Nihonkai Math. J.
vol. 3 (1992), 67-72

NORM INEQUALITIES RELATED TO MCINTOSH TYPE INEQUALITY

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ABSTRACT. We consider norm inequalities associated to McIntosh type inequality $\|AB\| \le \|\mathrm{Re}\, BA\|$ which is closely related to the Heinz one. Consequently, we give a simple and elementary proof of the Heinz inequality.

1. Introduction. This work is a continuation of preceding paper [2] in some sense. Throughout this note, a capital letter means a (bounded linear) operator on a Hilbert space. Our starting point is the following norm inequality due to Heinz [7]:

Theorem A. *If A and B are positive operators, then*

$$\|AQ + QB\| \ge \|A^r QB^{1-r} + A^{1-r}QB^r\| \tag{1}$$

for $0 \le r \le 1$.

To give an elementary proof to the Heinz inequality, McIntosh [9] showed the following inequality which is just the case $r = 1/2$ in Theorem A.

Theorem B. *For arbitrary operators P, Q and R,*

$$\|P^*PQ + QRR^*\| \ge 2\|PQR\|. \tag{2}$$

Very recently, we pointed out in [2] that both inequalities (1) and (2) are equivalent to an interesting inequality recently obtained by Corach, Porta and Recht [1] that

$$\|STS^{-1} + S^{-1}TS\| \ge 2\|T\| \tag{3}$$

1980 *Mathematics Subject Classification* (1985 *Revision*). 47A30, 47B15 and 47B20.

for selfadjoint invertible operators S. It is understood as a key in their study on differential geometry.

Now, to prove (2), McIntosh required the following:

Theorem C. *If AB is selfadjoint, then*

$$\|AB\| \le \|\mathrm{Re}\, BA\|, \tag{4}$$

where $\mathrm{Re}\, T = (T + T^*)/2$ is the real part of T.

In this note, we consider such norm inequalities as (4). Furthermore, we pose another norm inequality equivalent to the Heinz one (1) via the McIntosh one (2), which is an improvement of the work in [2]. As a result, we have a simple and elementary proof of (1).

2. Norm inequalities. First of all, we note that (4) is obtained by combining the following facts:

Lemma 1. *Let $\sigma(T)$ (resp. $r(T)$) be the spectrum (resp. spectral radius) of T. Then*

$$\sigma(XY) = \sigma(YX) \quad \textit{up to } 0 \quad \textit{and} \quad r(XY) = r(YX). \tag{5}$$

In particular, if either X or Y is invertible, then

$$\sigma(XY) = \sigma(YX). \tag{5'}$$

Lemma 2. *If $\sigma(T)$ is contained in the real axis $\mathbf{R}$, then*

$$r(T) \le \|\mathrm{Re}\, T\|. \tag{6}$$

Proof. Since $\sigma(T)$ is contained in the closed numerical range $\overline{W(T)}$ of T, we have $\sigma(T) \subseteq \mathrm{Re}\, \overline{W(T)} = \overline{W(\mathrm{Re}\, T)}$. Hence it follows that either $r(T)$ or $-r(T)$ belongs to $\overline{W(\mathrm{Re}\, T)}$, so that $r(T) \le \|\mathrm{Re}\, T\|$.

For the sake of convenience, we write down a proof of (4):

$$\|AB\| = r(AB) = r(BA) \le \|\mathrm{Re}\, BA\|.$$

Now, in this section, we discuss some norm inequalities based on the above lemmas. Following Wigner [12], an operator T is *weakly positive* if T is similar to a positive operator. We note that the invertibility of T is not assumed in our definition. It is known that T is weakly positive if and only if $T = AB$ for some positive operators A and B, one of which is invertible, cf. [12; Theorems 1 and 3]. Actually, suppose that $T = S^{-1}CS$ for some positive C and invertible S and $S = UD$ is the polar decomposition of S. Then we have $T = D^{-2}DU^*CUD$.

Theorem 3. *Let T be a weakly positive operator with factorization $T = AB$ for some positive operators A and B. Then the norm of every factor is evaluated as*

$$\|B\| \leq \|\mathrm{Re}\, TA^{-1}\| \quad (\textit{resp.} \quad \|A\| \leq \|\mathrm{Re}\, B^{-1}T\|)$$

provided that A (resp. B) is invertible.

Proof. It follows from Lemmas 1 and 2 that

$$\|B\| = r(B) = r(A^{-1}T) = r(TA^{-1}) \leq \|\mathrm{Re}\, TA^{-1}\|.$$

The other case is quite similar.

Remark. In [3], we obtained a factorization of an idempotent operator T such as $T = EA$ for some $A \geq 0$ and projection E. From our viewpoint, this factorization can be improved as follows : Suppose that T is expressed as

$$T = \begin{pmatrix} 1 & B \\ 0 & 0 \end{pmatrix} \text{ on } \overline{ranT} \oplus (ranT)^{\perp}.$$

Then we take

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } A = \begin{pmatrix} 1 & B \\ B^* & b \end{pmatrix}.$$

If $b > \|B\|^2$, then A is positive invertible with the inverse

$$A^{-1} = \begin{pmatrix} 1 + BCB^* & -BC \\ -CB^* & C \end{pmatrix}, \text{ where } C = (b - B^*B)^{-1}.$$

Therefore we have $T = EA$ and so $A^{1/2}TA^{-1/2} = A^{1/2}EA^{1/2} \geq 0$. Consequently, we can see idempotent operators as typical examples of (noninvertible) weakly positive operators.

Next we discuss a similar norm inequality in some general setting. To do this, we introduce the weak selfadjointness for operators; an operator T is *weakly selfadjoint* if T is similar to a selfadjoint operator. It is easily checked that T is weakly selfadjoint if and only if $T = AB$ for some selfadjoint operators A and B, one of which is positive invertible.

We here recall that an operator T on H is paranormal [4] if $\|T^2x\|\|x\| \geq \|Tx\|^2$ for $x \in H$, and T is hyponormal if $\|Tx\| \geq \|T^*x\|$ for $x \in H$, or equivalently $T^*T \geq TT^*$. It is clear that every hyponormal operator is paranormal because $\|T^2x\|\|x\| \geq \|T^*Tx\|\|x\| \geq (T^*Tx, x) = \|Tx\|^2$.

Theorem 4. *Let T be a weakly selfadjoint operator, which can be wrriten as follows: $T = B^{-1}AB$ where A is selfadjoint and B is positive invertible. Then*

(i) if AB is paranormal, then $\|BT\| \leq \|\mathrm{Re}\, TB\|$.

(ii) if AB is hyponormal, then $BT = \mathrm{Re}\, TB$, that is, BT is itself selfadjoint and so is AB.

Proof. Since A is selfadjoint and $AB = BT$, it follows from (5') that

$$\sigma(TB) = \sigma(BT) = \sigma(AB) = \sigma(B^{1/2}AB^{1/2}) \subseteq \mathbb{R}.$$

Suppose that AB is paranormal. Then it is normaloid by [4] and [8], i.e., $r(AB) = \|AB\|$. Since $r(BT) = \|BT\|$ by $AB = BT$, and $\sigma(TB) \subseteq \mathbb{R}$, it follows from Lemmas 1 and 2 that

$$\|BT\| = r(BT) = r(TB) \leq \|\mathrm{Re}\, TB\|$$

Next suppose that AB is hyponormal. Then it is convexoid by [5] and [11], i.e., its closed numerical range coincides with the convex hull of its spectrum. Therefore , since $\sigma(AB) \subseteq \mathbb{R}$, it follows that AB is selfadjoint.

Remark. The statements in Theorem 4 have a bit of contrast. As a matter of fact, Halmos says that normal operators have two faces; that is, normaloid and convexoid ones. There is no implication between them, actually, there exist convexoid operators that are not normaloid and vice versa [6]. It is known that paranormality does not imply convexoidness [10] but so does hyponormality, and that paranormality implies normaloidness. It is obvious that (i) remains valid under the assumption AB belongs to subclasses of (nonconvexoid) normaloid operators, e.g., quasihyponormal, semihyponormal and k-paranormal operators, whose definitions are as follows ; $\|T^*Tx\| \le \|T^2x\|$ for $x \in H, (TT^*)^{1/2} \le (T^*T)^{1/2}$ and $\|T^{k+1}x\| \ge \|Tx\|^{k+1}$ for unit vectors $x \in H$ respectively. Incidentally, as a consequence of (ii), T turns out to be itself selfadjoint because A and B commute.

3. The Heinz inequality. In the preceding note [2], we discuss some norm inequalities equivalent to the Heinz inequality (1). Thus we add to such an norm inequality, which is a restricted version of (2) due to McIntosh.

Theorem 5. *The inequalities* (1), (2) *and* (3) *are equivalent to the inequality*

$$\|\mathrm{Re}\, A^2Q\| \ge \|AQA\| \tag{7}$$

for $A \ge 0$ *and selfadjoint operators* Q.

Proof. Since (2) implies (7) trivially, it suffices to show the converse. For arbitrary P, Q and R, we put

$$T = \begin{pmatrix} P & 0 \\ 0 & R^* \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & Q \\ Q^* & 0 \end{pmatrix}.$$

Then it follows from the assumption (7) that

$$\|T^*TS + ST^*T\| \ge 2\||T|S|T|\|. \tag{8}$$

Moreover we have the following (9) and (10) since $\| \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \| = \|X\|$.

$$\|T^*TS + ST^*T\| = \|P^*PQ + QRR^*\| \tag{9}$$

and

$$\||T|S|T|\| = \|TST^*\| = \|PQR\|, \tag{10}$$

so that (2) is obtained by (8), (9) and (10).

Finally we give a short proof to this proposed inequality (7), by which we also have an alternative elementary proof of the Heinz inequality (1).

Proof of (7). We only use Lemmas 1 and 2. Suppose that $A \ge 0$ and $Q = Q^*$. Then, since $\sigma(A^2Q) \subseteq \mathbf{R}$, we have

$$\|AQA\| = r(AQA) = r(A^2Q) \le \|\mathrm{Re}\, A^2Q\|.$$

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Recieved Mar. 23,1992

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