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ON SOME VARIATIONAL INEQUALITIES

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ABSTRACT. In this paper, we introduce and study a new class of variational inequalities. Using the auxiliary principle technique, we prove the existence of a solution of this class of variational inequalities and suggest a new and novel iterative algorithm. Several special cases, which can be obtained from the main results, are also discussed.

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1. INTRODUCTION AND FORMULATION

An elegant theory of variational inequalities has been developed since the early sixties, which has greatly stimulated the research in pure and applied sciences. In the last thirty years remarkable progress has been made in the field of variational inequalities. Variational inequalities arise in models for a large number of mathematical, physical, regional, engineering and other problems. The theory of variational inequalities has led to exciting and important contributions to pure and applied sciences which includes work on differential equations, contact problems in elasticity, fluid flow through porous media, general equilibrium problems in economics and transportation, unilateral, obstacle, moving and free boundary problems, see, for example, [1,2,3,4,5,7,8,13,14]. Inspired and motivated by the recent research work going on in this field, we introduce and consider some new classes of variational inequalities. We remark that the projection method and its variant form cannot be applied to study the existence of a solution of these new variational inequalities. This fact motivated us to use the auxiliary principle technique of Glowinski, Lions and Tremolieres [5] and Noor [10,11] to study the problems of the existence of these variational inequalities. This technique deals with an auxiliary variational inequality problem and proving that the solution of the auxiliary problem is the solution of the original variational inequality problem. This technique is quite general and flexible. This technique is then used to suggest an iterative algorithm for variational inequalities.

To be more precise, let H be a real Hilbert space on which inner product and norm are denoted by $< .,. >$ and $\| . \|$ respectively. Let K be a nonempty closed convex set in H. Given $T, g : H \rightarrow H$ continuous operators, we consider the problem of finding $u \epsilon H$ such that $g(u) \epsilon K$ and

$$< Tu, v - g(u) > + b(u, v) - b(u, g(u)) \geq 0, \qquad for \ all \ \ v \epsilon K, \tag{1.1}$$

where the form $b(.,.) : H \times H \rightarrow R$ is non-differentiable and satisfies the following:

(i) $b(u, v)$ is linear in the first argument.

(ii) $b(u, v)$ is bounded, that is, there exists a constant $\gamma > 0$ such that

$$b(u, v) \leq \gamma \| u \| \| v \|, \qquad for \ all \ \ u, v \epsilon K, \tag{1.2}$$

(iii) $b(u, v) - b(u, w) \leq b(u, v - w)$, for all $u, v, w \epsilon H$.

The inequalities of the type (1.1) are called the mixed variational inequalities. We now discuss some special cases.

I. Note that, if $g \equiv I$, the identity operator, then problem (1.1) is equivalent to finding $u \epsilon K$ such that

$$< Tu, v - u > + b(u, v) - b(u, u) \geq 0, \qquad for \ all \ \ v \epsilon K, \tag{1.3}$$

a problem considered and studied by Kikuchi and Oden [7], and Noor [10] by using quite different techniques. For physical and mathematical formulations, see [1,2,7].

II. If $b(u, v) \equiv j(v)$, is a convex, lower semi- continuous, proper and non-differentiable functional, then the problem (1.1) is equivalent to finding $u \epsilon H$ such that $g(u) \epsilon K$ and

$$< Tu, v - g(u) > + j(v) - j(g(u)) \geq 0, \qquad for \ all \ \ v \epsilon K, \tag{1.4}$$

which is called the mixed variational inequality problem and appears to be new one.

III. If $b(v,u) \equiv 0$, then problem (1.1) reduces to the problem of finding $u \epsilon H$ such that $g(u) \epsilon K$ and

$$< Tu, v - g(u) > \geq 0, \qquad for\ all\ v \epsilon K, \qquad (1.5)$$

a problem introduced by Oettli[13], Isac[6] and Noor[9] independently in different contexts and applications.

IV. If $b(u,v) \equiv 0$, $K^{*} = \{u \epsilon H; < u, v > \geq 0$ for all $v \epsilon K\}$ is a polar cone of the convex cone K in H, and $K \subset g(K)$, then problem (1.1) is equivalent to finding $u \epsilon H$ such that

$$g(u) \epsilon K, \quad Tu \epsilon K^{*} \quad and \quad < Tu, g(u) > = 0, \qquad (1.6)$$

which is known as the general nonlinear complementarity problem. The problem (1.6) is mainly due to Oettli [13]. For the iterative methods, convergence analysis and extensions of the general nonlinear complementarity problem (1.3), see [13,12]. The problem (1.6) is quite general and includes many previously known classes of linear and nonlinear complementarity problems as special cases.

It is clear that problems (1.3) - (1.6) are special cases of the problem (1.1). In brief, the problem (1.1) is the most general and unifying ones, which is one of the main motivations of the paper.

DEFINITION 1.1 A mapping $T : H \to H$ is said to be:

(a) **Strongly monotone**, if there exists a constant $\alpha > 0$ such that

$$< Tu - Tv, u - v > \geq \alpha \| u - v \|^{2}, \qquad for\ all\ u, v \epsilon H.$$

(b) **Lipschitz continuous**, if there exists a constant $\beta > 0$ such that

$$\| Tu - Tv \| \leq \beta \| u - v \|, \qquad for\ all\ u, v \epsilon H.$$

In particular, it follows that $\alpha \leq \beta$.

2. MAIN RESULTS

In this section, we prove the existence of a solution of the mixed variational inequality problem (1.1) by using the auxiliary principle technique and this is the main motivation of our next result.

THEOREM 2.1. Let the operators $T, g : H \rightarrow H$ be both strongly monotone Lipschitz continuous and the form $b(u, v)$ satisfy the conditions (i)-(iii), then there exists a solution $u \epsilon H$ such that $g(u) \epsilon K$ and (1.1) holds.

PROOF. We now use the auxiliary principle technique to prove the existence of a solution of (1.1) using the ideas of Glowinski, Lions and Tremolieres [5] and Noor [10,11]. For a given $u \epsilon H$ such that $g(u) \epsilon K$, we consider the problem of finding a unique $w \epsilon H$ such that $g(w) \epsilon K$, (see[5]), satisfying the auxiliary variational inequality

$$\begin{aligned} < w, v - g(w) > + \rho b(u, v) - \rho b(u, g(w)) \geq & < u, v - g(w) > \\ - & \rho < Tu, v - g(w) >, \quad for \ all \ v \epsilon K, \end{aligned} \tag{2.1}$$

where $\rho > 0$ is a constant.

Let w_1, w_2 be two solutions of (2.1) related to $u_1, u_2 \epsilon H$ respectively. It is enough to show that the mapping $u \rightarrow w$ has a fixed point belonging to H satisfying (1.1). In other words, it is sufficient to show that for well chosen $\rho > 0$,

$$\| w_1 - w_2 \| \leq \theta \| u_1 - u_2 \|,$$

with $0 < \theta < 1$, where θ is independent of u_1 and u_2. Taking $v = g(w_2)$ (respectively $g(w_1)$) in (2.1) related to u_1 (respectively u_2), we have

$$\begin{aligned} < w_1, g(w_2) - g(w_1) > + \rho b(u_1, g(w_2)) - \rho b(u_1, g(w_1)) \geq & < u_2, g(w_2) - g(w_1) > \\ - & \rho < Tu_1, g(w_2) - g(w_1) > \end{aligned}$$

and

$$\begin{aligned} < w_2, g(w_1) - g(w_2) > + \rho b(u_2, g(w_1)) - \rho b(u_2, g(w_2)) \geq & < u_2, g(w_1) - g(w_2) > \\ - & \rho < Tu_2, g(w_1) - g(w_2) > \end{aligned}$$

Adding these inequalities and using (iii), we have

$$\begin{aligned}
< w_1 - w_2, g(w_1) - g(w_2) > &\leq < u_1 - u_2, g(w_1) - g(w_2) > + \rho b(u_1 - u_2, g(w_2) - g(w_1)) \\
&- \rho < Tu_1 - Tu_2, g(w_1) - g(w_2) > \\
&= < u_1 - u_2 - \rho(Tu_1 - Tu_2), g(w_1) - g(w_2) > \\
&+ \rho b(u_1 - u_2, g(w_2) - g(w_1) >,
\end{aligned}$$

from which using (1.2), we obtain

$$\begin{aligned}
\eta \| w_1 - w_2 \|^2 &\leq \{\| u_1 - u_2 - \rho(Tu_1 - Tu_2) \| + \rho\gamma \| u_1 - u_2 \|\|\} \| g(w_1) - g(w_2) \| \\
&\leq \xi\{\| u_1 - u_2 - \rho(Tu_1 - Tu_2) \| + \rho\gamma \| u_1 - u_2 \|\} \| w_1 - w_2 \|, \qquad (2.2)
\end{aligned}$$

where $\eta > 0$ and $\xi > 0$ are the strongly monotonicity and Lipschitz continuity constants of the operator g.

Since T is a strongly monotone Lipschitz continuous operator, so

$$\begin{aligned}
\| u_1 - u_2 - \rho(Tu_1 - Tu_2) \|^2 &\leq \| u_1 - u_2 \|^2 - 2\rho < u_1 - u_2, Tu_1 - Tu_2 > \\
&+ \rho \| Tu_1 - Tu_2 \|^2 \\
&\leq (1 - 2\rho\alpha + \rho^2\beta^2) \| u_1 - u_2 \|^2 . \qquad (2.3)
\end{aligned}$$

Combining (2.2) and (2.3), we obtain

$$\begin{aligned}
\| w_1 - w_2 \| &\leq \frac{\rho\gamma + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}}{k} \| u_1 - u_2 \|, \quad \textit{where} \quad k = \frac{\eta}{\xi} \neq 0 \\
&= \theta \| u_1 - u_2 \|,
\end{aligned}$$

with $\theta = \frac{\rho\gamma + t(\rho)}{k}$ and $t(\rho) = \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}$.

We have to show that $\theta < 1$. It is clear that $t(\rho)$ assumes its minimum value for $\bar{\rho} = \frac{\alpha}{\beta^2}$ with $t(\bar{\rho}) = \sqrt{1 - (\frac{\alpha^2}{\beta^2})}$. For $\rho = \bar{\rho}, \rho\gamma + t(\bar{\rho}) < k$ implies that $\rho\gamma < k$ and $\gamma k < \alpha$. Thus it follows that $\theta < 1$ for all ρ with

$$\left| \rho - \frac{\alpha - k\gamma}{\beta^2 - \gamma^2} \right| < \frac{\sqrt{(\alpha - k\gamma)^2 - (\beta^2 - \gamma^2)(1 - k^2)}}{\beta^2 - \gamma^2}, \quad \rho\gamma < k, \gamma k < \alpha,$$

$$k < 1 \quad \textit{and} \quad \alpha > k\gamma + \sqrt{(\beta^2 - \gamma^2)(1 - k^2)}$$

Since $\theta < 1$, so the mapping $u \to w$ defined by (2.1) has a fixed point, which is the solution of (1.1), the required result.

REMARK 2.1. If $g = I$, the identity operator, then problem (2.1) is equivalent to finding $u \epsilon H$ for a given $u \epsilon H$ such that

$$< w, v-w > +\rho b(u,v) - \rho b(u,w) \geq < u, v-w > -\rho < Tu, v-w >, \tag{2.4}$$

for all $v \epsilon K$ and $\rho > 0$, is a constant. From the proof of Theorem 2.1, we see that $k = 1$ and $\theta = \rho\gamma + t(\rho) < 1$ for $0 < \rho < 2\frac{\alpha-\gamma}{\beta^2-\rho^2}$, $\gamma < \alpha$ and $\rho\gamma < 1$, so the mapping $u \to w$ defined by (2.4) has a fixed point, which is the solution of the variational inequality (1.3) studied by Kikuchi and Oden [7] in elasticity. Similarly for appropriate choices of the operators T, g, the form $b(u,v)$ and the convex set K, we can apply Theorem 2.1 to prove the existence of a solution for various classes of variational inequalities studied previously.

REMARK 2.2. It is clear that if $w = u$, then w is the solution of the variational inequality (1.1). This observation enables us to suggest an iterative algorithm for finding the approximate solution of the variational inequality (1.1) and its various special cases.

ALGORITHM 2.1. Given the initial value w_0, solve the problem (2.1) with $u = w_n$. If $\| w_{n+1} - w_n \| \leq \varepsilon$, for given $\varepsilon > 0$, stop. Otherwise repeat the process with n=n+1 and so on.

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