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On self-dual almost Hermitian 4-manifolds *

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1. Introduction

For an oriented Riemannian 4-manifold M, the space $\wedge^2 M$ of 2-forms on M splits with respect to the star operator $*$ into $\wedge^2 M = \wedge^2_+ M \oplus \wedge^2_- M$, where $\wedge^2_\pm M$ are eigenspaces corresponding to the eigenvalues ± 1. The Weyl conformal curvature tensor W viewed as an $End(TM)$-valued 2-form decomposes into $W = W_+ \oplus W_-$ and we say that M is *self-dual* if $W_- = 0$.

An almost Hermitian 4-manifold (M, g, J) is said to be *of pointwise constant holomorphic sectional curvature* if the holomorphic sectional curvature of M is constant for every unit tangent vectors and depends only on points of M. S. Tachibana ([7]) introduced the notion of Ricci $*$-tensor on an almost Hermitian manifold M, and we say that M is *weakly $*$-Einstein* if the Ricci $*$-tensor ρ^* takes the form $\rho^* = \lambda^* g$ for some differentiable function λ^* on M. In particular, if λ^* is constant on M, then M is said to be *$*$-Eisenstein* (see also [5] and [6]).

The main purpose of this paper is to prove the followings

Theorem A. *An almost Hermitian 4-manifold (M, g, J) is self-dual, and the components of the Ricci tensor ρ of M satisfy*

$$\rho_{11} + \rho_{22} = \rho_{33} + \rho_{44}, \rho_{14} = \rho_{23}, \rho_{13} + \rho_{24} = 0, \tag{1.1}$$

or the components of the Ricci $*$-tensor ρ^* of M satisfy

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(1.2) $$\rho^*_{11} = \rho^*_{22} = \rho^*_{33} = \rho^*_{44}, \rho^*_{14} = \rho^*_{23}, \rho^*_{13} + \rho^*_{24} = 0$$

for any local orthonormal frame field $\{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ *if and only if* M *is of pointwise constant holomorphic sectional curvature.*

The conditions (1.1) and (1.2) are always satisfied on an almost Hermitian 4-manifold of pointwise constant holomorphic sectional curvature (see Lemma 3.3). The following is immediate from Theorem A, and the Einstein case is proved by T. Koda ([3]).

Corollary B. *Every self-dual almost Hermitian Einstein or weakly* $*$*-Einstein 4-manifold is of pointwise constant holomorphic sectional curvature.*

The following is somewhat interesting.

Theorem C. *Assume that an almost Hermitian 4-manifold* (M, g, J) *of pointwise constant holomorphic sectional curvature satisfies*

(1.3) $$g(R(JX, JY)JZ, JW) = g(R(X, Y)Z, W)$$

for any vector fields X, Y, Z *and* W *on* M*. Then* M *is both Einstein and weakly* $*$*-Einstein.*

A.Gray and L.Vanhecke ([2]) have constructed examples of Hermitian manifolds of pointwise constant holomorphic sectional curvature. We may show that their examples are weakly $*$-Einstein and not Einstein. In the last section, we shall give the proof when the dimension is equal to four.

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2. Self-dual almost Hermitian 4-manifolds

Let (M, g) be a Riemannian 4-manifold and ∇ be the Riemannian connection with respect to the metric g. The Riemannian curvature tensor R, Ricci tensor ρ, Ricci operator Q and scalar curvature τ of M are defined by

$$
(2.1) \qquad \begin{cases} R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \\ \rho(X,Y) = trace(Z \to R(Z,X)Y), g(QX,Y) = \rho(X,Y), \\ \tau = trace(X \to QX) \end{cases}
$$

for any vector fields X, Y and Z on M. The Weyl conformal curvature tensor W of M is given by

(2.2)

$$
W(X,Y)Z = R(X,Y)Z - \frac{1}{2}\{\rho(Y,Z)X - \rho(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} + \frac{\tau}{6}\{g(Y,Z)X - g(X,Z)Y\},
$$

and satisfies the identity

$$
(2.3) \qquad trace(Z \to W(Z,X)Y) = 0
$$

for any vector fields X, Y and Z on M. Let $\{e_1, e_2, e_3, e_4\}$ be any local frame field on M. With respect to the frame, we denote the components of R and ρ by $R_{kji}{}^h$ and ρ_{ji} respectively, where here and in the sequel the indices $k, j, i, \ldots$ run over the range $\{1, 2, 3, 4\}$, unless otherwise stated. Then it follows from (2.1) that

$$
(2.4) \qquad \begin{cases} R_{1212} - R_{3434} = \frac{\tau}{2} - (\rho_{11} + \rho_{22}), \\ R_{1313} - R_{2424} = \frac{\tau}{2} - (\rho_{11} + \rho_{33}), \\ R_{1414} - R_{2323} = \frac{\tau}{2} - (\rho_{11} + \rho_{44}), \end{cases}
$$

where $R_{kjih} = g(R(e_k, e_j)e_i, e_h)$.

Let (M, g, J) be an almost Hermitian 4-manifold oriented by the volume form $\frac{1}{2}\Omega^2$, where Ω is the Kaehler form defined by

$$
\Omega(X,Y) = g(X, JY)
$$

for any vector fields X and Y on M. Let $\{e_i\} = \{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ be a positively oriented orthonormal basis for the tangent space $T_p(M)$, $p \in M$, and $\{e^1, e^2, e^3, e^4\}$ be the dual basis. Then the set $\{e^1 \wedge e^2 - e^3 \wedge e^4, e^1 \wedge e^3 - e^4 \wedge e^2, e^1 \wedge e^4 - e^2 \wedge e^3\}$ is a basis for the eigenspace $(\wedge^2_- M)_p$. Therefore we see that M is self-dual if and only if the components of the Weyl conformal curvature tensor W satisfy

$$W_{12ih} = W_{34ih},\ W_{13ih} = W_{42ih},\ W_{14ih} = W_{23ih} \tag{2.5}$$

for the local orthonormal frame field $\{e_i\}$. We shall prove

Lemma 2.1. *Let (M, g, J) be an almost Hermitian 4-manifold. Then M is self-dual if and only if the components of the Weyl conformal curvature tensor W satisfy*

$$W_{123h} = W_{343h} \quad \text{and} \quad W_{132h} = W_{422h} \tag{2.6}$$

for any local orthonormal frame field $\{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$.

Proof. The relation (2.6) follows from (2.5). Conversely, by putting $h = 1$ into the first of (2.6), we obtain $W_{1213} = W_{3413}$. Since we have $W_{1231} = -W_{4234}$ and $W_{3431} = -W_{2421}$ by (2.3), we get $W_{1224} = W_{3424}$ and $W_{1242} = W_{3442}$. Similarly, by putting $h = 2, 4$ into the first of (2.6) and using (2.3), we also obtain $W_{1223} = W_{3423}$, $W_{1214} = W_{3414}$, $W_{1241} = W_{3441}$, $W_{1243} = W_{3443}$, $W_{1212} = W_{3412}$, $W_{1221} = W_{3421}$. Summing up these results, we see that the first of (2.6) yields the first of (2.5). By an argument similar to the above, we also see that the second of (2.6) yields the second of (2.5). The third of (2.5) follows from the first and second of (2.5).

It is immediate from (2.2) and Lemma 2.1

Proposition 2.2. *An almost Hermitian 4-manifold (M, g, J) is self-dual if and only if the components of the Riemannian curvature R and Ricci tensor ρ of M satisfy*

$$(2.7)\qquad \begin{cases} R_{1231} = R_{3431} + \dfrac{1}{2}(\rho_{23} + \rho_{14}), \\ R_{1232} = R_{3432} + \dfrac{1}{2}(\rho_{24} - \rho_{13}), \\ R_{1234} = R_{3434} + \dfrac{1}{2}(\rho_{33} + \rho_{44}) - \dfrac{\tau}{6}, \\ R_{1323} = R_{4223} - \dfrac{1}{2}(\rho_{12} + \rho_{34}), \\ R_{1324} = R_{4224} - \dfrac{1}{2}(\rho_{22} + \rho_{44}) + \dfrac{\tau}{6}, \end{cases}$$

for any local orthonormal frame field $\{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$, where τ is the scalar curvature of M.

A tensor field ρ^* of type (0,2) defined by

$$(2.8)\qquad \rho^*(X, Y) = g(Q^*X, Y) = \frac{1}{2} trace(Z \to R(X, JY)JZ)$$

for any vector fields X, Y and Z is called the ***Ricci *-tensor*** of M. The trace of the linear endomorphism Q^* is denoted by τ^* and called the ****-scalar curvature*** of M. It follows (2.8) that

$$(2.9)\qquad \rho^*(X, Y) = \rho^*(JY, JX)$$

for any vector fields X and Y on M. With respect to the local orthonormal frame field $\{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ on M, it is easily seen from (2.4), (2.8) and (2.9) that the components of the Ricci $*$-tensor ρ^* satisfy

$$\rho^*_{11} - \rho^*_{33} = \rho^*_{22} - \rho^*_{44} = \rho_{11} + \rho_{22} - \frac{\tau}{2}, \quad \rho^*_{12} = \rho^*_{21} = \rho^*_{34} = \rho^*_{43} = 0,$$
$$\rho^*_{13} + \rho^*_{24} = \rho^*_{31} + \rho^*_{42} = \rho_{13} + \rho_{24}, \quad \rho^*_{14} - \rho^*_{23} = \rho^*_{41} - \rho^*_{32} = \rho_{14} + \rho_{23}.$$

Therefore we can state

Lemma 2.3. *Let (M,g,J) be an almost Hermitian 4-manifold. Then the components of the Ricci tensor ρ of M satisfy (1.1) if and only if those of the Ricci $*$-tensor ρ^* of M satisfy (1.2) for any local orthonormal frame field $\{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$.*

3. Pointwise constant holomorphic sectional curvatures

The holomorphic sectional curvature $H(X)$ for a unit vector field X on an almost Hermitian manifold (M, g, J) is the sectional curvature $g(R(X, JX)JX, X)$. In the 4-dimensional case, we can state

Proposition 3.1. *An almost Hermitian 4-manifold (M, g, J) is of pointwise constant holomorphic sectional curvature if and only if the components of the Riemannian curvature tensor R of M satisfy*

$$(3.1)\qquad \begin{cases} R_{1212} = R_{3434}, \quad R_{1334} + R_{2434} = 0, \quad R_{1434} = R_{2334}, \\ R_{1313} + R_{2424} + 2(R_{1234} + R_{1324}) = 2R_{1212}, \\ R_{1414} + R_{2323} + 2(R_{1234} - R_{1423}) = 2R_{1212}, \\ R_{1314} + R_{1424} = R_{1323} + R_{2324}, \\ R_{1213} + R_{1224} = 0, \quad R_{1214} = R_{1223} \end{cases}$$

for any local orthonormal frame field $\{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ on M.

Proof. By definition, M is of pointwise constant holomorphic sectional curvature if and only if it satisfies

$$(3.2)\qquad g(R(X, JX)JX, X) = R_{3443}$$

for $X = \Sigma_i a_i e_i$, where a_1, a_2, a_3 and a_4 are any scalar functions such that $\Sigma_i a_i^2 = 1$. Let $x = a_1^2 + a_2^2$, $y = a_2 a_3 - a_1 a_4$ and $z = a_1 a_3 + a_2 a_4$. Then the functions x, y and z are linearly independent, and satisfy $x^2 = x - y^2 - z^2$. By a simple computation, we see that the equation (3.2) implies

$$
\begin{aligned}
&(R_{1212} - R_{3434})x + 2(R_{1334} + R_{2434})y + 2(R_{1434} - R_{2334})z \\
&+\{R_{1313} + R_{2424} + 2(R_{1234} + R_{1324}) - R_{1212} - R_{3434}\}y^2 \\
&+\{R_{1414} + R_{2323} + 2(R_{1234} - R_{1423}) - R_{1212} - R_{3434}\}z^2 \\
&+2(R_{1323} + R_{2324} - R_{1314} - R_{1424})yz + 2(R_{1334} + R_{2434} - R_{1213} - R_{1224})xy \\
&+2(R_{1223} + R_{1434} - R_{1214} - R_{2334})zx \\
&= 0.
\end{aligned}
$$

This completes the proof.

Remark 3.2. S.Tanno ([8]) has studied the curvature identities for an n-dimensional almost Hermitian manifold to be of pointwise constant holomorphic sectional curvature. By his results, Proposition 3.1 will be also obtained.

As for the Ricci tensor and Ricci $*$-tensor of M, we can state

Lemma 3.3. *Let (M, g, J) be an almost Hermitian 4-manifold of pointwise constant holomorphic sectional curvature. Then the components of the Ricci tensor ρ and Ricci $*$-tensor ρ^* of M satisfy (1.1) and (1.2) respectively for any local orthonormal frame field $\{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ on M.*

Proof. It follows from the first, fourth and fifth of (3.1) that

$$\rho_{11} + \rho_{22} = \rho_{33} + \rho_{44} = 6(R_{1234} + R_{1212}).$$

By the second and seventh of (3.1), we can verify that $\rho_{14} = \rho_{23}$. Since the third and eighth of (3.1) imply $\rho_{13} + \rho_{24}$, we see that (1.1) is satisfied on M. The remaining part of the Lemma follows from Lemma 2.3.

4. Proof of Theorems

Let (M, g, J) be a self-dual almost Hermitian 4-manifold. Assume that the components of the Ricci tensor ρ of M satisfy (1.1) for a local orthonormal frame field $\{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$. Then the first of (3.1) follows from the first relations of (1.1) and (2.4). From the first of (2.7) and the second of (1.1), we get the second and seventh of (3.1). We also obtain the third and eighth of (3.1) from the third of (1.1) and the second of (2.7). It is easily seen that the fourth of (2.7) is equivalent to the sixth of (3.1). Since we have

$$R_{1234} + R_{1324} = R_{3434} + R_{4224} + \frac{1}{2}(\rho_{33} - \rho_{22})$$

from the third and fifth of (2.7), then we get the fourth of (3.1) by use of the first of (3.1) and the definition of Ricci tensor. From the third of (2.7), we obtain

$$R_{1234} = -\frac{2}{3}R_{1212} + \frac{1}{6}(R_{1331} + R_{2442}) + \frac{1}{6}(R_{1441} + R_{2332})$$

by virtue of the first of (3.1). Applying the fourth of (3.1) and the identity $R_{1324} = R_{1234} - R_{1432}$ to the above equation, we have the fifth of (3.1). Therefore, by Proposition 3.1, M is of pointwise constant holomorphic sectional curvature.

Conversely, let (M, g, J) be an almost Hermitian 4-manifold of pointwise constant holomorphic sectional curvature. Then, by Lemma 3.3, we have (1.1). Since it follows from the fourth and fifth of (3.1) and the identity $R_{1324}+R_{1432} = -R_{1243}$ that

$$6R_{1234} = -6R_{1212} + \rho_{11} + \rho_{22},$$

then we get the third of (2.7) by taking account of the first of (1.1) and (3.1).

From the fourth of (3.1), we obtain

$$2R_{1324} = \frac{\tau}{3} - (\rho_{33} + \rho_{44}) + R_{1331} + R_{4224}$$

by virtue of the third of (2.7) and first of (3.1). The above equation and the first of (3.1) yield the fifth of (2.7). The first and second of (2.7) follows from the seventh and eighth of (3.1), respectively, by use of (1.1). We have already seen that the fourth of (2.7) is equivalent to the sixth of (3.1). Thus, by Proposition 2.2, M is self-dual.

The remaining part of Theorem A follows from Lemmas 2.3 and 3.3. This completes the proof of Theorem A.

Remark 4.1. T. Koda and K. Sekigawa ([4]) announced that an almost Hermitian 4-manifold of pointwise constant holomorphic sectional curvature is self-dual.

If M is Einstein, then M satisfies (1.1). Provided that M is weakly $*$-Einstein, we also have (1.2). Thus Corollary B is immediate from Theorem A.

Now we shall prove Theorem C. First of all, we have (1.1) and (1.2) by Lemma 3.3. It follows from (1.3) that

$$\rho(X, Y) = \rho(JX, JY) \tag{4.1}$$

for any vector fields X and Y on M. Thus we see from (1.1) and (4.1) that $\rho_{11} = \rho_{22} = \rho_{33} = \rho_{44}$ and 0 otherwise for the orthonormal frame field $\{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$, which means that M is Einstein.

It follows from (1.3), (2.8) and (2.9) that

$$\rho^*(X, Y) = \rho^*(Y, X) = \rho^*(JX, JY) \tag{4.2}$$

for any vector fields X and Y on M. By taking account of (1.2) and (4.2), we see that $\rho^*_{11} = \rho^*_{22} = \rho^*_{33} = \rho^*_{44}$ and 0 otherwise. Thus M is also a weakly $*$-Einstein manifold.

5. Examples

Let M be an open connected domain of $\mathbf{C}^2 \cong \mathbf{R}^4$, and $(z^1, z^2) = (x^1, y^1, x^2, y^2)$, $z^i = x^i + \sqrt{-1}y^i$, be the canonical coordinate system of M, where here and in the sequel the indices i, j run over the range $\{1, 2\}$. For the canonical basis $\{\partial/\partial x^1, \partial/\partial y^1, \partial/\partial x^2, \partial/\partial y^2\}$ for the tangent space $T_p(M)$, $p \in M$, we define an endomorphism $J : T_P(M) \to T_p(M)$ by

$$J\partial/\partial x^i = \partial/\partial y^i \quad \text{and} \quad J\partial/\partial y^i = -\partial/\partial x^i. \tag{5.1}$$

Then J is a complex structure on M. Let $<,>$ be the canonical metric on M, that is,

$$< \partial/\partial x^i, \partial/\partial x^j >= \delta_{ij}, \; < \partial/\partial x^i, \partial/\partial y^j >= 0, \; < \partial/\partial y^i, \partial/\partial y^j >= \delta_{ij}.$$

Then we see that $(M, <,>, J)$ is a Kaehlerian 4-manifold.

Let $f : M \to \mathbf{C}$ be a non-linear holomorphic function such that $Ref > -1$, and we put $\sigma = -log(1 + Ref)$. As a conformal change of $<,>$, we consider a Riemannian metric g on M such that

$$g(X, Y) = exp(2\sigma) < X, Y > \tag{5.2}$$

for any vector fields X and Y on M. Let ∇ and D be the Riemannian connections of M with respect to the metrics g and $<,>$ respectively. Then we have

$$\nabla_X Y = D_X Y + (X\sigma)Y + (Y\sigma)X - < X, Y > grad\,\sigma \tag{5.3}$$

for any vector fields X and Y, where $grad\,\sigma$ is the gradient vector field of σ given by

$$grad\,\sigma = \Sigma_i\left(\frac{\partial\sigma}{\partial x^i}\frac{\partial}{\partial x^i} + \frac{\partial\sigma}{\partial y^i}\frac{\partial}{\partial y^i}\right). \tag{5.4}$$

It is easily seen from (5.1) and (5.2) that (M, g, J) is a Hermitian 4-manifold.

Now we define a linear endomorphism $\Psi : T_p(M) \to T_p(M)$ by

$$\Psi(X) = -D_X grad\,\sigma + (X\sigma) grad\,\sigma \tag{5.5}$$

for any $X \in T_p(M)$. Then the 2-form ψ on M defined by

$$\psi(X, Y) = < \Psi(X), Y >$$

is bilinear and symmetric for X and Y, and is given by

$$\psi(X, Y) = (D_X Y)\sigma - XY\sigma + (X\sigma)(Y\sigma) \tag{5.6}$$

for any vector fields X and Y on M. By a straightforward computation, the Riemannian curvature tensor R of M with respect to the connection ∇ is given by

$$\begin{aligned} R(X,Y)Z &= \psi(Y,Z)X - \psi(X,Z)Y \\ &\quad + < Y, Z > \Psi(X) - < X, Z > \Psi(Y) \\ &\quad + \|grad\,\sigma\|^2 (< X, Z > Y - < Y, Z > X) \end{aligned} \tag{5.7}$$

for any vector fields X, Y and Z on M.

Let $f = u + \sqrt{-1}v$, where $u = Ref$ and $v = Imf$. Since f is a holomorphic function on M, we have

$$\frac{\partial^2 u}{\partial x^i \partial x^j} = -\frac{\partial^2 u}{\partial y^i \partial y^j}, \quad \frac{\partial^2 u}{\partial x^i \partial y^j} = -\frac{\partial^2 v}{\partial x^i \partial x^j}. \tag{5.8}$$

Since $\sigma = -log(1 + u)$, then, using (5.8), we obtain

$$
(5.9) \qquad \begin{cases} \dfrac{\partial^2\sigma}{\partial x^i\partial x^j} = \dfrac{\partial\sigma}{\partial x^i}\dfrac{\partial\sigma}{\partial x^j} - \dfrac{1}{1+u}\dfrac{\partial^2 u}{\partial x^i\partial x^j}, \\ \dfrac{\partial^2\sigma}{\partial x^i\partial y^j} = \dfrac{\partial\sigma}{\partial x^i}\dfrac{\partial\sigma}{\partial y^j} + \dfrac{1}{1+u}\dfrac{\partial^2 v}{\partial x^i\partial x^j}, \\ \dfrac{\partial^2\sigma}{\partial y^i\partial y^j} = \dfrac{\partial\sigma}{\partial y^i}\dfrac{\partial\sigma}{\partial y^j} + \dfrac{1}{1+u}\dfrac{\partial^2 u}{\partial x^i\partial x^j}. \end{cases}
$$

We choose an orthonormal basis$\{e_1 = exp(-\sigma)\partial/\partial x^1, e_2 = Je_1 = exp(-\sigma)\partial/\partial y^1$, $e_3 = exp(-\sigma)\partial/\partial x^2, e_4 = Je_3 = exp(-\sigma)\partial/\partial y^2\}$ for $T_p(M)$ with respect to the metric g. We put

$$
u_{ij} = \frac{\partial^2 u}{\partial x^i\partial x^j} \quad \text{and} \quad v_{ij} = \frac{\partial^2 v}{\partial x^i\partial x^j}.
$$

Then, by taking account of (5.4), (5.5) and (5.9), the linear endomorphism Ψ is given by

$$
(5.10) \qquad \Psi = \frac{1}{1+u}\begin{pmatrix} u_{11} & -v_{11} & u_{12} & -v_{12} \\ -v_{11} & -u_{11} & -v_{12} & -u_{12} \\ u_{12} & -v_{12} & u_{22} & -v_{22} \\ -v_{12} & -u_{12} & -v_{22} & -u_{22} \end{pmatrix}
$$

with respect to the basis $\{e_1, e_2, e_3, e_4\}$.

Since the Riemannian curvature tensor R of (M, g, J) can be written as

(5.11)
$$
\begin{aligned} g(R(X,Y)Z,W) = {} & \psi(Y,Z)g(X,W) - \psi(X,Z)g(Y,W) + g(Y,Z)\psi(X,W) \\ & - g(X,Z)\psi(Y,W) + exp(-2\sigma)\|grad\,\sigma\|^2\{g(X,Z)g(Y,W) \\ & - g(Y,Z)g(X,W)\} \end{aligned}
$$

for any vector fields X, Y, Z and W on M, then, by comparing (3.1) with (5.10) and (5.11), we see that (M, g, J) is a Hermitian 4-manifold of pointwise constant holomorphic sectional curvature $-exp(-2\sigma)\|grad\,\sigma\|^2$ (see also A. Gray and L. Vanhecke [2]).

Since (M, g, J) is conformally flat and not of constant sectional curvature, it is not Einstein. This fact can be also checked explicitly by use of (5.10) and (5.11). Since it follows from (2.8), (5.10) and (5.11) that

$$\rho^*_{11} = \rho^*_{22} = \rho^*_{33} = \rho^*_{44} = -exp(-2\sigma)||grad\,\sigma||^2,$$

$$\rho^*_{13} = -R_{1412} - R_{1434} = \psi_{24} + \psi_{13} = 0,$$

$$\rho^*_{14} = -\psi_{23} + \psi_{14} = 0$$

and so on, where ψ_{ij} are the components of ψ for the basis $\{e_1, e_2, e_3, e_4\}$ of $T_p(M)$, then we have

$$\rho^*(X, Y) = -exp(-2\sigma)||grad\,\sigma||^2 g(X, Y)$$

for any vector fields X and Y on M. Summing up these results, we see that (M, g, J) is a Hermitian 4-manifold of pointwise constant holomorphic sectional curvature, which is not Einstein but weakly $*$-Einstein.

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