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Almost Hermitian homogeneous manifolds and Lie groups

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Abstract

Comparing the classifications of almost Hermitian structures and almost Hermitian homogeneous structures, we obtain some geometrical results about different classes of almost Hermitian homogeneous manifolds. In particular we study the Lie groups endowed with left invariant metrics and compatible almost complex structures. Some examples are discussed in detail.

1. Introduction

Let (M, g, J) be an almost Hermitian homogeneous manifold, that is an almost Hermitian manifold which admits a transitive and effective Lie group G of holomorphic isometries acting on it.

In 1978, K. Sekigawa proved the following

THEOREM 1.1 [Se]. *A connected, simply–connected and complete almost Hermitian manifold (M, g, J) is homogeneous if and only if there exists a tensor field T*

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of type $(1,2)$ *on* M *which satisfies the following conditions:*

(1) $\widetilde{\nabla} g = 0$,

(2) $\widetilde{\nabla} R = 0$,

(3) $\widetilde{\nabla} T = 0$,

(4) $\widetilde{\nabla} J = 0$,

where $\widetilde{\nabla} = \nabla - T$, ∇ *is the Levi–Civita connection of* M *and* R *is the Riemannian curvature tensor of* ∇.

Such a tensor T is called an *almost Hermitian homogeneous structure* and Theorem 1.1 provides a characterization of the almost Hermitian homogeneous manifolds, extending a similar result obtained by W. Ambrose and I.M. Singer in the case of Riemannian manifolds [AS].

In 1988, we decomposed the vector space generated by the tensors with the same algebraic symmetries of the almost Hermitian homogeneous structures into eight subspaces, irreducible and invariant under the action of the unitary group (see [AG]). This decomposition is related to the classification of Riemannian homogeneous structures, given by F. Tricerri and L. Vanhecke [TV], and to the classification of almost Hermitian manifolds, due to A. Gray and L. Hervella [GH]. Comparing these three decompositions, some geometrical results about almost Hermitian homogeneous manifolds are obtained. For example, it is proved that a naturally reductive quasi–Kähler manifold is nearly Kähler, and a naturally reductive $\mathcal{G}_2$–manifold is Hermitian.

In section 4, we consider the case of a Lie group endowed with a left invariant metric and a compatible almost complex structure. Applying the previous results, we find intersting characterizations of some classes of almost Hermitian Lie groups. It is proved that an almost Hermitian Lie group with a bi–invariant metric is nearly Kähler if and only if the adjoint representation anti–commutes with the almost complex structure. As a direct consequence, we show that an almost Hermitian Lie group with a bi–invariant metric is Kähler if and only if it is abelian. In the last section we discuss some examples of Lie groups belonging to different classes.

We wish to thank S.M. Salamon and L. Vanhecke for several useful discussions and K. Sekigawa for showing us the examples of nearly Kähler Lie groups described at the end of section 5.

2. Almost Hermitian homogeneous structures

Let (M, g, J) be an almost Hermitian manifold of dimension $2n$, $n \geq 2$, that is an almost complex manifolds (M, J) endowed with a Riemannian metric g compatible with J. If F denotes the Kähler form of M, defined by

$$F(X, Y) = g(JX, Y)\,, \tag{2.1}$$

it is well known that the covariant derivative ∇F verifies the properties

$$(\nabla_X F)(Y, Z) = -(\nabla_X F)(Z, Y) = -(\nabla_X F)(JY, JZ)\,, \tag{2.2}$$

for all vector fields X, Y, Z on M.

In [GH] A. Gray and L. Hervella decomposed the vector space $\mathcal{W}$ of all tensors of type $(0,3)$ with the same algebraic symmetries of ∇F into four irreducible subspaces $\mathcal{W}_i$, $i = 1, 2, 3, 4$, which are invariant under the action of the unitary group $U(n)$. In this way, the almost Hermitian manifolds have been classified into sixteen classes. Here we recall the most interesting ones:

$\{0\}$	Kähler manifolds,
$\mathcal{W}_1$	nearly Kähler manifolds,
$\mathcal{W}_2$	almost Kähler manifolds,
$\mathcal{W}_3$	Hermitian semi–Kähler manifolds,
$\mathcal{W}_4$	locally conformal Kähler manifolds,
$\mathcal{W}_1 \oplus \mathcal{W}_2$	quasi–Kähler manifolds,
$\mathcal{W}_3 \oplus \mathcal{W}_4$	Hermitian manifolds,
$\mathcal{W}_2 \oplus \mathcal{W}_4$	locally conformal almost Kähler manifolds,
$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$	semi–Kähler manifolds,
$\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$	$\mathcal{G}_1$–manifolds,
$\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$	$\mathcal{G}_2$–manifolds.

If $\dim(M) = 4$, some of the above classes are trivial. For more details and examples, we refer to [GH], [Gr2] and their references.

As it is shown in [AS], a connected m–dimensional homogeneous Riemannian manifold (M, g) admits a *Riemannian homogeneous structure* T, i.e. a tensor field of type $(1,2)$ which satisfies the conditions (1), (2) and (3) of Theorem 1.1. In

1983, F. Tricerri and L. Vanhecke considered the vector space $\mathcal{T}$ of all tensors with the same algebraic symmetries of T and decomposed $\mathcal{T}$ into three irreducible components $\mathcal{T}_i$, $i = 1, 2, 3$, invariant under the action of the orthogonal group $O(m)$. In this way, they obtained a complete classification of Riemannian homogeneous structures into eight classes. For example, the class $\mathcal{T}_3$ characterizes the naturally reductive homogeneous manifolds (for more details, see [TV]).

If (M, g, J) is an almost Hermitian homogeneous manifolds, it is possible to relate the above two decompositions. In fact, from condition (4) of Theorem 1.1 it follows

$$(\nabla_X F)(Y, Z) = g(T_X JY, Z) + g(T_X Y, JZ) \tag{2.3}$$

and this suggests to consider the homomorphism $\Psi : \mathcal{T} \longrightarrow \mathcal{W}$ defined by

$$\Psi(T)_{XYZ} = g(T_X JY, Z) + g(T_X Y, JZ)\,. \tag{2.4}$$

Let $\Psi_i : \mathcal{T}_i \longrightarrow \mathcal{W}$ denote the restriction of Ψ to the subspace $\mathcal{T}_i$, $i = 1, 2, 3$. We recall the following result, proved in [AG].

THEOREM 2.1. *The homomorphisms*

$$\Psi_1 : \mathcal{T}_1 \longrightarrow \mathcal{W}_4\,, \quad \Psi_3 : \mathcal{T}_3 \longrightarrow \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \tag{2.5}$$

are isomorphisms and

$$\Psi_2 : \mathcal{T}_2 \longrightarrow \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \tag{2.6}$$

is surjective. Moreover, $\mathcal{T}$ splits into eight irreducible subspaces, invariant under the action of $U(n)$:

$$\mathcal{T} = \mathcal{T}_1 \oplus \ker \Psi_2 \oplus \mathcal{W}_2' \oplus \mathcal{W}_3' \oplus \mathcal{W}_4' \oplus \mathcal{W}_1'' \oplus \mathcal{W}_3'' \oplus \mathcal{W}_4''\,, \tag{2.7}$$

where $\Psi_2(\mathcal{W}_i') = \mathcal{W}_i$, $i = 2, 3, 4$; $\Psi_3(\mathcal{W}_j'') = \mathcal{W}_j$, $j = 1, 3, 4$.

For the explicit description of the above irreducible subspaces we refer to Theorem 4.4 of [AG].

3. Geometrical results

Let (M, g, J) be an almost Hermitian homogeneous manifold of dimension $2n$. For every point $p \in M$, the tangent space (T_pM, g_p, J_p) is a Hermitian vector space. Let $\mathcal{F}$ be an invariant subspace of $\mathcal{T}$. We say that M is of type $\mathcal{F}$ if $T_p \in \mathcal{F}$, for all $p \in M$, where T is the corresponding almost Hermitian homogeneous structure (see Theorem 1.1). Then we simply write $M \in \mathcal{F}$. In the same way, if $\mathcal{D}$ is an invariant subspace of $\mathcal{W}$, then we say that $M \in \mathcal{D}$, or M is of type $\mathcal{D}$, if $\nabla F \in \mathcal{D}$.

Now we review some results which follows from the three decompositions described in section 2.

THEOREM 3.1 [AG]. *The connected, simply–connected, almost Hermitian naturally reductive manifolds of dimension $2n$, $n \geq 2$, are classified into eight classes given by all the invariant subspaces of the decomposition*

$$\mathcal{T}_3 = \mathcal{W}_1'' \oplus \mathcal{W}_3'' \oplus \mathcal{W}_4''\,. \tag{3.1}$$

REMARK 1. From the classification of almost Hermitian manifolds it follows that $\mathcal{W}_1''$ is the class of naturally reductive nearly Kähler manifolds; $\mathcal{W}_4''$ is the class of naturally reductive locally conformal Kähler manifolds and $\mathcal{W}_3'' \oplus \mathcal{W}_4''$ is the class of naturally reductive Hermitian manifolds.

THEOREM 3.2 [AG]. *The almost Hermitian homogeneous structure of type $\mathcal{T}_2$ are classified into sixteen classes given by all the invariant subspaces of the decomposition*

$$\mathcal{T}_2 = \ker \Psi_2 \oplus \mathcal{W}_2' \oplus \mathcal{W}_3' \oplus \mathcal{W}_4'\,. \tag{3.2}$$

REMARK 2. Note that if an almost Hermitian homogeneous manifold belongs to $\ker \Psi_2$, then it is a Kähler manifold.

Comparing the decompositions described in the previous section, we get the following results in addition to the ones obtained in [AG].

THEOREM 3.3. *A naturally reductive quasi–Kähler manifold is nearly Kähler.*

PROOF: A naturally reductive manifold M admits a homogeneous structure T of class $\mathcal{T}_3$. From Theorem 2.1, it follows that, for a naturally reductive quasi–Kähler manifold, $\Psi_3(T) = \Psi(T) = \nabla F \in \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$. Since a quasi–Kähler manifold belongs to the class $\mathcal{W}_1 \oplus \mathcal{W}_2$, then necessarily $\Psi_3(T) \in \mathcal{W}_1$, i.e. M must be nearly Kähler.

REMARK 3. The above result is an extension of Theorem 5.7 of [AG] and it has also been obtained by M. Djorić and L. Vanhecke in [DV], by means of a direct computation. Note that in this case M is a 3–symmetric space with J as canonical almost complex structure.

The following properties of the naturally reductive almost Hermitian manifolds are proved in a similar way.

THEOREM 3.4. *An almost Hermitian naturally reductive manifold is a $\mathcal{G}_1$–manifold.*

A naturally reductive $\mathcal{G}_2$–manifold is Hermitian.

A naturally reductive locally conformal almost Kähler manifold manifold is locally conformal Kähler.

A naturally reductive semi–Kähler manifold belongs to the class $\mathcal{W}_1 \oplus \mathcal{W}_3$.

About the almost Hermitian homogeneous manifold of type $\mathcal{T}_2$, we have the following

THEOREM 3.5. *An almost Hermitian homogeneous manifold of type $\mathcal{T}_2$ is a $\mathcal{G}_2$–manifold. In particular, a homogeneous $\mathcal{G}_1$–manifold of type $\mathcal{T}_2$ is Hermitian.*

A quasi–Kähler manifold of type $\mathcal{T}_2$ is almost Kähler.

A semi–Kähler homogeneous manifold of type $\mathcal{T}_2$ belongs to the class $\mathcal{W}_2 \oplus \mathcal{W}_3$.

A nearly Kähler homogeneous manifold of type $\mathcal{T}_2$ is locally Hermitian symmetric.

4. Applications to Lie groups

Let G be a connected Lie group of dimension m, endowed with a left invariant metric g, and let $\mathfrak{g}$ denotes its Lie algebra. The left invariant tensor field T defined by

(4.1) $$2g(T_X Y, Z) = g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y), \quad X,Y,Z \in \mathfrak{g},$$

is a Riemannian homogeneous structure on G and the metric connection $\widetilde{\nabla} = \nabla - T$ coincides with the (–) connection of Cartan–Schouten (see [TV], p. 83).

In this section, we consider only this Riemannian homogeneous structure.

From the definition, it follows that this T is a naturally reductive homogeneous structure (i.e. $T \in \mathcal{T}_3$) on a Lie group G if and only if

(4.2) $$g(X,[Y,Z]) + g(Y,[X,Z]) = 0, \quad X,Y,Z \in \mathfrak{g},$$

which means that the metric g must be bi–invariant.

Moreover such a T is of type $\mathcal{T}_2$ if and only if

(4.3) $$\mathfrak{S}_{XYZ}\, g([X,Y],Z) = 0, \quad \sum_{i=1}^{m} g([e_i,X],e_i) = 0, \quad X,Y,Z \in \mathfrak{g},$$

where $(e_1,\dots,e_m)$ is an arbitrary orthonormal basis of $\mathfrak{g}$ and $\mathfrak{S}_{XYZ}$ denotes the cyclic sum with respect to X,Y and Z (see [TV]).

Finally, we recall that a connected Lie group G is unimodular if and only if the endomorphism ad_X has null trace, for all $X \in \mathfrak{g}$. But if we define $c_{12}(T) = \sum_{i=1}^{m} g([X,e_i],e_i)$, from (4.1) it follows

(4.4) $$tr\, ad_X = \sum_{i=1}^{m} g([X,e_i],e_i) = c_{12}(T).$$

Hence G is unimodular if and only if $c_{12}(T) = 0$ or, in other words, if and only if $T \in \mathcal{T}_2 \oplus \mathcal{T}_3$. A similar result has been proved in [TV, section 8], for three–dimensional unimodular Lie groups.

Let (G,g,J) be al almost Hermitian Lie group of dimension $2n$. It is easy to see that the tensor field defined by (4.1) is an almost Hermitian homogeneous structure on G. Moreover, it follows from (2.3) that the covariant derivative of the Kähler form F of (G,g,J) can be written as

(4.5) $$\begin{aligned} 2(\nabla_X F)(Y,Z) &= g([X,JY],Z) - g([JY,Z],X) + g([Z,X],JY) \\ &+ g([X,Y],JZ) - g([Y,JZ],X) + g([JZ,X],Y), \quad X,Y,Z \in \mathfrak{g}. \end{aligned}$$

Using the above expression, it is possible to characterize some classes of almost Hermitian homogeneous structures on G.

THEOREM 4.2. *Let (G, g, J) be an almost Hermitian Lie group with a bi–invariant metric g. Then G is nearly Kähler if and only if*

$$ad_X \circ J = -J \circ ad_X\,, \quad X \in \mathfrak{g}\,. \tag{4.6}$$

PROOF: We first show that a bi–invariant metric on G is nearly Kähler if and only if $[X, JX] = 0$, for $X \in \mathfrak{g}$. From formula (4.2), it follows that $T \in \mathcal{T}_3$. Theorem 3.1 implies that G is nearly Kähler if and only if $T \in \mathcal{W}_1''$ that is

$$T_X Y = -T_Y X = -T_{JX} JY\,, \quad X, Y \in \mathfrak{g}\,, \tag{4.7}$$

(see [AG], Theorem 4.4). Because of (4.1), this condition is equivalent to $[X, JY] = [JX, Y]$, which can be rewritten as $ad_X(JY) = ad_{JX}(Y)$. Now assume that G has a bi–invariant nearly Kähler metric g. Then

$$\begin{aligned} g(ad_X(JY), Z) &= g(ad_{JX}Y, Z) = -g(Y, ad_{JX}Z) = -g(Y, ad_X(JZ)) \\ &= g(ad_X Y, JZ) = -g(J(ad_X Y), Z)\,, \quad X, Y, Z \in \mathfrak{g}\,, \end{aligned} \tag{4.8}$$

so (4.6) holds. The converse follows easily.

REMARK 1. This property has also been found, in a more complicated way, by J.A. Wolf and A. Gray (see [WG], Theorem 8.11).

REMARK 2. In the next section we shall give an example of a Lie group such that (4.6) holds but which is not of class $\mathcal{W}_1''$ because it admits no bi–invariant metric (see example 1.b).

If (G, g, J) is a complex Lie group, that is if

$$[JX, Y] = [X, JY] = J[X, Y]\,, \quad X, Y \in \mathfrak{g}\,, \tag{4.9}$$

the following relation holds (see [AGr], formula (2.5))

$$(\nabla_X F)(Y, Z) = g(ad_{JZ}Y, X)\,, \quad X, Y, Z \in \mathfrak{g}\,. \tag{4.10}$$

Note that if G is a complex Lie group then J is integrable and (G, J) is a complex manifold, but the converse is not necessarily true. In other words, (G, J) can be a complex manifold even if (4.9) does not hold.

A direct consequence of (4.10) is that for a complex Lie group we have

$$\sum_{i=1}^{2n} g(T_{e_i}e_i, X) = \sum_{i=1}^{2n} [-g(T_{e_i}Je_i, JX) + g(T_{e_i}e_i, X)], \tag{4.11}$$

where $(e_1, \dots, e_{2n})$ is an orthonormal basis of $\mathfrak{g}$.

If we define $\bar{c}_{12}(T)(X) = \sum_{i=1}^{2n} g(T_{e_i}Je_i, X)$, $X \in \mathfrak{g}$ (see [AG], pag. 387), it follows from (4.11) that $T \in \ker \bar{c}_{12}$. Moreover, if G is also unimodular then $T \in \ker c_{12} \cap \ker \bar{c}_{12} = (\mathcal{T}_2 \oplus \mathcal{T}_3) \cap \ker \bar{c}_{12}$. But in [AGr], Theorem 2.2, it has been proved that a complex Lie group is unimodular if and only if G is Hermitian semi–Kähler, i.e. G belongs to the class $\mathcal{W}_3$. Finally, from the proof of Theorem 4.4 of [AG], we have that a complex unimodular Lie group belongs to the class $\ker \Psi_2 \oplus \mathcal{W}_3' \oplus \mathcal{W}_3''$.

If G is a (not necessarily complex) Lie group endowed with a bi–invariant nearly Kähler metric, then the Nijenhuis tensor N, defined by $N(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$, becomes $N(X,Y) = 4[X,Y]$. If in addition G is Kähler, then it is also abelian. Conversely, If (G, g, J) is an abelian Lie group with a bi–invariant metric, we have that G is a nearly Kähler manifold (Theorem 4.2) with $N = 0$. This implies that (G, g, J) is Kähler. Hence the following Corollary holds.

COROLLARY 4.3. *Let (G, g, J) be an almost Hermitian Lie group with a bi–invariant metric g. Then G is Kähler if and only if G is abelian.*

REMARK 3. A similar result has been obtained by M. Goto and K. Uesu in [GU], where they assume that G is a complex Lie group.

5. Examples

1) Let H be the complex Heisenberg Lie group of real dimension six, defined as

$$H = \left\{ \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix}, \ z_j = a_j + ib_j \in \mathbb{C}, \ j = 1,2,3 \right\}. \tag{5.1}$$

We shall put two different almost Hermitian structures on H, described in the following subsections a) and b).

a) Let g be the left invariant metric on H such that

$$\alpha = dz_1\,, \quad \beta = dz_3\,, \quad \gamma = dz_2 - z_1 dz_3 \tag{5.2}$$

is an orthonormal basis for the complex 1–forms. Let J be the almost complex structure given by $g(JX,Y) = F(X,Y)$, where

$$F = i\,(\alpha\wedge\overline{\alpha} + \beta\wedge\overline{\beta} + \gamma\wedge\overline{\gamma})\,, \tag{5.3}$$

and $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$ are the complex conjugated forms of α, β, γ, respectively. It can be shown that the almost Hermitian manifold (H,g,J) belongs to the class $\mathcal{W}_3$ (for more details see [FG]). It is possible to identify H with $\mathbf{R}^6$ in the following way (see [AGr])

$$H = \left\{ \begin{pmatrix} 1 & a_1 & a_2 & 0 & -b_1 & -b_2 \\ 0 & 1 & a_3 & 0 & 0 & -b_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & b_1 & b_2 & 1 & a_1 & a_2 \\ 0 & 0 & b_3 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},\ a_j, b_j \in \mathbf{R}\,,\ j = 1,2,3 \right\}. \tag{5.4}$$

The Lie algebra $\mathfrak{h}$ of H can be easily computed from the above representation. If $(e_1,e_2,e_3,e_4,e_5,e_6)$ is the canonical basis of $\mathfrak{h} \cong \mathbf{R}^6$, we get

$$Je_1 = e_4\,, \quad Je_2 = e_5\,, \quad Je_3 = e_6\,, \tag{5.5}$$

and

$$[e_1,e_3] = -[e_4,e_6] = e_2\,, \quad [e_1,e_6] = -[e_3,e_4] = e_5\,, \tag{5.6}$$

all other brackets being zero. Hence the non vanishing components of the almost Hermitian homogeneous structure T defined by (4.1) are (taking into account the symmetries of T)

$$\begin{aligned} &T_{132} = T_{165} = T_{231} = T_{246} = T_{321} = T_{354} = T_{426} \\ &= T_{435} = T_{534} = T_{561} = T_{642} = T_{651} = \frac{1}{2}\,, \end{aligned} \tag{5.7}$$

where $T_{ijk} = g(T_{e_i}e_j, e_k)$. Because H is unimodular, $T \in \mathcal{T}_2 \oplus \mathcal{T}_3$, but $T \notin \mathcal{T}_2$ and $T \notin \mathcal{T}_3$ since, for example, $\mathfrak{S}_{123}\, T_{123} \neq 0$ and $T_{123} \neq -T_{213}$. Then from Theorem 2.1 we get that $T \in \ker \Psi_2 \oplus \mathcal{W}_3' \oplus \mathcal{W}_3''$. It is also possible to check that T does not belong to any invariant subspace of $\ker \Psi_2 \oplus \mathcal{W}_3' \oplus \mathcal{W}_3''$.

b) Following the same steps as in the previous example, let us consider the basis $(\alpha, \beta, \gamma' = \overline{\gamma})$ of left invariant 1–forms on H. The Kähler form $F' = i(\alpha \wedge \overline{\alpha} + \beta \wedge \overline{\beta} + \gamma' \wedge \overline{\gamma}')$ defines a quasi–Kähler structure (H, g', J'), that is a structure which belongs to the class $\mathcal{W}_1 \oplus \mathcal{W}_2$. The almost complex structure J' is obtained from J by reversing the sign when applied to γ; moreover note that $F' = F - 2\,\gamma\wedge\overline{\gamma}$. In the same way, J. Eells and S. Salamon [ES, Theorem 9.1] constructed a quasi–Kähler structure on $\mathbf{C}P^3$, starting from the standard Kähler structure.

In this case, the suitable identification of H with $\mathbf{R}^6$ is

$$(5.8) \qquad H = \left\{ \begin{pmatrix} 1 & a_1 & a_2 & 0 & -b_1 & b_2 \\ 0 & 1 & a_3 & 0 & 0 & -b_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & b_1 & -b_2 & 1 & a_1 & a_2 \\ 0 & 0 & b_3 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; a_j, b_j \in \mathbf{R},\; j = 1,2,3 \right\}.$$

If $(e_1', e_2', e_3', e_4', e_5', e_6')$ denotes the canonical basis of the Lie algebra $\mathfrak{h}$ of H, we have similar expressions as (5.5) and (5.6), with the exception

$$(5.9) \qquad [e_1', e_6'] = -[e_3', e_4'] = -e_5'\,.$$

Note that in this case $[X, JX] = 0$, for all $X \in \mathfrak{h}$, but the metric g' is not bi–invariant. Actually, H does not admit any bi–invariant metric since it is diffeomorphic to $\mathbf{R}^6$ and is not abelian. This example shows that the assumption of Theorem 4.2 is indeed necessary. The non–vanishing components of the almost Hermitian homogeneous structure T' defined by (4.1) are

$$(5.10) \qquad \begin{aligned} &T_{132}' = T_{156}' = T_{231}' = T_{246'} = T_{321}' = T_{345}' = T_{426}' \\ &= T_{453}' = T_{516}' = T_{543}' = T_{615}' = T_{642}' = \frac{1}{2}, \end{aligned}$$

where $T_{ijk}' = g'(T_{e_i'}' e_j', e_k')$. As before, we have that $T' \notin \mathcal{T}_2$ and $T' \notin \mathcal{T}_3$ and it is possible to show that $T' \in \ker \Psi_2 \oplus \mathcal{W}_2' \oplus \mathcal{W}_1''$ but it does not belong to any other invariant subspace.

2) The special unitary group $SU(3)$ is an example of an eight–dimensional Hermitian Lie group which is not a complex Lie group since, in general, $J[X,Y] \neq [JX,Y]$ for $X,Y \in \mathfrak{su}(3)$, the Lie algebra of $SU(3)$. It has a standard metric related to the Killing form such that the following basis of $\mathfrak{su}(3)$ is orthonormal

$$(5.11)\quad \begin{gathered} e_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \\ e_4 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad e_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \\ e_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \quad e_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 2i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{pmatrix}. \end{gathered}$$

The almost complex structure J defined by

$$(5.12)\qquad Je_1 = e_2\,, \quad Je_3 = e_4\,, \quad Je_5 = e_6\,, \quad Je_7 = -e_8\,,$$

is integrable and $(SU(3), g, J)$ belongs to the class $\mathcal{W}_3 \oplus \mathcal{W}_4$. Moreover

$$(5.13)\quad \begin{aligned} &[e_1,e_2] = -e_7 + \sqrt{3}e_8, && [e_1,e_3] = [e_2,e_4] = -e_5, \\ &[e_1,e_4] = [e_3,e_2] = -e_6, && [e_1,e_5] = [e_6,e_2] = [e_4,e_7] = e_3, \\ &[e_1,e_6] = [e_2,e_5] = [e_7,e_3] = e_4, && [e_1,e_7] = [e_3,e_6] = [e_5,e_4] = e_2, \\ &[e_1,e_8] = -\sqrt{3}e_2, && [e_2,e_7] = [e_3,e_5] = [e_4,e_6] = -e_1, \\ &[e_2,e_8] = \sqrt{3}e_1, && [e_3,e_4] = e_7 + \sqrt{3}e_8, \\ &[e_3,e_8] = -\sqrt{3}e_4, && [e_4,e_8] = \sqrt{3}e_3, \\ &[e_5,e_6] = 2e_7, && [e_5,e_7] = -2e_6, \\ &[e_6,e_7] = 2e_5, && [e_5,e_8] = [e_6,e_8] = [e_7,e_8] = 0. \end{aligned}$$

Using this, the almost Hermitian homogeneous structure T defined by (4.1) is a 3–form and has the following non–vanishing components

$$(5.14)\quad \begin{aligned} &T_{127} = T_{135} = T_{146} = T_{236} = T_{245} = -T_{347} = -\frac{1}{2} \\ &T_{128} = T_{348} = \frac{\sqrt{3}}{2}\,, \quad T_{567} = 1\,. \end{aligned}$$

Since the metric g is bi–invariant and J is integrable, the structure T belongs to the class $\mathcal{W}_3'' \oplus \mathcal{W}_4''$.

3) The following examples, suggested to us by K. Sekigawa, seem to be, at least to our knowledge, the only family of nearly Kähler Lie groups explicitly described.

Let G be a Lie group, with Lie algebra $\mathfrak{g}$, endowed with a bi–invariant metric g. Let us consider the following vector fields on the product Lie group $G \times G$

$$X^v = (0, X), \quad X^h = \left(\frac{2}{\sqrt{3}}X, \frac{1}{\sqrt{3}}X\right), \quad X \in \mathfrak{g}. \tag{5.15}$$

We define an almost complex structure J and a compatible left invariant metric g' on $G \times G$ as follows

$$J(X^v) = X^h, \quad J(X^h) = -X^v, \quad X \in \mathfrak{g}; \tag{5.16}$$

$$\begin{aligned} g'(X^v, Y^v) &= g(X, Y), \quad g'(X^v, Y^h) = 0, \\ g'(X^h, Y^h) &= g(X, Y), \qquad X, Y \in \mathfrak{g}. \end{aligned} \tag{5.17}$$

Since

$$\begin{aligned} &[X^v, Y^v] = [X, Y]^v, \quad [X^v, Y^h] = [X^h, Y^v] = \frac{1}{\sqrt{3}}[X, Y]^v, \\ &[X^h, Y^h] = -\frac{1}{3}[X, Y]^v + \frac{2}{\sqrt{3}}[X, Y]^h, \qquad X, Y \in \mathfrak{g}, \end{aligned} \tag{5.18}$$

the almost Hermitian homogeneous structure T of (4.1) is given by

$$\begin{aligned} T_{X^v}Y^v &= \frac{1}{2}[X, Y]^v, \quad T_{X^v}Y^h = \frac{1}{6}[X, Y]^h, \\ T_{X^h}Y^v &= \frac{1}{\sqrt{3}}[X, Y]^v - \frac{1}{6}[X, Y]^h, \\ T_{X^h}Y^h &= -\frac{1}{6}[X, Y]^v + \frac{1}{\sqrt{3}}[X, Y]^h, \quad X, Y \in \mathfrak{g}. \end{aligned} \tag{5.19}$$

It is not difficult to see that $\Psi(T) \in \mathcal{W}_1$ (Ψ is the homomorphism defined by (2.4)), so $(G \times G, g', J)$ is a nearly Kähler Lie group. Note that the metric g' is not bi–invariant. Now we want to prove that $(G \times G, g', J)$ is a *locally 3–symmetric space* (see [Gr1] for the definition and main properties). According to the main result of [GV], it is enough to show that the tensor field

$$\widetilde{T}_{X'}Y' = \frac{1}{2}J(\nabla_{X'}J)Y', \quad X', Y' \in \mathfrak{g} \times \mathfrak{g}, \tag{5.20}$$

is an almost Hermitian homogeneous structure on $G \times G$ (here ∇ denotes the Levi Civita connection of g'). Since g' is left invariant, $\nabla_{X'}Y' = T_{X'}Y'$, for all $X', Y' \in \mathfrak{g} \times \mathfrak{g}$, and (5.20) can be rewritten as

$$\widetilde{T}_{X'}Y' = \frac{1}{2}(T_{X'}Y' + JT_{X'}JY'), \quad X', Y' \in \mathfrak{g} \times \mathfrak{g}. \tag{5.21}$$

Using (5.19) and the fact that the metric g is bi–invariant, a long computation shows that $\widetilde{T}$ is a 3–form and $\widetilde{\nabla}\widetilde{T} = \widetilde{\nabla}R' = 0$, where $\widetilde{\nabla} = \nabla - \widetilde{T}$ and R' is the Riemannian curvature tensor of g'. Hence the previous claim is true. Finally, one can see that g' is Einstein if and only if g is Einstein.

We end with a couple of open problems related to the last example:
i) to find a nearly Kähler Lie group with a bi–invariant metric, which is not Kähler;
ii) to find a nearly Kähler Lie group which is not locally 3–symmetric.

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