a generic submanifold of $\tilde{M}$ [4]. For example, hypersurfaces of a Sasakian manifold are generic.

The purpose of the present paper is to investigate generic submanifolds of an odd-dimensional sphere with nonvanishing parallel mean curvature vector.

In §1, we state general formulas on generic submanifolds of a Sasakian manifold. §2 is devoted to the study a generic submanifold of a Sasakian manifold, which is not tangent to the structure vector. Moreover, we suppose that the shape operator in the direction of the mean curvature vector commutes with the structure tensor induced on the submanifold. In §3 we study generic submanifolds which is not tangent to the structure vector of an odd-dimensional unit sphere with nonvanishing parallel mean curvature vector. In §4, we consider generic submanifolds, tangent to the structure vector, of an odd-dimensional sphere and compute the restricted Laplacian

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for the shape operator in the direction of the mean curvature vector. As applications of these, in the last §5 we prove our main results.

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1. Generic submanifolds of a Sasakian manifold

In this section, the fundamental properties of generic submanifolds of a Sasakian manifold are recalled [4], [7].

Let $\tilde{M}$ be a Sasakian manifold of dimension $2m+1$ with almost contact metric structure (ϕ, G, V). Then for any vector fields X and Y on $\tilde{M}$, we have

$$\phi^2 X = -X + v(X)V, \ G(\phi X, \phi Y) = G(X,Y) - v(X)v(Y),$$
$$v(\phi X) = 0, \ \phi V = 0, v(V) = 1, \ G(X,V) = v(X).$$

Since $\tilde{M}$ is a Sasakian manifold, we then have

$$\tilde{\nabla}_X V = \phi X, \ (\tilde{\nabla}_X \phi)Y = -G(X,Y)V + v(Y)X, \tag{1.1}$$

where $\tilde{\nabla}$ denotes the Riemannian connection of $\tilde{M}$.

Let M be an (n+1)-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$ and isometrically immersed in $\tilde{M}$ by the immersion $i : M \longrightarrow \tilde{M}$. When the argument is local, M need not be distinguished from $i(M)$ itself. Throughout this paper the indices $i, j, k, \cdots$ run from 1 to $n+1$. We represent the immersion i locally by

$$y^A = y^A(x^h), \ (A = 1, \cdots, n+1, \cdots, 2m+1)$$

and put $B_j{}^A = \partial_j y^A$, $(\partial_j = \frac{\partial}{\partial x^j})$ then $B_j = (B_j{}^A)$ are (n+1)-linearly independent local tangent vector fields of M. We choose $2m-n$ mutually orthogonal unit normals $C_x = (C_x{}^A)$ to M. Hereaafter the indices $u, v, w, x, \cdots$

run from $n+2$ to $2m+1$ and the summation convention will be used. The immersion being isometric, the induced Riemannian metric tensor g with compoents g_{ji} and the metric tensor δ with components δ_{yx} of the normal bundle are respectively obtained :

$$g_{ji} = G(B_j, B_i), \quad \delta_{yx} = G(C_y, C_x).$$

By denoting ∇_j the operator of van der Waerden-Bortolotti covariant differentiation with respect to g and G, the equations of Gauss and Weingarten for the submanifold M are respectively given by

$$\text{(1.2)} \qquad \nabla_j B_i = A_{ji}{}^x C_x, \quad \nabla_j C_x = -A_j{}^h{}_x B_h,$$

where $A_{ji}{}^x$ are components of the second fundamental tensors and the shape operator A^x in the direction of C_x are related by

$$A^x = (A_j{}^{hx}) = (A_{jiy} g^{ih} \delta^{yx}), \quad g^{ji} = (g_{ji})^{-1}.$$

An $(n+1)$-dimensional submanifold M of a Sasakian manifold $\tilde{M}$ is called a *generic submanifold* if

$$\phi N_p(M) \subset T_p(M)$$

at each point $p \in M$, where $T_p(M)$ is the tangent space of M at p and $N_p(M)$ the normal space at p, [4], [6].

From now on, we have only to consider generic submanifolds of a Sasakian manifold. Then the transforms of B_i and C_x by ϕ are respectively represented in each coordinate neighborhood as follows :

$$\text{(1.3)} \qquad \phi B_j = f_j{}^h B_h - J_j{}^x C_x, \quad \phi C_x = J_x{}^h B_h,$$

where we have put $f_{ji} = G(\phi B_j, B_i)$, $J_{jx} = -G(\phi B_j, C_x)$, $J_{xj} = G(\phi C_x, B_j)$, $f_j{}^h = f_{ji} g^{ih}$ and $J_j{}^x = J_{jy} \delta^{yx}$. From these definitions we verify that $f_{ji} + f_{ij} = 0$ and $J_{jx} = J_{xj}$.

Also, we can put the Sasakian structure vector V of the form

$$(1.4) \qquad V = \xi^h B_h + w^x C_x,$$

where $\xi_i = G(B_i, V)$ and $w_x = G(C_x, V), \xi^h$ being the associated vector with ξ_i.

By the properties of the Sasakian structure tensor, it follows from (1.3) and (1.4) that we obtain

$$(1.5) \qquad f_j{}^t f_t{}^h = -\delta_j^h + \xi_j \xi^h + J_j{}^x J_x{}^h,$$

$$(1.6) \qquad J_y{}^t J_t{}^x = \delta_y^x - w_y w^x,$$

$$(1.7) \qquad f_t{}^h J_x{}^t = w_x \xi^h, \quad f_{jt} \xi^t = J_{jx} w^x,$$

$$(1.8) \qquad J_t{}^x \xi^t = 0,$$

$$(1.9) \qquad w_x w^x = 1 - \xi_t \xi^t.$$

Differentiating (1.3) and (1.4) covariantly along M and making use of (1.1), (1.2) and these equations, we easily find (cf. [4])

$$(1.10) \qquad \nabla_k f_j{}^h = -g_{kj} \xi^h + \delta_k^h \xi_j + A_{kj}{}^x J_x{}^h - A_k{}^{hx} J_{jx},$$

$$(1.11) \qquad \nabla_k J_{jx} = g_{kj} w_x + A_{krx} f_j{}^r,$$

$$(1.12) \qquad A_{jrx} J^{ry} = A_j{}^{ry} J_{rx},$$

$$(1.13) \qquad \nabla_j \xi_i = f_{ji} + A_{jix} w^x,$$

(1.14) $$\nabla_j w_x = -J_{jx} - A_{jrx}\xi^r.$$

Remark 1. We notice here that if $\xi_j = 0$ on some open set, then $f_{ji} = 0$ and $A_{jvx}w^x = 0$ because of (1.13).

Promise. In the following, we define

(1.15) $$\lambda_x = A_{jix}\xi^j\xi^i/|\xi|^2, \quad P_{xyz} = A_{jix}J_y{}^j J_z{}^i.$$

Then it is clear that P_{xyz} is symmetric for all indices because of (1.12). In the sequel the index $n+2$ will be denoted by the symbol $*$.

For the shape operator A^* a function $h_{(m)}$ for any integer $m \geq 2$ is introduced as follows :

(1.16) $$h_{(m)} = tr(A^*)^m.$$

2. The structure tensor of generic submanifolds

Let M be a generic submanifold of a Sasakian manifold and denote by $\alpha = w_x w^x$. Suppose that the function $\alpha(1-\alpha)$ does not vanish almost everywhere, and that $A^* f = f A^*$, namely

(2.1) $$A_{jr}{}^* f_i{}^r + A_{ir}{}^* f_j{}^r = 0$$

holds on M. By transvecting $f_k{}^i$ and making use of (1.5), we then have

$$A_{jk}{}^* - (A_{jr}{}^* J_z{}^r)J_k{}^z - (A_{jr}{}^*\xi^r)\xi_k - A_{sr}{}^* f_j{}^s f_k{}^r = 0,$$

and hence taking the skew-symmetric part with respect to k and j,

(2.2) $$(A_{jr*}J_z{}^r)J_k{}^z - (A_{kr*}J_z{}^r)J_j{}^z + (A_{jr*}\xi^r)\xi_k - (A_{kr*}\xi^r)\xi_j = 0.$$

If we transvect this by ξ^k and make use of (1.8) and (1.9), then we get

$$(2.3)\qquad (1-\alpha)A_{jr*}\xi^r - (A_{sr*}\xi^s\xi^r)\xi_j - (A_{sr*}\xi^s J_z{}^r)J_j{}^z = 0.$$

Transvecting this with $J_y{}^j$ and using (1.6) and (1.8), we find

$$\alpha A_{sr*}\xi^r J_y{}^s - w_y(A_{sr*}\xi^s J_z{}^r w^z) = 0,$$

which joined with (1.7) gives

$$\alpha A_{sr*}\xi^r J_y{}^s + w_y(A_{jr*}f_i{}^r\xi^j\xi^i) = 0.$$

Thus it is, taking account of (2.1), clear that $A_{sr*}\xi^r J_y{}^s = 0$ because the function α is not vanish almost everywhere. Hence (2.3) implies $(1-\alpha)(A_{jr*}\xi^r - \lambda_*\xi_j) = 0$ because of (1.15) and consequently

$$(2.4)\qquad A_{jr*}\xi^r = \lambda_*\xi_j$$

with the aid of (1.9). Therefore (2.2) is reduced to

$$(2.5)\qquad (A_{jr*}J_z{}^r)J_i{}^z - (A_{ir*}J_z{}^r)J_j{}^z = 0.$$

Because of (1.7), (2.1) and (2.4), we have

$$A_{jr*}J_z{}^r w^z = -A_{jr*}f_s{}^r\xi^s = \lambda_* w_z J_j{}^z.$$

Thus, by transvecting J_w^i to (2.5) and using (1.6) and (1.15), we find

$$(2.6)\qquad A_{jr*}J_x{}^r = Q_{xz*}J_j{}^z,$$

where we have put

$$(2.7)\qquad Q_{xz*} = \lambda_* w_x w_z + P_{xz*}.$$

Transforming (2.6) by $f_i{}^j$ and taking account of (1.7), (2.1) and (2.4), we find

$$(\lambda_* w_x - Q_{xz*}w^z)\xi_j = 0$$

and hence

$$Q_{xz*}w^z = \lambda_* w_x \tag{2.8}$$

because the function $1-\alpha$ does not vanish almost everywhere.

Differentiating (2.8) covariantly along M and making use of (1.14), we find

$$\begin{aligned} w^z\nabla_k Q_{yz*} &- (J_k{}^z + A_{kr}{}^z\xi^r)Q_{yz*} \\ &= (\nabla_k\lambda_*)w_y - \lambda_*(J_{ky} + A_{kry}\xi^r). \end{aligned}$$

By transvecting ξ^k and using (1.8) and (1.15), we can get

$$\xi^k w^z\nabla_k Q_{yz*} = (\xi^k\nabla_k\lambda_*)w_y + (1-\alpha)(Q_{yz*}\lambda^z - \lambda_*\lambda_y). \tag{2.9}$$

From (2.7) we have

$$Q_{***} = \lambda_* w_*{}^2 + A_{ji}{}^* J_*{}^j J_*{}^i$$

by virtue of the second equation of (1.15). Differentiating the last equation covariantly and using (1.11), (1.14) and (2.6), we find

$$\begin{aligned} \nabla_k Q_{***} &= (\nabla_k\lambda_*)w_*{}^2 - 2\lambda_* w_*(J_{k*} + A_{kr*}\xi^r) \\ &\quad + (\nabla_k A_{ji}{}^*)J_*{}^j J_*{}^i + 2Q_{z**}(w_* J_k{}^z + J^{jz}A_{kr*}f_j{}^r), \end{aligned}$$

which together with (1.7) and (2.8) gives

$$\begin{aligned} \nabla_k Q_{***} &= (\nabla_k\lambda_*)w_*{}^2 + 2w_*(Q_{z**}J_k{}^z - \lambda_* J_{k*}) \\ &\quad + (\nabla_k A_{ji}{}^*)J_*{}^j J_*{}^i. \end{aligned} \tag{2.10}$$

3. Generic submanifolds with parallel mean curvature vector

In this section, we consider that a generic submanifold M of an odd-dimensional unit sphere S^{2m+1}. Then, the equations of Gauss, Codazzi and Ricci for M are respectively obtained :

$$(3.1)\qquad R_{kjih} = g_{kh}g_{ji} - g_{jh}g_{ki} + A_{kh}{}^{x}A_{jix} - A_{jh}{}^{x}A_{kix},$$

$$(3.2)\qquad \nabla_k A_{ji}{}^{x} - \nabla_j A_{ki}{}^{x} = 0,$$

$$(3.3)\qquad R_{jiyx} = A_{jr}{}^{x}A_{i}{}^{r}{}_{y} - A_{ir}{}^{x}A_{j}{}^{r}{}_{y}.$$

where R_{kjih} and R_{jiyx} are components of the Riemannian curvature tensor of M and that with respect to the connection induced in the normal bundle of M, respectively.

Let H be a mean curvature vector of a generic submanifold M. Namely, it is defined by

$$H = g^{ji}A_{ji}{}^{x}C_x/(n+1) = h^x C_x/(n+1),$$

which is independent of the choice of the local field of orthonormal frames $\{C_x\}$.

In what follows we suppose that the mean curvature vector H of M is nonzero and is parallel in the normal bundle. Then we may choose a local filed $\{C_x\}$ in such a way that $H = \sigma C_{n+2} = \sigma C_*$, where $\sigma = |H|$ is nonzero constant. Because of the choice of the local field, the parallelism of H yields

$$(3.4)\qquad \begin{cases} h^x = 0, \quad x \geq n+3 \\ h^* = (n+1)\sigma. \end{cases}$$

Then by (3.1), the Ricci tensor S with components S_{ji} of M is given by

$$(3.5)\qquad S_{ji} = ng_{ji} + h^* A_{ji*} - A_{jr}{}^{x}A_{i}{}^{r}{}_{x}.$$

Here we notice that the condition (2.1) does not depend on the choice of the local field because of (3.4).

The parallelism of the mean curvature vector yields that the restricted Laplacian $\Delta A_{ji}{}^*$ of A^* is given by

$$(3.6) \qquad \Delta A_{ji}{}^* = S_{jr} A_i{}^{r*} - R_{kjih} A^{kh*}$$

because of (3.2), and that $R_{ji*x} = 0$ shows

$$(3.7) \qquad A_{jr}{}^x A_i{}^{r*} - A_{ir}{}^x A_j{}^{r*} = 0$$

with the aid of (3.3).

Lemma 3.1. Let M be an $(n+1)$-dimensional generic submanifold satisfying (2.1) of S^{2m+1} with nonvanishing parallel mean curvature vector. If the function $\alpha(1-\alpha)$ does not vanish almost everywhere, then we have

$$(3.8) \qquad A_{jiy} A^{ji*} = h^* Q_{y**}.$$

Proof. Transforming (3.7) by $J_y{}^i J_z{}^j$ and using (1.15) and (2.6), we find

$$(3.9) \qquad Q_{yu*} P_z{}^{ux} = Q_{zu*} P_y{}^{ux},$$

which enable us to obtain

$$(3.10) \qquad Q_{yz*} P_x{}^{yz} = P^z Q_{xz*}$$

for any index x, where we have defined $P^z = P_x{}^{xz}$. Thus, it is clear that

$$(3.11) \qquad Q_{yz*} P^{yz*} = P^z Q_{z**}.$$

Differentiating (2.6) covariantly and substituting (1.12), we find

$$(3.12)\qquad (\nabla_k A_{jr*})J_y{}^r + A_{kj*}w_y + A_{jr*}A_{ksy}f^{rs}$$
$$= (\nabla_k Q_{yz*})J_j{}^z + Q_{yz*}w^z g_{kj} + Q_{yz*}A_{kr}{}^z f_j{}^r,$$

from which taking the skew-symmetric part with respect to indices k and j, and making use of (2.1) and (3.2),

$$A_r{}^{s*}A_{ksy}f_j{}^r - A_r{}^{s*}A_{jsy}f_k{}^r$$
$$= (\nabla_k Q_{yz*})J_j{}^z - (\nabla_j Q_{yz*})J_k{}^z + Q_{yz*}(A_{kr}{}^z f_j{}^r - A_{jr}{}^z f_k{}^r).$$

If we transvect f^{jk} to this and take account of (1.5) and (1.7), then we obtain

$$A_r{}^{s*}A_{ksy}(g^{kr} - \xi^k\xi^r - J^{kz}J_z{}^r)$$
$$= \xi^k(\nabla_k Q_{yz*})w^z + Q_{yz*}A_{kr}{}^z(g^{kr} - \xi^k\xi^r - J^{ku}J_u{}^r),$$

which together with (1.15), (2.4), (2.6), (2.8), (2.9) and (3.4) yields

$$A_{jiy}A^{ji*} = (\xi^k\nabla_k\lambda_*)w_y + h^*Q_{y**} + Q_{uz*}P^{uz}{}_y - P^zQ_{zy*}.$$

Because of (3.10), it is seen that

$$(3.13)\qquad A_{jiy}A^{ji*} = (\xi^k\nabla_k\lambda_*)w_y + h^*Q_{y**}.$$

On the other hand, differentiating (2.4) covariantly and substituting (1.13), we obtain

$$(3.14)\quad (\nabla_k A_{jr*})\xi^r + A_j{}^{r*}(f_{kr} + A_{krx}w^x) = (\nabla_k\lambda_*)\xi_j + \lambda_*(f_{kj} + A_{kjx}w^x).$$

Transvecting g^{kj} to this and taking account of (3.2) and (3.4), we get

$$w^x A_{jix}A^{ji*} = \xi^k\nabla_k\lambda_* + \lambda_*h^*w_*,$$

which together with (2.8) and (3.13) give raise to $(1-\alpha)\xi^k\nabla_k\lambda_* = 0$ and hence

$$(3.15) \qquad \xi^k \nabla_k \lambda_* = 0$$

because the function $1 - \alpha$ is not zero almost everywhere. By (3.13) our assertion is thus proved.

Lemma 3.2. Under the same hypothesis as that in Lemma 3.1, we have

$$(3.16) \qquad h^* = (n+1)\lambda^*,$$

$$(3.17) \qquad A_{ji*} = \lambda_*(g_{ji} - J_j{}^z J_{iz}) + Q_{yz*} J_j{}^y J_i{}^z.$$

Proof. If we take the skew-symmetric part with respect to k and j in (3.14), and make use of (2.1), (3.2) and (3.7), then we find

$$2A_{jr*} f_k{}^r = (\nabla_k \lambda_*)\xi_j - (\nabla_j \lambda_*)\xi_k + 2\lambda_* f_{kj}.$$

By transvecting ξ^j and using (2.4) and (3.15), we see that $\lambda_* = const.$ Therefore we have

$$(3.18) \qquad A_{jr*} f_k{}^r = \lambda_* f_{kj}.$$

Accordingly (3.14) is reduced to

$$(\nabla_k A_{jr*})\xi^r = -A_{jr*} A_k{}^{rx} w_x + \lambda_* A_{kjx} w^x,$$

from which, transforming $f_l{}^j$ and taking account of (3.18),

$$(3.19) \qquad (\nabla_k A_{jr*})\xi^r f_l{}^j = 0.$$

Differentiating (3.18) covariantly and using (1.10), (2.4) and (2.6), we obtain

$$(3.20) \quad (\nabla_k A_{ir*}) f_j{}^r = (A_{krx} A_i{}^{r*} - \lambda_* A_{kix}) J_j{}^x - (A_{ki*} - \lambda_* g_{ki})\xi_j - (Q_{xz*} J_i{}^z - \lambda_* J_{ix}) A_{kj}{}^x,$$

which together with (3.2) and (3.7) implies that

$$A_{kj}{}^{x}(Q_{xz*}J_i{}^z - \lambda_* J_{ix}) = A_{ki}{}^{x}(Q_{xz*}J_j{}^z - \lambda_* J_{jx}).$$

Therefore (3.20) implies

$$(\nabla_k A_{ir*})f_j{}^r - (\nabla_k A_{jr*})f_i{}^r + (A_{ki*} - \lambda_* g_{ki})\xi_j - (A_{kj*} - \lambda_* g_{kj})\xi_i$$
$$= (A_{krx}A_i{}^{r*} - \lambda_* A_{kix})J_j{}^x - (A_{krx}A_j{}^{r*} - \lambda_* A_{kjx})J_i{}^x.$$

If we transvect ξ^j to this and take account of (1.8), (2.4) and (3.19), then we get

$$(\nabla_k A_{ir*})f_j{}^r\xi^j + (1-\alpha)(A_{ki*} - \lambda_* g_{ki}) = 0$$

and hence $(1-\alpha)\{h^* - (n+1)\lambda_*\} = 0$. Thus (3.16) is obtained.

Transforming (3.18) by $f_i{}^k$ and using (1.5), (2.4) and (2.6), we can verify that (3.17) is valid. This completes the proof of Lemma 3.2.

Lemma 3.3. Under the same hypothesis as that in Lemma 3.1, we have

$$h_{(3)} = h^* Q_{z**}Q^{z**}. \tag{3.21}$$

Proof. Since we have (3.17), it follows that we obtain

$$A_{jiy}A^{ji*} = \lambda_*(h_y - P_y) + Q_{zx*}P^{zx}{}_y,$$

which joined with (3.8) and (3.10) gives

$$P^z Q_{yz*} = h^* Q_{y**} + \lambda_*(P_y - h_y), \tag{3.22}$$

which implies

$$P^z Q_{z**} = h^* Q_{***} + \lambda_*(P^* - h^*). \tag{3.23}$$

Thus, (3.11) turns out to be

$$Q_{yz*}P^{yz*} = h^*Q_{***} + \lambda_*(P^* - h^*). \tag{3.24}$$

Making use of (3.9) and (3.22), it is seen that

$$Q_{uz*}P^{xz*}Q_x{}^{u*} = h^*Q_{z**}Q^{z**} + (\lambda*)^2(P^* - h^*). \tag{3.25}$$

On the other hand, we have from (3.17)

$$A_{jr}{}^*A_i{}^{r*} = \lambda_*(A_{ji*} - Q_{yz*}J_j{}^yJ_i{}^z) + Q_{yz*}Q_w{}^{y*}J_j{}^wJ_i{}^z$$

by virtue of (2.6). Transvecting A^{ji*} to the last equation and taking account of (1.15) and (1.16), we get

$$h_{(3)} = \lambda_*(h_{(2)} - Q_{yz*}P^{yz*}) + Q_{yz*}Q_w{}^{y*}P^{wz*}.$$

Since we have from (3.8)

$$h_{(2)} = h^*Q_{***}, \tag{3.26}$$

it follows, using (3.24) and (3.25), that $h_{(3)} = h^*Q_{z**}Q^{z**}$. Hence Lemma 3.3 is proved .

Lemma 3.4. Under the same assumptions as that in Lemma 3.1, we have

$$\Delta h_{(2)} = (n-1)\{h_{(2)} - \frac{1}{n+1}(h^*)^2\}. \tag{3.27}$$

Proof. By (3.6) we have

$$\Delta A_{ji}{}^* = (n+1)A_{ji}{}^* - h^*g_{ji} + h^*(A_{ji}{}^*)^2 - h^*Q_{x**}A_{ji}{}^x, \tag{3.28}$$

where we have used (3.1), (3.5), (3.7) and (3.8). If we transvect (3.28) with $J_*{}^j J_*{}^i$ and make use of (1.6), (1.15), (2.6) and (2.7), we obtain

$$(3.29)\qquad (\Delta A_{ji}{}^*) J_*{}^j J_*{}^i = (n+1) P_{***} - h^*(1 - w_*{}^2).$$

As is, in the proof of Lemma 3.2, already shown that λ_* is constant, (2.10) turns out to be

$$\nabla_k Q_{***} = 2w_*(Q_{z**} J_k{}^z - \lambda_* J_{k*}) + (\nabla_k A_{ji}{}^*) J_*{}^j J_*{}^i.$$

We then have

$$\Delta Q_{***} = -2(J_{j*} + \lambda_* \xi_j)(Q_{z**} J^{jz} - \lambda_* J^{j*}) + (\Delta A_{ji}{}^*) J_*{}^j J_*{}^i$$
$$+ 2w_*\{(\nabla_j Q_{z**}) J^{jz} + (n+1)(Q_{z**} w^z - \lambda_* w_*)\}$$

because of (1.11), (1.14), (2.4) and (3.2), which joined with (1.6), (1.8), (2.7), (2.8), (3.16) and (3.29) yields

$$(3.30)\qquad \Delta Q_{***} = (n-1)\{Q_{***} - \frac{1}{n+1} h^*\} + 2w_*(\nabla_j Q_{z**}) J^{jz}.$$

On the other hand, transvecting (3.12) with g^{kj} and taking account of (3.2), (3.4), (3.7) and (3.16), the last term of (3.30) vanishes identically. Thus we arrive at (3.27) because of (3.26). This completes the proof.

By (3.28) we have

$$(3.31)\qquad A^{ji*} \Delta A_{ji}{}^* = (n+1) h_{(2)} - (h^*)^2$$

by virtue of Lemma 3.1 and Lemma 3.3.

Since we have in general

$$(3.32)\qquad \frac{1}{2} \Delta h_{(2)} = A^{ji*} \Delta A_{ji*} + |\nabla A^*|^2,$$

it is seen that

$$(n+3)|A^* - \frac{h^*}{n+1} I|^2 + 2|\nabla A^*|^2 = 0$$

by (3.31) and Lemma 3.4., where I denotes the unit tensor. Therefore M is pseudo-umbilical and hence A^* is parallel. Thus we have

Theorem 3.5. Let M be a generic submanifold satisfying (2.1) of S^{2m+1} with nonvanishing parallel mean curvature vector. If the function $\alpha(1-\alpha)$ is not zero almost everywhere, then M is pseudo-umbilical.

Remark 2. The hypersurface satisfying (2.1) in a Sasakian space form is totally umbilical (cf. [5]).

4. Generic submanifolds tangent to the structure vector

Let M be a generic submanifold satisfying (2.1) of an odd-dimensional sphere S^{2m+1}. Suppose that the mean curvature vector of M is nonzero and is parallel in the normal bundle. Moreover, we suppose that the submanifold M is tangent to the structure vector V. Then by (1.4) we have

$$w_x = 0 \tag{4.1}$$

and hence $\xi_t \xi^t = 1$. Thus, (1.6) and (1.7) become respectively

$$J_x{}^t J_t{}^y = \delta_x{}^y, \tag{4.2}$$

$$f_t{}^h J_x{}^t = 0, \quad f_{jt}\xi^t = 0. \tag{4.3}$$

We also obtain from (1.14)

$$A_{jrx}\xi^r = -J_{jx}. \tag{4.4}$$

Consequently (2.2) leads to

$$(A_{jr*}J_z{}^r)J_i{}^z - (A_{ir*}J_z{}^r)J_j{}^z + J_{i*}\xi_j - J_{j*}\xi_i = 0.$$

Transvecting this with $J_y{}^i$ and using (4.2) and (4.3), we get

$$A_{jr*}J_y{}^r = P_{yz*}J_j{}^z - \delta_{y*}\xi_j. \tag{4.5}$$

If we transvect (3.7) with $J_y{}^iJ_u{}^j$ and use (4.4) and (4.5), then we obtain

$$P_{yz*}P_{xu}{}^z - P_{zu*}P_{xy}{}^z = \delta_{u*}\delta_{xy} - \delta_{y*}\delta_{xu} \tag{4.6}$$

and thus

$$P_{xz*}P^{xz}{}_y - P^zP_{zy*} = (p-1)\delta_{y*}, \tag{4.7}$$

where $p = 2m - n$.

Lemma 4.1. Let M be a generic submanifold satisfying (2.1) of S^{2m+1} with nonvanishing parallel mean curvature vector. If M is tangent to the structure vector, then we have

$$A_{jiy}A^{ji*} = h^*P_{y**} + (n+1)\delta_{y*}, \tag{4.8}$$

$$h_{(3)} = h^*P_{z**}P^{z**} + (n+1)P_{***} + h^*. \tag{4.9}$$

Proof. Differentiating (4.5) covariantly along M and making use of (1.11), (1.13), (2.1) and (4.1), we find

$$(\nabla_kA_{jr*})J_y{}^r + A_r{}^s{}_*A_{ksy}f_j{}^r = (\nabla_kP_{yz*})J_j{}^z + P_{yz*}A_{kr}{}^zf_j{}^r - \delta_{y*}f_{kj}.$$

Hence, if we take the skew-symmetric part with respect to k and j, then we obtain

$$(4.10)\ A_r{}^s{}_* A_{ksy} f_j{}^r - A_r{}^s{}_* A_{jsy} f_k{}^r$$
$$= (\nabla_k P_{yz*}) J_j{}^z - (\nabla_j P_{yz*}) J_k{}^z + P_{yz*}(A_{kr}{}^z f_j{}^r - A_{jr}{}^z f_k{}^r) + 2\delta_{y*} f_{jk}.$$

Transvecting (4.10) with f^{jk} and using (1.5) and (4.3), we can get

$$A_r{}^s{}_* A_{ksy}(g^{kr} - \xi^k \xi^r - J^{kw} J_w{}^r)$$
$$= P_{yz*} A_{kr}{}^z (g^{kr} - \xi^k \xi^r - J^{kw} J_w{}^r) + \delta_{y*}(n-p),$$

which joined with (3.4), (4.4) and (4.5) yields

$$A_{jiy} A^{ji*} + J_*{}^s A_{ksy} \xi^k - (P_{wz*} J^{sw} - \delta_{z*} \xi^s) A_{ksy} J^{kz}$$
$$= h^* P_{y**} - P^z P_{zy*} + (n-p)\delta_{y*}.$$

Hence we have

$$A_{jiy} A^{ji*} = h^* P_{y**} + P_{wz*} P^{wz}{}_y - P^z P_{yz*} + 2 J_{sy} J^{s*} + (n-p)\delta_{y*},$$

where we have used (1.12) and (4.4), which together with (4.2) and (4.7) implies that (4.8) is valid.

When $y = *$ in (4.10), we have

$$(4.11) \qquad -2 A_{js}{}^* A_r{}^{s*} f_k{}^r = (\nabla_k P_{z**}) J_j{}^z - (\nabla_j P_{z**}) J_k{}^z$$
$$+ P_{z**}(A_{kr}{}^z f_j{}^r - A_{jr}{}^z f_k{}^r) + 2 f_{jk}$$

because of (2.1).

By the way we have

$$A^{jr*} f_r{}^t (\nabla_t P_{z**}) J_j{}^z = 0$$

with the aid of (4.3) and (4.5). Hence, by transvecting $A_s{}^{j*} f^{ks}$ to (4.11) and using (1.5), we find

$$A_r{}^{t*} A_{ts}{}^* A_i{}^{s*}(g^{ri} - \xi^r \xi^i - J^{rz} J_z{}^i)$$
$$= P_{z**} A_{rs}{}^z A_i{}^{r*}(g^{si} - \xi^s \xi^i - J^{sz} J_z{}^i) + A_{ri}{}^*(g^{ri} - \xi^r \xi^i - J^{rz} J_z{}^i),$$

or, using (4.2), (4.3), (4.4) and (4.5)

$$h_{(3)} - A^{st*}(P_{yz*}J_t{}^y - \delta_{z*}\xi_t)(P_x{}^{z*}J_s{}^x - \delta^{z*}\xi_s)$$
$$= P_{z**}A_{ji}{}^z A^{ji*} - P_{z**}A_{js}{}^z J^{sw}(P_{yw*}J^{jy} - \delta_{w*}\xi^j) + h^* - P^*.$$

Thus it follows that we obtain

$$h_{(3)} - P_{yz*}P_x{}^{z*}P^{xy*}$$
$$= h^* P_{z**}P^{z**} + (n+2)P_{***} - P_{z**}P_{yw*}P^{zwy} + h^* - P^*$$

because of (4.2), (4.4) and (4.8). Since we have from (4.6)

$$P_{yz*}P_x{}^{z*}P^{xy*} - P_{z**}P_{xy*}P^{xyz} = P^* - P_{***}, \tag{4.11}$$

above equation can be written as

$$h_{(3)} = h^* P_{z**}P^{z**} + (n+1)P_{***} + h^*,$$

which proves (4.9). Hence, Lemma 4.1 is proved.

Lemma 4.2. Under the same hypothesis as that in Lemma 4.1, the function $h_{(2)}$ is harmonic.

Proof. By definition, we have $P_{***} = A_{ji}{}^* J_*{}^j J_*{}^i$. Differentiating this covariantly and making use of (1.11), (4.1) and (4.5), we find

$$\nabla_k P_{***} = (\nabla_k A_{ji}{}^*)J_*{}^j J_*{}^i + 2(P_{z**}J_j{}^z - \xi_j)A_{kr}{}^* f^{jr},$$

which gives $\nabla_k P_{***} = (\nabla_k A_{ji}{}^*)J_*{}^j J_*{}^i$ because of (4.3). Thus the Laplacian of the function P_{***} is given by

$$\Delta P_{***} = (\Delta A_{ji}{}^*)J_*{}^j J_*{}^i + 2(\nabla_k A_{ji}{}^*)J_*{}^i A_r{}^{k*} f^{jr},$$

which together with (2.1) and (3.2) yields

$$\Delta P_{***} = (\Delta A_{ji}{}^{*}) J_{*}{}^{j} J_{*}{}^{i}.$$

Hence, by (3.6) we have

$$\Delta P_{***} = S_{jr} A_{i}{}^{r*} J_{*}{}^{j} J_{*}{}^{i} - R_{kjih} A^{kh*} J_{*}{}^{j} J_{*}{}^{i}. \tag{4.12}$$

On the other hand, we verify, taking account of (3.5), (4.4) and (4.5), that

$$S_{ji} J_{*}{}^{i} = (n-1) J_{j*} + h^{*}(P_{z**} J_{j}{}^{z} - \xi_{j}) - P_{wx}{}^{*} A_{jr}{}^{x} J^{rw}.$$

Thus, using (1.8), (4.2), (4.3) and (4.4) we obtain

$$\begin{aligned} &S_{jr} A_{i}{}^{r*} J_{*}{}^{i} J_{*}{}^{j} \\ &= (n-1) P_{***} + h^{*} P_{z**} P^{z**} - P_{y**} P_{wx*} P^{yxw} + h^{*} - P^{*}. \end{aligned} \tag{4.13}$$

We also have from (3.1)

$$\begin{aligned} R_{kjih} A^{kh*} J_{*}{}^{j} J_{*}{}^{i} = h^{*} - P_{***} + P_{z**} A^{kh*} A_{kh}{}^{z} \\ - A^{kh*}(P_{wx*} J_{h}{}^{w} - \delta_{x*}\xi_{h})(P_{y}{}^{x*} J_{k}{}^{y} - \delta^{x*}\xi_{k}) \end{aligned}$$

because of (4.2), which joined with (4.3), (4.4), (4.8) and (4.11) yields

$$\begin{aligned} &R_{kjih} A^{kh*} J_{*}{}^{j} J_{*}{}^{i} \\ &= h^{*} - P^{*} + h^{*} P_{z**} P^{z**} + (n-1) P_{***} - P_{z**} P_{xy*} P^{xyz}. \end{aligned}$$

Hence the right hand side of (4.12) vanishes identically by virtue of (4.13). Since we have from (4.8)

$$h_{(2)} = h^{*} P_{***} + n + 1, \tag{4.14}$$

it follows that we have $\Delta h_{(2)} = 0$ because the mean curvature vector is parallel. This completes the proof.

On the other hand, we have from (3.5)

$$S_{js}A_i^{\ s*}A^{ji*} = nh_{(2)} + h^* h_{(3)} - A_{jr}^{\ \ x}A_s^{\ r}{}_x A_i^{\ s*}A^{ji*},$$

which together with (3.7), (4.9) and (4.14) implies that

$$\begin{aligned} S_{js}A_i^{\ s*}A^{ji*} = {} & (h^*)^2 P_{z**}P^{z**} + (2n+1)h^* P_{***} + n(n+1) \\ & + (h^*)^2 - A_{jr}^{\ \ x}A_{isx}A^{rs*}A^{ji*}. \end{aligned}$$

We also have from (3.1)

$$\begin{aligned} R_{kjih}A^{kh*}A^{ji*} = {} & (h^*)^2 - h^* P_{***} - (n+1) + (h^*)^2 P_{z**}P^{z**} \\ & + 2(n+1)h^* P_{***} + (n+1)^2 - A_{jr}^{\ \ x}A_{isx}A^{rs*}A^{ji*} \end{aligned}$$

because of (4.8) and (4.14). From the last two equations, it follows that we obtain

$$S_{js}A_i^{\ s*}A^{ji*} - R_{kjih}A^{kh*}A^{ji*} = 0.$$

Therefore, by (3.6) we have $A^{ji*}\Delta A_{ji*} = 0$ and consequently $\nabla A^* = 0$ by virtue of (3.32) and Lemma 4.2. Thus we have

Theroem 4.3. Let M be a generic submanifold satisfying (2.1) of an odd-dimensional unit sphere with nonvanishing parallel mean curvature vector. If the structure vector V is tangent to M, then the shape operator in the direction of the mean curvature vector is parallel.

Remark 3. An example satisfying all conditions of Theorem 4.3 is given in [2].

5. Theorems

Let M be a generic submanifold of an odd-dimensional unit sphere. Suppose that the mean curvature vector of M is nonzero and is parallel in the normal bundle, and that $A^* f = f A^*$ holds on M.

If we now put $M_0 = \{p \in M | \alpha(p) = 0\}$, $M_1 = \{p \in M | \alpha(p) = 1\}$ and $\bar{M} = M - M_0 \cup M_1$, then we have $M = \bar{M} \cup M_0 \cup M_1$, where $\alpha = |w_x|^2$. The sets M_0 and M_1 are then geometrically characterized as follows: The structure vector field V in the ambient space is tangent to the generic submanifold M at any point in the set M_0, and the vector field V is orthogonal to M at each point in M_1 because of (1.4).

If we suppose that there is an open subset U contained in M_1, then we have $A_{jix}w^x = 0$ in U by Remark 1 and hence $h^*w_* = 0$ because of (3.4). Consequently we see that $w_* = 0$ on U since the mean curvature vector is not zero, and therefore $J_{j*} = 0$ on U by (1.14). This contradicts (1.6). Thus the set M_1 is bordered. Hence we may discuss properties of the covariant derivative of the second fundamental tensor A^* only on $\bar{M} \cup M_0$ since it is continuous.

Suppose that there exists a connected component W of the set M_0. Then, by Theorem 4.3, we have $\nabla A^* = 0$ on W. When $\bar{M}$ is not empty, as a consequence of Theorem 3.5, we see that $\nabla A^* = 0$. Hence $\nabla A^* = 0$ on each component of $\bar{M}$ and M_0. Since ∇A^* is continuous it follows that we obtain $\nabla A^* = 0$ on M. When $\bar{M}$ is empty, Theorem 4.3 implies $\nabla A^* = 0$. Thus we have

Theorem 5.1. Let M be a generic submanifold of an odd-dimensional unit sphere with nonvanishing parallel mean curvature vector. If the shape operator A^* in the direction of the mean curvature vector commutes with the structure tensor f induced on M, then A^* is parallel.

We will consider the case where A^* is parallel on M. For any point in M we can choose a local orthonormal frame $\{E_i\}$ so that the shape operator A^* is diagonalizable at that point q, say $A_{ji}{}^* = \lambda_j \delta_{ji}$. Then $h_{(m)}$ can be written as

$$h_{(m)} = \sum_i \lambda_i{}^m, \quad (m = 1, 2, \cdots).$$

Because of $\nabla A^* = 0$, we see that $h_{(m)} = const.$ for any integer $m \geq 1$, it is seen that λ_i constant, namely all eigenvalues of A^* are constant. Taking accout of the Ricci formula for A^* and the facts that $R_{ji*x} = 0$ and $\nabla A^* = 0$, we find

$$(\lambda_j - \lambda_i)\sigma_{ji} = 0$$

for any fixed j and i, where σ_{ji} denotes the sectional curvature spanned by E_j and E_i. If eigenvalues of A^* are mutually distinct, we then obtain $\sigma_{ji} = 0$. Because each eigenvector of A^* is parallel, it follows that we have

Theorem 5.2. Let M be a generic submanifold of S^{2m+1} with nonvanishing parallel mean curvature vector. If $A^*f = fA^*$ and if the eigenvalues of A^* are mutually distinct, then M is flat.

Now, let $\mu_1, \cdots, \mu_s$ be mutually distinct eigenvalues of A^* and $n_1, \cdots, n_s$ their multiplicities. Since A^* is parallel, the smooth distribution $T_a(a = 1, \cdots, s)$ which consists of all eigenvalues accociated with the eigenvalue can be defined and is parallel. M is assumed to be simply connected and complete, then by means of the de Rham decomposition theorem, the submanifold is a product of Riemannian manifolds $M_1 \times \cdots \times M_s$, where the tangent bundle of M_a correspond to T_a. Since the shape operator A^* restricted to T_a is proprotional to the identity transformation of T_a and each submanifold M_a is totally geodesic in M, the mean curvature of M is an umbilical section of M_a in S^{2m+1}. Thus, by means of the above arguments and Theorem 5.1, we have

Theorem 5.3. Let M be a complete and simply connected generic submanifold of S^{2m+1} with nonvanishing parallel mean curvature vector. If the shape operator A^* in the direction of the mean curvature vector commutes with the structure tensor f induced on M, then M is a product of Riemannian manifolds $M_1 \times \cdots \times M_s$, where s is the number of distinct eigenvalues of A^*, and the mean curvature vector of M is an umbilical section of M_a $(a = 1, \cdots, s)$.

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