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A SUFFICIENT CONDITION FOR A GRAPH TO BE TRACEABLE *

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ABSTRACT

We prove that a 2-connected graph of order p is traceable if for all distinct vertices u and v, dist(u,v)=2 implies that $|N(u)\cup N(v)|\geqslant(p-1)/2$. This result was once conjectured by T. E. Lindquester.

INTRODUCTION

A path in a graph G is called a hamiltonian path in G if it contains all the vertices of G. A graph is traceable if it has a hamiltonian path. The neighborhood of a vertex v, denoted N(v), is the set of all vertices adjacent to v. We define the distance, denoted dist(u,v), between two vertices u and v as the minimum of the lengths of all u-v paths. Let $NC2=\min|N(u)\cup N(v)|$, where the minimum is taken over all pairs of vertices u,v that are at distance two in the graph. Refer to [2] for other terminology.

T.E.Lindquester has given the following theorem in [1]:

Theorem 1. Let G be a 2-connected graph of order p. If

$$NC2>(2p-5)/3 ,$$

then G is traceable.

He also raised the following conjecture:

Conjecture. Let G be a 2-connected graph of order p. If

$$NC2\geqslant(p-1)/2 ,$$

then G is traceable.

He also pointed out that the 2-connected bipartite graph K(n-2,n), $n\geqslant 4$, is nontraceable. If $|K(n-2,n)|=p$, then $NC2\geqslant(p-2)/2$. Thus, the conjecture is the best possible result of this nature that can be obtained.

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PROOF OF THE CONJECTURE

In the following lemmas, we suppose that $G=(V,E)$ is a 2-connected graph of order p, $NC2 \geqslant (p-1)/2$ and G is nontraceable.

Lemma 1. If P: $a_1 a_2 \ldots a_m$ be a path of maximum length in G, then

$$\mathrm{dist}(a_1, a_m) \neq 2 \quad .$$

Proof. Since G is nontraceable, there exists a vertex x not on P but adjacent to vertices on P. Assume x is adjacent to a_i. Since G is 2-connected, there exists at least one other path from x to P, besides the edge xa_i, that is vertex disjoint from P except at the end point, Let a_j be the end vertex of such a path P'. Without loss of generality, assume $i<j$. Since P is of maximum length, we have $xa_{i+1} \bar{\in} E$. Thus,

$$\mathrm{dist}(x, a_{i+1})=2 \quad .$$

Now, we suppose

$$\mathrm{dist}(a_1, a_m)=2.$$

We define the function f: $N(x) \cup N(a_{i+1}) \longrightarrow V$ by

$$f(a_k)=a_{k+1}, \text{ for } 2 \leqslant k \leqslant i \; ;$$

$$f(a_k)=a_{k-1}, \text{ for } i+2 < k \leqslant m-1 \; ;$$

$$f(a_{i+2})=x \; ;$$

$$f(y)=y \quad \text{for} \quad y \bar{\in} P \; .$$

It is easily verified that for $\forall z \in N(x) \cup N(a_{i+1})$, $f(z)$ is well defined and f is injective. Thus, we have

$$|f(N(x) \cup N(a_{i+1}))| = |N(x) \cup N(a_{i+1})| \geqslant (p-1)/2 \; .$$

Since P is of maximum length, we have $N(a_1) \cup N(a_m) \subseteq P$. We assert that

$$f(N(x) \cup N(a_{i+1})) \cap (N(a_1) \cup N(a_m)) = \phi \; .$$

For if $\quad a_k \in f(N(x) \cup N(a_{i+1})) \cap (N(a_1) \cup N(a_m))$, then

(1). $k \neq 1, m$. For $a_1 a_m \bar{\in} E$ or the following path

$$xa_i\ldots a_1a_m\ldots a_{i+1}$$

is longer than P. Hence $a_1,\ a_m \bar{\in} N(a_1)\cup N(a_m)$.

(2). $k\neq 2,3,\ldots,i$. For (2.1) if $a_{k-1}a_{i+1}\in E,\ a_ka_1\in E$, then

$$xa_i\ldots a_ka_1\ldots a_{k-1}a_{i+1}\ldots a_m$$

is a path longer than P. (2.2). if $a_{k-1}a_{i+1}\in E,\ a_ka_m\in E$, then

$$xa_i\ldots a_ka_m\ldots a_{i+1}a_{k-1}\ldots a_1$$

is a path longer than P. (2.3). if $a_{k-1}x\in E,\ \ a_ka_1\in E$, then

$$xa_{k-1}\ldots a_1a_k\ldots a_m$$

is a path longer than P.(2.4). if $a_{k-1}x\in E,\ a_ka_m\in E$, then

$$a_1\ldots a_{k-1}xP'a_j\ldots a_ka_m\ldots a_{j+1}$$

is a path longer than P.

(3). $k\neq i+1$. This is evident.

(4). $k\neq i+2,\ldots,m-1$. The reason is the same as in (2).

It is clear that $a_1,\ a_m \bar{\in} f(N(x)\cup N(a_{i+1}))\cup(N(a_1)\cup N(a_m))$, we have $p-1\leqslant |f(N(x)\cup N(a_{i+1}))|+|N(a_1)\cup N(a_m)|=|f(N(x)\cup N(a_{i+1}))\cup (N(a_1)\cup N(a_m))|\leqslant p-2$.

A contradiction. The lemma has been proved.

Lemma 2. If P: $a_1a_2\ldots a_m$ is a path of maximum length in G, then there is no i,j such that

$$1< i\leqslant j< m,\quad a_1a_i\in E,\quad a_1a_{i-1}\bar{\in} E,\quad a_ja_m\in E,\quad a_{j+1}a_m\bar{\in} E. \tag{1}$$

Proof. If there is i,j such that (1) holds, then

P1: $a_{i-1}\ldots a_1a_i\ldots a_ja_m\ldots a_{j+1}$, P2: $a_{i-1}\ldots a_1a_i\ldots a_m$

and P3: $a_{j+1}\ldots a_ma_j\ldots a_1$

are all of paths of maximum length and $V(P1)=V(P2)=V(P3)=V(P)$. Thus,

$$N(a_1)\cup N(a_{i-1})\subseteq P, \quad N(a_m)\cup N(a_{j+1})\subseteq P .$$

By lemma 1, we have $(N(a_1)\cup N(a_{i-1}))\cap(N(a_m)\cup N(a_{j+1}))=\emptyset$. But $a_1, a_m\overline{\in}(N(a_1)\cup N(a_{i-1}))\cup(N(a_m)\cup N(a_{j+1}))$, Hence $p-1\leqslant|N(a_1)\cup N(a_{i-1})|+|N(a_m)\cup N(a_{j+1})|=|(N(a_1)\cup N(a_{i-1}))\cup(N(a_m)\cup N(a_{j+1}))|\leqslant p-2$ (Since $dist(a_1,a_{i-1})=dist(a_m,a_{j+1})=2$).

A contradiction. The lemma has been proved.

Lemma 3. There exists a path $P: a_1a_2\ldots a_m$ of maximum length and i,j such that

$$i<j, \quad a_i\in N(a_m), \quad a_j\in N(a_1) .$$

Proof. For any path $P: a_1a_2\ldots a_k$, let $i^*=\max[i|a_i\in N(a_1)]$, $j^*=\min[j|a_j\in N(a_k)]$. If for a path of maximum length, $i^*>j^*$ holds, then let $i^*=j$, $j^*=i$. The path of lemma 3 has been found. Now, suppose for any path of maximum length $i^*\leqslant j^*$ holds. We choose a path $P: a_1a_2\ldots a_m$ such that j^*-i^* is taken minimum. By lemma 2, without loss of generality, we suppose that $a_i\in N(a_1)$ for $2\leqslant i\leqslant i^*$. Since G is 2-connected, there exists e,f such that $e<i^*$, $f>i^*$, $a_ea_f\in E$. If $f\leqslant j^*$, then

$$a_{f-1}\ldots a_{e+1}a_1\ldots a_ea_f\ldots a_m$$

is also a path of maximum length. But the j^*-i^* of this path ia less than that of P. A contradiction. If $f>j^*$, then

$$a_e\ldots a_1a_{e+1}\ldots a_m$$

is the required path of maximum length with $i=j^*$, $j=f$.

Lemma 4. For any path $P: a_1a_2\ldots a_m$ of lemma 3, if $k\neq i,j$, then $N(a_k)\subseteq P$.

Proof. Let $i' = \min[k \mid a_t \in N(a_m)$ for $k \leq t \leq i]$, $j' = \min[k \mid a_t \in N(a_1)$ for $k \leq t \leq j]$. Since P is of maximum length, $G[a_1, a_2, \ldots a_m]$ can not be a hamiltonian graph. Hence $i' > 1$, $j' > i+1$, $\mathrm{dist}(a_1, a_{j'-1}) = 2$, $\mathrm{dist}(a_{i'-1}, a_m) = 2$.

1. If $a_{i'-1} \in N(a_1)$, then since $G[a_1, a_2, \ldots a_m]$ is not a hamiltonian graph, there exists t such that $i < t < j'$ and $a_t \notin N(a_m)$. By lemma 2 we know $a_s \in N(a_1)$ for $2 \leq s \leq i'-1$. Hence

$a_t \ldots a_1 a_{t+1} \ldots a_m$ is a path of maximum length and

$$N(a_t) \subseteq P \quad \text{for} \quad 1 \leq t \leq i'-2 \ .$$

2. If $a_{i'-1} \notin N(a_1)$, then since $G[a_1, \ldots a_m]$ is not a hamiltonian graph, we have $a_1, a_{i'-1}, a_{j'-1}, a_m \notin (N(a_1) \cup N(a_{j'-1})) \cup (N(a_{i'-1}) \cup N(a_m))$. Thus,

$$p-1 \leq |N(a_1) \cup N(a_{j'-1})| + |N(a_{i'-1}) \cup N(a_m)| \leq p-4 + |(N(a_1) \cup N(a_{j'-1})) \cap (N(a_{i'-1}) \cup N(a_m))| \ .$$

Hence

$$|(N(a_1) \cup N(a_{j'-1})) \cap (N(a_{i'-1}) \cup N(a_m))| \geq 3 \ .$$

Let $a_e, a_f \in (N(a_1) \cup N(a_{j'-1})) \cap (N(a_{i'-1}) \cup N(a_m))$ Since $a_{j'-1} \ldots a_1 a_{j'} \ldots a_m$; $a_{j'-1} \ldots a_{i'} a_m \ldots a_{j'} a_1 \ldots a_{i'-1}$; $a_1 \ldots a_m$ are all paths of maximum length, we have $a_e, a_f \in N(a_1) \cap N(a_{i'-1})$ by lemma 1. Without loss of generality, assume $e < f$. But if $f > i'$, then

$$a_{f-1} \ldots a_{i'} a_m \ldots a_f a_{i'-1} \ldots a_1$$

is a path of maximum length and $\mathrm{dist}(a_{f-1}, a_1) = 2$. This is a contradiction to lemma 1. Evidently, f is not equal to $i'-1, i'$.

We have $f < i'-1$. From the previous discussion we know $N(a_t) \subseteq P$ for $1 \leq t \leq f-1$. Again, since

$$a_{i'-1} \ldots a_1 a_{j'} \ldots a_m a_{i'} \ldots a_{j'-1}$$

is a path of maximum length and $a_{j'-1}a_{j'} \in E$, $a_{j'-1}a_m \bar{\in} E$, we have $a_t \in N(a_{i'-1})$ for $e \leq t \leq i'-2$, and

$$a_t \ldots a_{i'-1}a_{t-1} \ldots a_1 a_{j'} \ldots a_m a_{i'} \ldots a_{j'-1}$$

is a path of maximum length and $N(a_t) \subseteq P$ for $e+1 \leq t \leq i'-1$. Since $e<f$, hence $N(a_t) \subseteq P$ for $1 \leq t \leq i'-1$.

3. Since

$$a_t \ldots a_1 a_{j'} \ldots a_m a_{t+1} \ldots a_{j'-1}$$

is a path of maximum length for $i'-1 \leq t \leq i-1$. Thus, $N(a_t) \subseteq P$ holds for $i'-1 \leq t \leq i-1$.

From the above three aspects we know $N(a_t) \subseteq P$ for $1 \leq t \leq i-1$.

Apply the above discussion to the paths $a_{j-1} \ldots a_1 a_j \ldots a_m$ and $a_m \ldots a_1$, we will get $N(a_t) \subseteq P$ for $i+1 \leq t \leq j-1$ and $N(a_t) \subseteq P$ for $j+1 \leq t \leq m$. This completes the proof of the lemma.

Lemma 5. Let P: $a_1 a_2 \ldots a_m$ be a path of maximum length, $a_1 a_j \in E$. If there exists i such that $1 < i < j$ and $a_i \in N(a_m)$, then $a_t \bar{\in} N(a_m)$ for $1 \leq t \leq j-1$, $t \neq i$.

Proof. For $\forall x \bar{\in} P$, if $xa_k \in E$, then k=i or k=j by lemma 4. If there exists t such that $t<j$, $t \neq i$, $a_t \in N(a_m)$, then $N(a_i) \subseteq P$ by lemma 4. Hence a_j is a cut point in G. This is a contradiction to the fact that G is 2-connected. The proof of the lemma has been completed.

Now, we give the proof of the conjecture.

Theorem 2. Let G be a 2-connected graph of order p. If

$$NC2 \geq (p-1)/2$$

then G is traceable.

Proof. Suppose that G has no hamiltonian path. By lemma 3, we can choose a path P: $a_1a_2\ldots a_m$ of maximum length and i,j such that $i<j$, $a_i\in N(a_m)$, $a_j\in N(a_1)$. Let

$$A=\{a_1,a_2,\ldots,a_{i-1}\}\ ;\quad B=\{a_{i+1},\ldots,a_{j-1}\}\ ;\quad C=\{a_{j+1},\ldots,a_m\}$$

and suppose $|A|=\min\{|A|,|B|,|C|\}$ (or, we may substitute

$$a_m\ldots a_1 \qquad \text{or} \qquad a_{j-1}\ldots a_1a_j\ldots a_m$$

for P in discussion). Since G is 2-connected, by lemma 4, we can choose a vertex x of G such that $x\bar{\in}P$ and $xa_i\in E$.

Hence, we have

$$|G|=p\geqslant|A|+|B|+|C|+d(x)\ .$$

Since

$$a_{j+1}\ldots a_ma_i\ldots a_ja_1\ldots a_{i-1}$$

is of maximum length and $a_ja_{j+1}\in E$, $a_{i-1}a_i\in E$, by lemma 5, we know

$$N(a_{i-1})\subseteq(A\cup\{a_i\})\setminus\{a_{i-1}\}\ ,\ \text{similarly,}\ N(a_{i+1})\subseteq(B\cup\{a_i\})\setminus\{a_{i+1}\}.$$

Since $a_{i-1}a_{i+1}\bar{\in}E$, or

$$xa_ia_m\ldots a_{i+1}a_{i-1}\ldots a_1$$

is a path longer than P, we have $\mathrm{dist}(a_{i-1},a_{i+1})=2$. and $|N(a_{i-1})\cup N(a_{i+1})|\leqslant|A|+|B|-1$. Hence

$$|A|+|B|-1\geqslant|N(a_{i-1})\cup N(a_{i+1})|\geqslant NC2\geqslant(p-1)/2\geqslant(|A|+|B|+|C|+d(x)-1)/2$$

that is

$$|A|+|B|-2\geqslant|C|+d(x)-1\geqslant|A|+d(x)-1$$

thus

$$|B|\geqslant d(x)+1\ .$$

Sinçe P is of maximum length, $xa_{i-1}\bar{\in} E$ and $dist(x,a_{i-1})=2$ must hold . But $N(x)\cap N(a_{i-1})=\{a_i\}$, so $|N(x)\cup N(a_{i-1})|\leq d(x)+|A|-1$. Hence

$$d(x)+|A|-1\geqslant (|A|+|B|+|C|+d(x)-1)/2 \quad .$$

Thus

$$2d(x)+2|A|-2\geqslant |A|+|B|+|C|+d(x)-1\geqslant 2|A|+|B|+d(x)-1$$

that is

$$d(x)\geqslant |B|+1 \quad .$$

This is a contradiction to $|B|\geqslant d(x)+1$. This completes the proof of the theorem .

It is easily seen that theorem 1 is a corollary of theorem 2.

References

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