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ALMOST COMPLEX METRIC CONNECTIONS ON ALMOST HERMITIAN MANIFOLDS

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It is well known that the classical Kähler manifolds can be characterized in terms of the Hermitian geometry by some torsionless connections defined by the metric. More precisely, an almost Hermitian manifold (M,g,J) with an almost complex structure J, Hermitian metric g and fundamental 2-form F, admits a torsion-free almost complex connection if and only if the fundamental form F is closed and the almost complex structure J is a torsionless structure ($N=0$, where N is the Nijenhuis tensor), i.e. when (M,g,J) is a Kähler manifold.

In this note we give an example of a linear family (stratification) of almost complex metric connections on an almost Hermitian manifold M dependent on two real parameters. The well known characteristic connection of A.Lichnerowicz [6],[7]corresponds to one strata of the introduced family.

We develop an analogous characteristic for an almost Kähler manifold (M,g,J,F) $(dF=0)$ with the help of a given almost complex connection with a torsion ($\nabla J=0$ but $N\neq 0$).

1. Connections on almost Hermitian manifolds.

Let (M,g,J) be an almost Hermitian manifold endowed with an almost complex structure J and a Riemannian metric g compatible with the almost complex structure J. By F is denoted the fundamental 2-form of M, defined by $F(X,Y)=g(X,JY)$ for each pair $X,Y\in\mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of C^∞ vector fields on M. Let N denote the torsion tensor of the almost complex structure J, i.e.

$$N(X,Y)=2([JX,JY]-J[JX,Y]-J[X,JY]-[X,Y])$$

for each pair $X,Y\in\mathfrak{X}(M)$ (see [5]).

It is well known that there exists unique metric connection D on M with a given torsion tensor T. This connection is defined by the following formula (see [4]):

$$2g(D_XY,Z)=X(g(Y,Z))+Y(g(Z,X))-Z(g(X,Y))$$
$$+g([X,Y]+T(X,Y),Z)+g([Z,X]+T(Z,X),Y)-g([Y,Z]+T(Y,Z),X)$$

and if ∇ is the Riemannian connection on M then

$$(1.1)\quad 2g(D_XY,Z)=2g(\nabla_XY,Z)+g(T(X,Y),Z)+g(T(Z,X),Y)-g(T(Y,Z),X).$$

We put

$$(1.2)\qquad P(X,Y,Z)=g(T(JY,Z)+T(Y,JZ),X)$$
$$-g(T(X,Y)+JT(X,JY),JZ)-g(T(Z,X)+JT(JZ,X),JY).$$

Note that P is a skew-symmetric tensor with respect to the last two arguments and

$$P(X,JY,JZ)=-P(X,Y,Z).$$

We put also

$$(1.3)\qquad GN(X,Y,Z) = g(N(X,Y),Z)+g(N(Y,Z),X)+g(N(Z,X),Y),$$

$$(1.4)\qquad GNJ(X,Y,Z) = g(N(X,Y),JZ)+g(N(Y,Z),JX)+g(N(Z,X),JY),$$

where GN and GNJ are skew-symmetric tensors and the 3-forms

$$(1.5)\ \alpha(X,Y,Z) = dF(JX,JY,JZ)-dF(X,Y,JZ)-dF(X,JY,Z)-dF(JX,Y,Z),$$

$$(1.6)\ \beta(X,Y,Z) = dF(X,Y,Z)-dF(X,JX,JZ)-dF(JX,JY,Z)-dF(JX,Y,JZ).$$

Remind that from $F(X,Y)=g(X,JY)$ and the formula for dF it follows

$$3dF(X,Y,Z) = X(g(Y,JZ))+Y(g(Z,JX))+Z(g(X,JY))$$
$$-g([X,Y],JZ)-g([Z,X],JY)-g([Y,Z],JX).$$

Lemma 1.1. Let D be a metric connection with torsion tensor T on the almost Hermitian manifold (M,g,J). Then

$$2g(D_XJY-JD_XY,Z)=3dF(X,JY,JZ)-3dF(X,Y,Z)+\frac{1}{2}g(N(Y,Z),JX)-P(X,Y,Z)$$

is satisfied for every $X,Y,X\in\mathfrak{X}(M)$.

Proof. From $g(D_XY,Z)=-g(D_XY,JZ)$ we have

$$g(D_XJY-JD_XY,Z)=g(D_XJY,Z)+g(D_XY,JZ)$$

and using the above formula for D we obtain that

$$
\begin{aligned}
(1.7)\qquad 2g(D_X JY - JD_X Y, Z) &= X(g(JY,Z)) + JY(g(Z,X)) - Z(g(X,JY)) \\
&+ g([X,JY] + T(X,JY),Z) + g([Z,X] + T(Z,X),JY) \\
&- g([JY,Z] + T(JY,Z),X) + X(g(Y,JZ)) + Y(g(JZ,X)) \\
&+ g([JZ,X] + T(JZ,X),Y) - g([Y,JZ] + T(Y,JZ),X) \\
&- JZ(g(X,Y)) + g([X,Y] + T(X,Y),JZ).
\end{aligned}
$$

Taking account of the formula for dF we have

$$
\begin{aligned}
(1.8)\qquad 3dF(X,JY,JZ) - 3dF(X,Y,Z) &= X(g(Y,JZ)) + JY(g(Z,X)) - JZ(g(X,JY)) \\
&+ g(J[X,JY],JZ) + g(J[JZ,X],JY) - g([JY,JZ],JX) \\
&- X(g(Y,JZ)) - Y(g(Z,JX)) - Z(g(X,JY)) \\
&+ g([X,Y],JZ) + g([Z,X],JY) + g([Y,Z],JX).
\end{aligned}
$$

Substituting (1.8) into the right hand side of (1.7). We have the required equality. Q.E.D.

Lemma 1.2. A metric connection D with torsion tensor T on the almost Hermitian manifold (M,g,J) is an almost complex connection if and only if the torsion tensor T satisfies

$$P(X,Y,Z) = 3dF(X,JY,JZ) - 3dF(X,Y,Z) + \frac{1}{2}g(N(Y,Z),JX)$$

for each $X,Y,Z \in \mathfrak{X}(M)$.

Proof. Let metric connection D be an almost complex connection, i.e.

$$D_X JY - JD_X Y = 0 \quad \text{for every} \quad X,Y \in \mathfrak{X}(M).$$

Then $g(D_X JY - JD_X Y, Z) = 0$ for every $X,Y,Z \in \mathfrak{X}(M)$ and the equation for the torsion tensor T follows from Lemma 1.1.

Let now the equation for the torsion tensor T be satisfied. Then

$$g(D_X JY - JD_X Y, Z) = 0 \quad \text{for every} \quad Z \in \mathfrak{X}(M),$$

so we obtain that $D_X JY - JD_X Y = 0$ for every $X,Y \in \mathfrak{X}(M)$. The last condition is equivalent to $DJ = 0$. Q.E.D.

Lemma 1.3. Let (M,g,J) be an almost Hermitian manifold with a fundamental 2-form F. Then the following identities are satisfied:

a) $$GN(X,Y,Z) = 6\alpha(X,Y,Z);$$

b) $$GNJ(X,Y,Z) = 6\beta(X,Y,Z).$$

Proof. By using the formulas for F and dF we calculate $\alpha(X,Y,Z)$ to obtain the identity a). The proof of the identity b) is similar.

Theorem 1.1. Let (M,g,J) be an almost Hermitian manifold and let F be the fundamental 2-form of M. There exists a family

$$\mathcal{D} = \left\{D_{pq} : (p,q) \in \mathbf{R}^2\right\}$$

of almost complex metric connection D_{pq} on M dependent on two parameters $p,q \in \mathbf{R}$. The connections $D_{pq} = D$ can be defined by the following equation:

$$\begin{aligned} 2g(D_XY,Z) &= 2g(\nabla_XY,Z) + \frac{3}{2}\alpha(X,Y,Z) - \frac{1}{4}g(N(Y,Z),X) \\ &\quad +(2q-12p)dF(JX,JY,JZ) + (2q-12p+3)dF(JX,Y,Z) \end{aligned}$$

for $X,Y,Z \in \mathfrak{X}(M)$, where ∇ is the Riemannian connection on M.

Proof. Let D be a connection with a torsion tensor T, defined by

(1.9) $$\begin{aligned} g(T(X,Y),Z) &= \lambda_1 g(N(X,Y),Z) + \lambda_2 g(N(Y,Z),X) \\ &\quad + \lambda_2 g(N(Z,X),Y) + \mu_1 dF(JX,JY,JZ) + \mu_2 dF(X,Y,JZ) \\ &\quad + \mu_3 dF(X,JY,Z) + \mu_3 dF(JX,Y,Z) \end{aligned}$$

for every $X,Y,Z \in \mathfrak{X}(M)$, where $\lambda_1,\lambda_2,\mu_1,\mu_2,\mu_3 \in \mathbf{R}$ are arbitrary parameters.
By (1.2) and (1.9), we have

$$\begin{aligned} P(X,Y,Z) &= (2\lambda_1 - 4\lambda_2)g(N(Y,Z),JX) - 2\lambda_1 g(N(X,Y),JZ) \\ &\quad - 2\lambda_1 g(N(Z,X),JY) + 2\mu_2 dF(X,Y,Z) - 2\mu_2 dF(X,JY,JZ) \\ &\quad + (\mu_1+\mu_2-2\mu_3)dF(JX,JY,Z) + (\mu_1+\mu_2-2\mu_3)dF(JX,Y,JZ). \end{aligned}$$

If the parameters $\lambda_1,\lambda_2,\mu_1,\mu_2,\mu_3$ satisfy the equations

$$2\lambda_1 - 4\lambda_2 - \frac{1}{2} = -2\lambda_1, \quad 2\mu_2 - 3 = 12\lambda_1,$$
$$2\mu_2 + 3 = -(\mu_1+\mu_2-2\mu_3),$$

then the condition of Lemma 1.2 reduces to

$$-\lambda_1 GNJ(X,Y,Z) + 6\lambda_1\beta(X,Y,Z) = 0$$

for every $X,Y,Z \in \mathfrak{X}(M)$ and $\lambda_1 \in \mathbf{R}$.
It follows form Lemma 1.3 that if we put

$$\lambda_1 = p, \quad \lambda_2 = p - \frac{1}{8}, \quad \mu_2 = 6p - \frac{3}{2},$$

$$\mu_3 = q, \quad \mu_1 = 2q - 18p + \frac{3}{2},$$

where $p, q \in \mathbf{R}$ are arbitrary parameters, then the connection D with the torsion tensor T, defined by

$$\begin{aligned} g(T(X,Y),Z) = &-\frac{1}{8}(g(N(Y,Z),X) + g(N(Z,X),Y)) \\ &+ pGN(X,Y,Z) + \left(2q - 18p + \frac{3}{2}\right) dF(JX,JY,JZ) \\ &+ \left(6p - \frac{3}{2}\right) dF(X,Y,JZ) + q(dF(X,JY,Z) + dF(JX,Y,Z)) \end{aligned} \tag{1.10}$$

is an almost complex connection.

The theorem follows after substituting (1.10) into the formula (1.1) of D and taking account of Lemma 1.3. a). Q.E.D.

We have defined the family

$$\mathcal{D} = \left\{D_{pq} : (p,q) \in \mathbf{R}^2\right\}$$

dependent in a linear manner on a point $(p,q) \in \mathbf{R}^2$. The parameter space $\mathbf{R}^2$ admits an elementary stratification defined as follows

$$(p,q) \sim (r,s) \quad \text{iff} \quad g(T_{pq}(X,Y),Z) = g(T_{rs}(X,Y),Z)$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

An almost Hermitian manifold (M,g,J) is called a quasi-Kähler manifold if $(\nabla_X J)Y + (\nabla_{JX} J)JY = 0$ for all $X, Y \in \mathfrak{X}(M)$. It is easily observed that (M,g,J) is a quasi-Kähler manifold iff $dF(X,JY,JZ) + dF(X,Y,Z) = 0$ for all $X,Y,Z \in \mathfrak{X}(M)$. As it is well-known an almost Kähler manifold ($dF = 0$) and a nearly Kähler manifold ($(\nabla_X J)Y + (\nabla_Y J)X = 0$ for all $X, Y \in \mathfrak{X}(M)$) are quasi-Kähler manifolds. For a quasi-Kähler manifold (M,g,J), we may see that $D_{p,q} = D_{r,s}$ for all $(p,q),(r,s) \in \mathbf{R}^2$. When (M,g,J) is not quasi-Kähler, then it follows that $D_{p,q} = D_{r,s}$ iff

$$q - s - 6(p - r) = 0 \quad \text{or} \quad q - 6p = s - 6r$$

We herewith write a proof for

"(A) $(\nabla_X J)Y + (\nabla_{JX} J)JY = 0$ iff (B) $dF(X,JY,JZ) + dF(X,Y,Z) = 0$ "

It is evident that (A) implies (B). We shall show the converse. Indeed, $dF(X,Y,Z) + dF(X,JY,JZ) = 0$ implies

$$g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y) + g((\nabla_{JY} J)JZ, X) + g((\nabla_{JZ} J)X, JY) = 0 \qquad \text{(a)}$$

Putting $Z = JX$ in (a), we obtain

$$(\nabla_X J)X + (\nabla_{JX} J)JX = 0 \qquad \text{(b)}$$

Linearizing (b), we have

$$(\nabla_X J)Y + (\nabla_Y J)X + (\nabla_{JX} J)JY + (\nabla_{JY} J)JX = 0 \qquad \text{(c)}$$

Replacing X by JX in (c), we get

$$(\nabla_{JX} J)Y + (\nabla_Y J)JX - (\nabla_X J)JY - (\nabla_{JY} J)X = 0,$$

from which

$$J\{(\nabla_X J)Y - (\nabla_Y J)X + J(\nabla_{JY} J)X - J(\nabla_{JX} J)Y\} = 0,$$

and hence

$$(\nabla_X J)Y - (\nabla_Y J)X + (\nabla_{JX} J)JY - (\nabla_{JY} J)JX = 0. \qquad \text{(d)}$$

Thus, from (c) and (d), we have finally

$$(\nabla_X J)Y + (\nabla_{JX} J)JY = 0 \qquad \text{Q.E.D.}$$

The almost complex metric connection D on an almost Hermitian manifold (M, g, J) is called a characteristic connection, if its torsion tensor T satisfies the condition

$$T(X, Y) + T(JX, JY) = 0$$

for all $X, Y \in \mathfrak{X}(M)$ (cf [2]). Moreover, in the same paper it is proved that there exists unique characteristic connection on M.

Now we proved that the characteristic connection belongs to the family $\mathcal{D}$.

Theorem 1.2. Let (M, g, J) be an almost Hermitian manifold and let F be the fundamental 2-form of M. Then the characteristic connection D belongs to the family $\mathcal{D}$ and is given by the following equation:

$$8g(D_X Y, Z) = 8g(\nabla_X Y, Z) - g(N(Y, Z), X)$$
$$+ 6\,(dF(JX, JY, JZ) - dF(X, Y, JZ) - dF(X, JY, Z) + dF(JX, Y, Z))$$

for every $X, Y, Z \in \mathfrak{X}(M)$, where ∇ is the Riemannian connection on M.

Proof. Using the formula for $g(T(X, Y), Z)$ in Theorem1.1 we obtain that

$$g(T(X, Y) + T(JX, JY), Z) = (2q - 12p)\,(dF(X, Y, JZ) + dF(JX, JY, JZ))$$

for every $X, Y, Z \in \mathfrak{X}(M)$.

From the above result, it follows immediately that D is characteristic if (M,g,J) is quasi-Kähler and that if (M,g,J) is not quasi-Kähler, D is characteristic if and only if $q-6p=0$.

This means that the characteristic connection corresponds to these points $(p,q)\in \mathbf{R}^2$ which varie on the line through the origin.

Now the torsion tensor T is defined by

$$\begin{aligned} g(T(X,Y),Z) &= \frac{1}{4}g(N(X,Y),Z)+\frac{1}{8}\{g(N(Z,X),Y)+g(N(Y,Z),X)\} \\ &\quad +\frac{2}{3}\{dF(Z,X,JY)+dF(Z,JX,Y)\} \end{aligned}$$

and D in the desired connection.

We herewith write the proof of the above equality. By Lemma 1.2. we have

$$\begin{aligned} &2g(JT(JY,Z)-T(Y,Z),X) \\ &=3\{dF(JX,Y,Z)-dF(X,Y,JZ)-dF(X,JY,Z)-dF(JX,JY,JZ)\} \\ &=3\{-dF(X,Y,Z)-2(dF(X,JY,Z)+dF(X,Y,JZ))\} \\ &=3\Big\{-\frac{1}{6}(g(N(X,Y),Z)+g(N(Y,Z),X)+g(N(Z,X),Y)) \\ &\quad -2(dF(X,JY,Z)+dF(X,Y,JZ))\Big\} \end{aligned}$$

On one hand, we have obtained the following

$$JT(JY,Z)+T(Y,Z)=\frac{1}{4}N(Y,Z).$$

We have immediately

$$\begin{aligned} g(T(Y,Z),Z) &= \frac{1}{4}g(N(Y,Z),X)+\frac{1}{8}\{g(N(X,Y),Z)+g(N(Z,X),Y)\} \\ &\quad +\frac{3}{2}\{dF(X,Y,JZ)+dF(X,JY,Z)\} \end{aligned}$$

Q.E.D.

Theorem 1.3. Let (M,g,J) be an almost Hermitian manifold and let F be the fundamental 2-form of M. Then the characteristic connection D is uniquely determined by the tensor fields N and F.

Proof. As we know (see [7]) the torsion tensor T of an almost complex connection satisfies the following identity:

$$T(JX,JY)-JT(JX,Y)-JT(X,JY)-T(X,Y)=-\frac{1}{2}N(X,Y).$$

It follows from this identity and Gray condition that

$$JT(JY,Z)+T(Y,Z)=\frac{1}{4}N(Y,Z).$$

Taking account of (1.2), we have

$$P(X,JY,Z)-P(JX,Y,Z)=2g(JT(JY,Z)-T(Y,Z),X).$$

Thus by using Lemma 1.2 we have finally

$$\begin{aligned} g(T(X,Y),Z) &= -\frac{1}{8}(g(N(Y,Z),X)+g(N(Z,X),Y)) \\ &\quad +\frac{3}{2}(dF(JX,JY,JZ)-dF(X,Y,JZ)). \end{aligned}$$

Thus we see that the characteristic connection D is uniquely determined by the tensor fields N and F. **Q.E.D.**

2. Connections on almost Kähler manifolds.

Let now (M,g,J,F) be an almost Kähler manifold, i.e. (M,g,J) is an almost Hermitian manifold and the fundamental 2-form F is closed ($dF=0$).

Lemma 2.1. **Let D be a metric connection with torsion tensor T on the almost Kähler manifold (M,g,J,F). Then**

$$2g(D_XJY-JD_XY,Z)=\frac{1}{2}g(N(Y,Z),JX)-P(X,Y,Z)$$

is satisfied for each $X,Y,Z\in\mathfrak{X}(M)$.

The proof follows from Lemma1.1 and $dF=0$.

Lemma 2.2. **The metric connection D with torsion tensor T on an almost Kähler manifold (M,g,J,F) is an almost complex connection if and only if T satisfies**

$$P(X,Y,Z)=\frac{1}{2}g(N(Y,Z),JX)$$

for each $X,Y,Z\in\mathfrak{X}(M)$.

The proof follows immediately from Lemma1.2 and $dF=0$.

Lemma 2.3. **Let (M,g,J,F) be an almost Kähler manifold. Then the following identities are satisfied:**

a) $$GN(X,Y,Z)=0;$$

b) $$GNJ(X,Y,Z)=0.$$

The proof follows from Lemma1.3 and $dF=0$.

Theorem 2.1. **The metric connection D with the torsion tensor**

$$T(X,Y)=\frac{1}{8}N(X,Y)$$

on the almost Kähler manifolds M is an almost complex connection.

The proof follows from Theorem 1.1 and Lemma 2.3. a).

Theorem 2.2. If an almost Hermitian manifold (M, g, J) admits an almost metric connection D with the torsion tensor $T(X,Y) = \frac{1}{8}N(X,Y)$, then (M, g, J) is a quasi-Kähler manifold and vice versa.

Proof. First we give the following

Lemma 2.4. Under the conditions of the above theorem we have that the fundamental 2-form F satisfies the following equation

$$dF(X,Y,Z) - dF(X,JY,JZ) + dF(JX,JY,Z) + dF(JX,Y,JZ) = 0$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Proof of the Lemma 2.4. We see that

$$P(X,Y,Z) = \frac{1}{4}\left(g(N(Y,Z),JX) - g(N(X,Y),JZ) - g(N(Z,X),JY)\right). \tag{2.1}$$

From Lemma 1.2 it follows that

$$\frac{1}{4}GNJ(X,Y,Z) = 3dF(X,Y,Z) - 3dF(X,JY,JZ). \tag{2.2}$$

The proof of the Lemma 2.4 can be finished using the Lemma 1.3.

Proof of the Theorem 2.2. Putting $Y = JX$ in (2.2) and taking account of (1.4), we have

$$dF(X,JX,Z) = 0 \quad \text{for all} \quad X, Z \in \mathfrak{X}(M)$$

By linearizing the above equality, we have

$$dF(X,JY,Z) + dF(Y,JX,Z) = 0$$

and hence

$$dF(X,Y,Z) + dF(JX,JY,Z) = 0.$$

Therefore, (M, g, J) is a quasi-Kähler manifold.

Conversely, we assume that (M, g, J) is a quasi-Kähler manifold. Then, by the last equality in the proof of theorem 1.2. we have

$$g(T(X,Y),Z) = -\frac{1}{8}\{g(N(Y,Z),X) + g(N(Z,X),Y)\} - 3dF(X,Y,JZ). \tag{2.3}$$

On one hand, by (2.2) and the hypothesis, we get

$$\frac{1}{4}GNJ(X,Y,Z) = 6dF(X,Y,Z). \tag{2.4}$$

By (1.4) and (2.4), we get

$$3dF(X,Y,JZ) = -\frac{1}{8}\{g(N(X,Y),Z) + g(N(Y,Z),X) + g(N(Z,X),Y)\}. \tag{2.5}$$

Thus, by (2.3) and (2.5), we have finally

$$g(T(X,Y),Z) = \tfrac{1}{8}g(N(X,Y),Z).$$ Q.E.D.

Remark 1. It is known that 4-dimensional quasi-Kähler manifold is necessarily an almost Kähler manifold [3]

Thus, form theorem 2.2. we have the following

Corollary 2.3. If a 4-dimensional almost Hermitian manifold (M,g,J) admits an almost complex metric connection D with the torsion tensor $N(X,Y) = \frac{1}{8}N(X,Y)$, then (M,g,J) is an almost Kähler manifold and vice versa.

Example. Let $H \times S^1$ be the cartesian product of the Heisenberg group H and the circle S^1. The Lie algebra $L(H \times S^1)$ of all left invariant vector fields has a base $\{f_1, f_2, f_3, f_4\}$ with

$$[f_1, f_2] = f_3 \quad \text{and} \quad [f_p, f_q] = 0 \quad \text{for} \quad (p,q) \neq (1,2).$$

Let J be a left invariant almost complex structure of $H \times S^1$. As we know (see [1]) the left invariant vector fields

$$f_3, \quad f_4, \quad Jf_3, \quad Jf_4$$

are linearly independent for every non-integrable J.

We put

$$e_1 = f_3, \quad e_2 = f_4, \quad e_3 = Jf_3, \quad e_4 = Jf_4$$

and now $[e_3, e_4] = \mu e_1$, where μ dependent on the structure J and $\mu \neq 0$ for every non-integrable J.

We consider the left invariant metric g, defined by

$$g(e_p, e_p) = 1 \quad \text{and} \quad g(e_p, e_q) = 0.$$

If the left invariant forms $\{e_1^*, e_2^*, e_3^*, e_4^*\}$ form the dual base of $\{e_1, e_2, e_3, e_4\}$ then the fundamental 2-form F is defined by

$$F = -(e_1^* \wedge e_2^* + e_3^* \wedge e_4^*)$$

and $dF = 0$.

We calculate $N(e_1, e_2) = 2\mu e_1$ and the Riemannian almost complex connection D is defined by

$$D_1e_1 = -\frac{\mu}{4}e_2, \quad D_1e_2 = -\frac{\mu}{4}e_1, \quad D_2e_2 = 0.$$

Remark 2. Another approach was proposed by Prof. K. Sekigawa. It is based on a non-metric connection defined as follows

$$D_X Y := \nabla_X Y - \frac{1}{2}J(\nabla_X J)Y + \frac{1}{4}J(\nabla_Y J)X + \frac{1}{4}(\nabla_{JY} J)X,$$

$$(D_X J)Y := D_X(JY) - JD_X Y.$$

The proposed connection D satisfies $DJ = 0$, but $Dg \neq 0$. The condition $Dg = 0$ is equivalent to the following equation

$$dF(JX, JY, JZ) + dF(X, Y, Z) = 0$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

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