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A note on the differentiability of the distance function to regular submanifolds of Riemannian manifolds

Kazuko MATSUMOTO

Introduction. Let M be a C^∞ Riemannian manifold with a metric g, let S be a submanifold of M and denote by $d(x)$ the distance from $x \in M$ to S induced by the metric g. In the study of various problems of analysis, the function $d = d(x)$ is a useful tool and one must ensure that it is sufficiently differentiable (on some open subset of M) for one's purpose.

In this paper we prove that if S is a C^k regular submanifold of M and $2 \leq k \leq \infty$, then there exists an open subset Δ of M such that $S \subset \Delta$ and the function $h = h(x) = d(x)^2$ is of class C^k on Δ. Here we say that S is a C^k regular submanifold of M if each point x_0 of S has a C^k coordinate neighborhood (U, ψ), $\psi = (\psi_1, \ldots, \psi_n)$, such that $S \cap U = \{p \in U : \psi_{r+1}(p) = \cdots = \psi_n(p) = 0\}$, where $n = \dim M$ and $0 \leq r \leq n-1$. In particular, the set S has no boundary but it needs not be closed or connected.

When S is a hypersurface of the Euclidean space $\mathbf{R}^n$, it is easy by the implicit function theorem to see that if S is of class C^k, $k \geq 2$, then there exists an open set including S where h is of class C^{k-1}. In this case, Gilbarg-Trudinger ([2], Lemma 1 of Appendix) showed, as the strict result of Serrin ([5], Lemma 1 of Chapter I, §3), that h is further of class C^k on some open set including S. Their proofs depend on the geometric method, but later Krantz-Parks [3] showed it by elementary means (see also Krantz [4], pp. 136–137). Our proof in this paper is the extension of Krantz-Parks' one.

We note here that the statement above is false in the case $k = 1$. In fact, there is a C^1 curve S in the Euclidean space $\mathbf{R}^2$ which contains a point without positive reach (see, for example, [3]). It follows from the general result of Federer ([1], Theorem 4.8) that the function $h = d^2$ is then not differentiable near the point of S without positive reach.

1. Let M be a C^∞ Riemannian manifold of dimension n and let g be a metric on M. For two points x and y of M, we denote by $\delta(x,y)$ the distance between x and y induced by the metric g.

It is well-known that each $x_0 \in M$ has a coordinate neighborhood U where any two points x and y can be joined by a unique minimizing geodesic $\xi = \xi(s)$, $s \in [0,1]$, in M. If the neighborhood U of x_0 is sufficiently small, the geodesic ξ has the expression $\xi(s) = \exp_x sv$ for some $v = v(x,y) \in T_x(M)$ and the mapping $v = v(x,y)$ is of class C^∞ on $U \times U$. Then we can write

$$\delta(x,y) = \delta(x, \exp_x v) = \sqrt{g_x(v,v)}$$

for $x, y \in U$.

Regarding the coordinate neighborhood U as an open subset of $\mathbf{R}^n$, we put $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ for $x, y \in U$. Moreover, we put

$$g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right), \qquad 1 \le i, j \le n,$$

and write

$$v = \sum_{i=1}^n v_i \left(\frac{\partial}{\partial x_i}\right)_x .$$

Then the functions $g_{ij} = g_{ij}(x)$ and $v_i = v_i(x,y)$ are of class C^∞ on U and $U \times U$ respectively, and the matrix (g_{ij}) is positive definite symmetric at each point of U. Further, it follows from the property of the exponential mapping $y = \exp_x v$ that the functions $v_i = v_i(x,y)$, $1 \le i \le n$, satisfy the conditions

$$v_i(x_0, x_0) = 0, \quad \frac{\partial v_i}{\partial y_j}(x_0, x_0) = \delta_{ij}, \qquad 1 \le i, j \le n.$$

Thus we obtain the following:

LEMMA. *For each point x_0 of a C^∞ Riemannian manifold M, there exist a coordinate neighborhood U of x_0 and C^∞ functions $v_i = v_i(x,y)$, $1 \le i \le n$, on $U \times U$ such that*

(i) $$\delta(x,y)^2 = \sum_{i,j=1}^n g_{ij}(x)\, v_i(x,y)\, v_j(x,y),$$

(ii) $$v_i(x_0, x_0) = 0, \quad \frac{\partial v_i}{\partial y_j}(x_0, x_0) = \delta_{ij}, \qquad 1 \le i, j \le n.$$

2. For a given submanifold S of M, we define the function $d = d(x)$ by

$$d(x) = \delta(x, S) = \inf\{\delta(x,y) : y \in S\}, \qquad x \in M.$$

We shall now prove the following:

THEOREM. *If S is a C^k regular submanifold of a C^∞ Riemannian manifold M and $2 \leq k \leq \infty$, then there exists an open subset Δ of M such that $S \subset \Delta$ and the restriction to Δ of the function $h = d^2$ is of class C^k.*

PROOF: Let x_0 be a point of S and let r be the dimension of the connected component of S containing x_0. Then we can take a coordinate neighborhood U ($\subset M$) of x_0, so that the set $S \cap U$ is written by

$$S \cap U = \{\varphi(t) = (\varphi_1(t), \dots, \varphi_n(t)) : t = (t_1, \dots, t_r) \in E\}$$

for some open subset $E \subset \mathbf{R}^r$ and some C^k mapping $\varphi : E \longrightarrow U$ such that the Jacobian matrix

$$\Phi = \frac{D(\varphi_1, \dots, \varphi_n)}{D(t_1, \dots, t_r)} = \left(\frac{\partial \varphi_i}{\partial t_\mu}\right)_{1 \leq i \leq n, 1 \leq \mu \leq r}$$

has the rank r at $t = t_0$ if $x_0 = \varphi(t_0)$ for $t_0 \in E$. (When $r = 0$, we take U so that $S \cap U = \{x_0\}$.)

Now if U' ($\subset M$) is a neighborhood of x_0 and

$$U' \subset \{x \in M : \delta(x, x_0) < \varepsilon\} \subset \{x \in M : \delta(x, x_0) < 2\varepsilon\} \subset U$$

for some $\varepsilon > 0$, then it follows that $d(x) = \delta(x, S) = \delta(x, S \cap U)$ for $x \in U'$. Therefore, it is sufficient for the proof of Theorem to show that the point x_0 has a neighborhood U_0 ($\subset U'$) where the function $h = d(x)^2 = \delta(x, S \cap U)^2$ is of class C^k. Moreover, we may, by shrinking the neighborhoods U and U' of x_0 if necessary, assume that for this U there exist C^∞ functions $v_i = v_i(x, y)$, $1 \leq i \leq n$, on $U \times U$ satisfying the conditions (i) and (ii) of Lemma.

First if $r = 0$, that is, if $S \cap U = \{x_0\}$, it follows immediately from the condition (i) that $h = d(x)^2 = \delta(x, x_0)^2$ is of class C^∞ on U'. Hence we suppose that $1 \leq r \leq n - 1$.

For $x = (x_1, \dots, x_n) \in U$ and $t = (t_1, \dots, t_r) \in E$, we put

$$f(x, t) = \delta(x, \varphi(t))^2 = \sum_{i,j=1}^{n} g_{ij}(x)\, v_i(x, \varphi(t))\, v_j(x, \varphi(t))$$

and

$$F_\mu(x, t) = \frac{\partial f}{\partial t_\mu}(x, t), \qquad 1 \leq \mu \leq r.$$

Then the mapping $F = (F_1, \ldots, F_r)$ is of class C^{k-1}, $k \geq 2$, on $U \times E$. Moreover, we can verify that

$$(*) \qquad \det \frac{D(F_1, \ldots, F_r)}{D(t_1, \ldots, t_r)}(x_0, t_0) \neq 0.$$

In fact, it follows from the condition (ii) and the symmetry of the matrix $G = (g_{ij})$ that

$$\begin{aligned}
\frac{\partial F_\mu}{\partial t_\nu}(x_0, t_0) &= \frac{\partial^2 f}{\partial t_\mu \partial t_\nu}(x_0, t_0) \\
&= 2 \sum_{i,j=1}^{n} g_{ij}(x_0) \left\{ \sum_{\alpha=1}^{n} \frac{\partial v_i}{\partial y_\alpha}(x_0, x_0) \frac{\partial \varphi_\alpha}{\partial t_\mu}(t_0) \right\} \\
&\qquad\qquad \times \left\{ \sum_{\beta=1}^{n} \frac{\partial v_j}{\partial y_\beta}(x_0, x_0) \frac{\partial \varphi_\beta}{\partial t_\nu}(t_0) \right\} \\
&= 2 \sum_{i,j=1}^{n} g_{ij}(x_0) \frac{\partial \varphi_i}{\partial t_\mu}(t_0) \frac{\partial \varphi_j}{\partial t_\nu}(t_0)
\end{aligned}$$

for $1 \leq \mu, \nu \leq r$, and hence

$$\frac{D(F_1, \ldots, F_r)}{D(t_1, \ldots, t_r)}(x_0, t_0) = 2\, {}^t\Phi(t_0)\, G(x_0)\, \Phi(t_0).$$

Now since $G(x_0)$ is positive definite symmetric and $\Phi(t_0)$ has the rank r, the matrix ${}^t\Phi(t_0)\, G(x_0)\, \Phi(t_0)$ is also positive definite symmetric and so its determinant does not vanish. This implies (*). Therefore, we can by the implicit function theorem find a neighborhood U_0 ($\subset U'$) of x_0, so that each $x \in U_0$ has a unique solution $t = t(x) \in E$ of the system of equations $F_\mu(x, t) = 0$, $1 \leq \mu \leq r$, and the mapping $t = t(x) = (t_1(x), \ldots, t_r(x))$ is of class C^{k-1} on U_0. Then for each $x \in U_0$ there exists at least one point $t' \in E$ such that $d(x) = \delta(x, S \cap U) = \delta(x, \varphi(t'))$. Further, the point t' is uniquely determined by x and it must coincide to $t(x)$ because $f = f(x, t) = \delta(x, \varphi(t))^2$ is minimal at $t = t'$ for each x.

Hence we can write

$$h(x) = \delta(x, S \cap U)^2 = \delta(x, \varphi(t(x)))^2 = f(x, t(x))$$

for $x \in U_0$, and first see that h is of class C^{k-1} on U_0. Then the partial derivatives of h are expressed by

$$\frac{\partial h}{\partial x_i}(x) = \frac{\partial f}{\partial x_i}(x, t(x)) + \sum_{\mu=1}^{r} \frac{\partial f}{\partial t_\mu}(x, t(x)) \frac{\partial t_\mu}{\partial x_i}(x).$$

Since $t = t(x)$ is the solution of $F_\mu(x,t) = (\partial f/\partial t_\mu)(x,t) = 0$ for $1 \le \mu \le r$, we further obtain

$$\frac{\partial h}{\partial x_i}(x) = \frac{\partial f}{\partial x_i}(x, t(x)), \qquad 1 \le i \le n,$$

and see that they are also of class C^{k-1} on U_0. Therefore, we can conclude that the function $h = d^2$ is of class C^k on $U_0 = U(x_0)$ and hence on the open set $\Delta = \cup_{x_0 \in S} U(x_0)$ including S, which proves the theorem. □

3. When $M = \mathbf{R}^n$ and the metric g of M is Euclidean, the distance $\delta(x,y)$ between $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ is given by $\delta(x,y)^2 = \sum_{i=1}^n (y_i - x_i)^2$. Then the functions $g_{ij} = g_{ij}(x)$ and $v_i = v_i(x,y)$ in Lemma are written by $g_{ij}(x) \equiv \delta_{ij}$ and $v_i(x,y) = y_i - x_i$. Finally we note that the calculation above of our proof of Theorem is simpler than that of Krantz ([4], pp. 136–137) in this case.

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Kazuko MATSUMOTO
Department of Mathematical Science
Graduate School of Science and Technology
Niigata University
Niigata, 950-21
Japan

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